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RE-DERIVATION OF LAPLACE OPERATOR ON CURVILINEAR COORDINATES USED FOR THE COMPUTATION OF FORCE ACTING IN SOLENOID VALVES

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Abstract. This article presents two mathematical methods of derivation of the Laplace operator in a given curvilinear co-ordinate system. This co-ordinate system is defined in the area between the armature and the yoke of a high-speed solenoid valve (HSV). The Laplace operator can further be used for the numerical solving of the Laplace's equation in order to determine the electromagnetic force acting on the armature of the HSV. In further steps the author derived an expression for the gradient and the vector surface element of the armature side surface in this co-ordinate system. The solution of the derivation was compared with one other solution derived in the past for the computational investigations on HSVs.

Keywords: *Laplace operator, solenoid valve, armature, electromagnetic force*

1. Introduction

On account of the growing globalization and the rising competition in the industry, the enterprises must develop economically and use sophisticated calculation algorithms [1]. Electromagnetic HSVs can be pre-calculated with suitable mathematical models already at an early time of the construction phase. Solenoid valves (SV) are used in fluid power pneumatic and hydraulic systems to control cylinders, fluid power motors or larger industrial valves. Domestic washing machines and dishwashers use SVs to control water entry into the machine. SVs can be used for a wide array of industrial applications, including general on-off control, calibration and test stands, pilot plant control loops and process control systems. They are also set very widely in the automotive industry. A number of numerical algorithms concerning computation of high-speed solenoid valves were published in [1]. The aim of the paper is a re-derivation and further development of some of them. The special interest lies in the derivation of the Laplace operator. This operator can be used for numerical computation of an electromagnetic force (EMF) acting on the armature of the HSV. In [2] the authors carried out research of key factors on EMF of HSV. Further EMF is an input to the computation of e.g. armature eccentricity, which is needed for building models of HSV similar to [3]. Most studies on HSVs assumed that the armature is concentrically positioned in the sleeve. Under this assumption the transversal component of the magnetic force is equal to zero. Using the derived Laplace operator one can compute the armature eccentricity as a function of the sleeve thickness or as a function of hydraulic clearance between the armature and the sleeve. After finding the eccentricity one can compute the permeance of the radial air gap, which has a direct impact on the drop of the magnetomotive force and finally influences the driving component of the magnetic force. The presided determination of the EMF is also useful for the controlling of HSVs. In [4] the authors described the method of closed loop control for the closure time and hold current, which strongly depends on the EMF.

2. Preface to derivations

The region Ω_{xy} (see Fig. 1) is the computation domain for the solution to the Laplace's differential equation that is going to be solved using the method of finite differences. The inner border defined by the function $\zeta(\varphi)$ is the contour of the solenoid valve armature where: $0 \leq \varphi < 2\pi$. The outer border given by the function $\zeta + h$ is the inner side of the magnet yoke. It should be noticed that following transformations are valid only for the case of no contact between the armature and the inner side of the magnet yoke neither at $(y = 0, x = r)$ nor at any other circumferential position φ. The room between the borders has the permittivity compared to the permittivity of a vacuum.

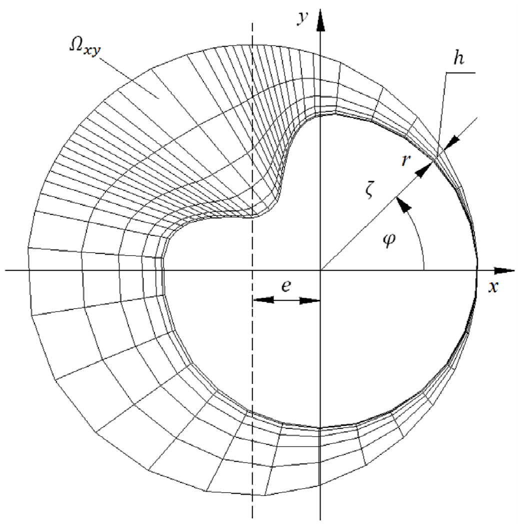

Fig. 1. Computation room of the Laplace's differential equation

The room Ω_{xy} is supposed to be discretized in a particular manner. Mesh lines in a radial direction are distorted in such a way that they exactly fit the inner and outer border. The number of mesh lines in a radial direction is kept constant independently from the distance h between the borders. That means that at the circumferential position $\varphi = 0$ there is still a positive distance $h(0) > 0$. In order to visualise the capacity of the transformation this distance at $\varphi = 0$ was purposely set in Figure 1 to a very small value. In general there is: $\bigvee_{0 \leq \varphi < 2\pi} h\,(\varphi) > 0$. This discretizing method has the advantage that regions with small h (in which the solution contributes dominantly to the sought force) are meshed more densely than regions with big h. That means that - in the difference to equidistant meshing methods - one can avoid an unnecessary fine mesh in the case of big nonmagnetic gaps. The density of mesh lines is variable both in the radial and peripheral direction. This method of discretizing allows for increase of computation precision with a simultaneous reduction of the mesh node numbers.

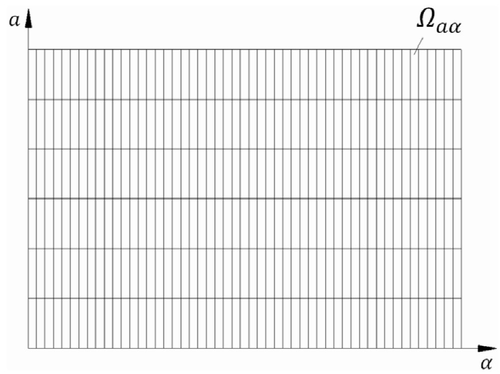

Fig. 2. Computation room of the Laplace's differential equation in a, α co-ordinate system

The aim of the work is to re-derive the Laplace operator to a co-ordinates system in which the computation domain gets a rectangle $\Omega_{a\alpha}$ shown in Figure 2. The derivation of the Laplace operator in a, α co-ordinate system was done using the following transformation:

$$n(a) = h(\varphi)^{-1}\big(r - \zeta(\varphi)\big) \tag{1}$$

$$\alpha = \alpha(\varphi) \tag{2}$$

Functions h and ζ in (1) are restricted only to functions which allow n being isomorphic and bijective in the whole domain $\Omega_{a\alpha}$. Furthermore, the range of validity of the function n is restricted to: $0 \leq n \leq 1$. In [1] this transformation was done using the transformation shoal

$$\Delta = (\Delta a)\partial_a + (\Delta\alpha)\partial_\alpha + (\nabla a)^2\partial_{aa} + (\nabla\alpha)^2\partial_{\alpha\alpha} + 2(\nabla a)(\nabla\alpha)\partial_{a\alpha} \tag{3}$$

with the operators:

$$\nabla a = a_r e^r + r^{-1} a_\varphi e^\varphi \tag{4}$$

$$\nabla \alpha = r^{-1} \alpha_\varphi e^\varphi \tag{5}$$

$$\Delta a = r^{-1} a_r + a_{rr} + r^{-2} a_{\varphi\varphi} \tag{6}$$

$$\Delta \alpha = r^{-2} \alpha_{\varphi\varphi} \tag{7}$$

The re-derivation of the Laplace operator will now be done in two ways: using the differential operators and using the differential geometry. In Table 1 the overview of derivation ways is presented.

Table 1

Overview of derivation methods of Laplace operator

Author	Vogel	Goraj
derivation using differential operators	$n = a, \alpha = \varphi$	$n = n(a), \alpha = \alpha(\varphi)$
derivation using the transformation shoal	$n = n(a), \alpha = \alpha(\varphi)$	not done
derivation using the differential geometry	not done	$n = n(a), \alpha = \alpha(\varphi)$

3. Derivation of Laplace operator using differential operators

The Laplace operator in polar co-ordinate system is given by [5]:

$$\mathit{\Delta} = \partial_{rr} + r^{-1}\partial_r + r^{-2}\partial_{\varphi\varphi} \tag{8}$$

The differential operator ∂_r can be split using rules of partial differentiation:

$$\partial_r = a_r \partial_a + \alpha_r \partial_\alpha \tag{9}$$

For the differential operator ∂_{rr} can be written:

$$\partial_{rr} = \partial_r(\partial_r) = \underbrace{\partial_r(a_r \partial_a)}_{A} + \underbrace{\partial_r(\alpha_r \partial_\alpha)}_{B} \tag{10}$$

The application of the product rules on the term A yields:

$$A = \partial_r(a_r)\partial_a + a_r \partial_r(\partial_a) \tag{11}$$

For the partial differential operator ∂_{ra} can be written with the use of (9)

$$\partial_r(\partial_a) = a_r \partial_a \partial_a + \alpha_r \partial_\alpha \partial_a = a_r \partial_{aa} + \alpha_r \partial_{\alpha a} \tag{12}$$

With the use of (12) the formula (11) can be simplified to:

$$A = a_{rr}\partial_a + a_r{}^2\partial_{aa} + a_r\alpha_r\partial_{\alpha a} \tag{13}$$

Analogously to *A* hold for *B*:

$$B = \partial_r(\alpha_r)\partial_\alpha + \alpha_r\partial_r(\partial_\alpha) \tag{14}$$

For the partial differential operator $\partial_{r\alpha}$ one can use (9) again, which gives:

$$\partial_r(\partial_\alpha) = a_r\partial_a(\partial_\alpha) + \alpha_r\partial_\alpha(\partial_\alpha) = a_r\partial_{a\alpha} + \alpha_r\partial_{\alpha\alpha} \tag{15}$$

The use of (15) in (14) yields:

$$B = \partial_r(\alpha_r)\partial_\alpha + \alpha_r a_r\partial_{a\alpha} + \alpha_r{}^2\partial_{\alpha\alpha} \tag{16}$$

The summation of *A* and *B* yields the operator ∂_{rr}

$$\partial_{rr} = a_{rr}\partial_a + \alpha_{rr}\partial_\alpha + 2\alpha_r a_r\partial_{a\alpha} + a_r{}^2\partial_{aa} + \alpha_r{}^2\partial_{\alpha\alpha} \tag{17}$$

Analogously to (17), one obtains $\partial_{\varphi\varphi}$:

$$\partial_{\varphi\varphi} = a_{\varphi\varphi}\partial_a + \alpha_{\varphi\varphi}\partial_\alpha + 2\alpha_\varphi a_\varphi\partial_{a\alpha} + a_\varphi{}^2\partial_{aa} + \alpha_\varphi{}^2\partial_{\alpha\alpha} \tag{18}$$

The unknowns in equations (17) and (18) are the first and the second derivative of a and α in r and φ – direction. From the definition (2) one obtains:

$$\alpha_r = 0 \tag{19}$$

$$\alpha_{rr} = 0 \tag{20}$$

The desolation of (1) for *r* leads to:

$$r = \zeta(\varphi) + n(a)h(\varphi) \tag{21}$$

The application of ∂_r in (21) gives:

$$1 = n_a a_r h \tag{22}$$

The application of ∂_r in (22) yields:

$$0 = n_{aa}a_r{}^2 h + n_a a_{rr} h \tag{23}$$

The desolation of (22) for a_r and the desolation of (23) for a_{rr} results in:

$$a_r = n_a{}^{-1}h^{-1} \tag{24}$$

$$a_{rr} = -n_{aa}n_a{}^{-3}h^{-2} \tag{25}$$

The application of ∂_φ in (21) yields:

$$0 = \zeta_\varphi + n_a a_\varphi h + n h_\varphi \tag{26}$$

The application of ∂_φ in (26) yields:

$$0 = \zeta_{\varphi\varphi} + n_{aa} {a_\varphi}^2 h + n_a a_{\varphi\varphi} h + 2 n_a a_\varphi h_\varphi + n h_{\varphi\varphi} \tag{27}$$

The desolation of (26) for a_φ and the desolation of (27) for $a_{\varphi\varphi}$ results in:

$$a_\varphi = -\frac{\zeta_\varphi + n h_\varphi}{n_a h} \tag{28}$$

$$a_{\varphi\varphi} = -\frac{1}{n_a h}\left(\zeta_{\varphi\varphi} + n_{aa} {a_\varphi}^2 h + 2 n_a a_\varphi h_\varphi + n h_{\varphi\varphi}\right) \tag{29}$$

Setting of (28) in (29) yields:

$$a_{\varphi\varphi} = -\frac{1}{n_a h}\left(\zeta_{\varphi\varphi} + n_{aa}\left(\frac{\zeta_\varphi + n h_\varphi}{n_a h}\right)^2 h - 2 n_a \left(\frac{\zeta_\varphi + n h_\varphi}{n_a h}\right) h_\varphi + n h_{\varphi\varphi}\right) \tag{30}$$

Setting of (9) and (17) in (19) and (20) yields:

$$\partial_r = a_r \partial_a \tag{31}$$

$$\partial_{rr} = a_{rr}\partial_a + {a_r}^2 \partial_{aa} \tag{32}$$

After the replacement of a_r and a_{rr} from (24) and (25) in (31) and (32), one obtains the transformed differential operators ∂_r and ∂_{rr}:

$$\partial_r = {n_a}^{-1} h^{-1} \partial_a \tag{33}$$

$$\partial_{rr} = -n_{aa} {n_a}^{-3} h^{-2} \partial_a + (n_a h)^{-2} \partial_{aa} \tag{34}$$

Setting now the relations (28) and (30) in (18) one gets the transformed differential operator $\partial_{\varphi\varphi}$:

$$\begin{aligned}\partial_{\varphi\varphi} = -\frac{1}{n_a h}\Bigg(\zeta_{\varphi\varphi} &+ n_{aa}\left(\frac{\zeta_\varphi + n h_\varphi}{n_a h}\right)^2 h - 2 n_a \left(\frac{\zeta_\varphi + n h_\varphi}{n_a h}\right) h_\varphi \\ &+ n h_{\varphi\varphi}\Bigg)\partial_a + \alpha_{\varphi\varphi}\partial_\alpha - 2\alpha_\varphi \frac{\zeta_\varphi + n h_\varphi}{n_a h}\partial_{a\alpha} \\ &+ \left(\frac{\zeta_\varphi + n h_\varphi}{n_a h}\right)^2 \partial_{aa} + {\alpha_\varphi}^2 \partial_{\alpha\alpha}\end{aligned} \tag{35}$$

After setting of the operators (33) to (35) in (8) one obtains - under the usage of (1) - the Laplace operator in a, α co-ordinate system:

$$\begin{aligned}\Delta = \Bigg(& \frac{1}{(\zeta + nh)n_a h} - \frac{n_{aa}}{{n_a}^3 h^2} \\ & - \frac{1}{(\zeta + nh)^2 n_a h}\left(\zeta_{\varphi\varphi} + \frac{n_{aa}}{h}\left(\frac{\zeta_\varphi + nh_\varphi}{n_a}\right)^2 \right. \\ & \left. \left. - 2\frac{\zeta_\varphi + nh_\varphi}{h} h_\varphi + nh_{\varphi\varphi}\right)\right)\partial_a + \frac{\alpha_{\varphi\varphi}}{(\zeta + nh)^2}\partial_\alpha \\ & + \left(\frac{1}{(n_a h)^2} + \left(\frac{\zeta_\varphi + nh_\varphi}{(\zeta + nh)n_a h}\right)^2\right)\partial_{aa} + \frac{{\alpha_\varphi}^2}{(\zeta + nh)^2}\partial_{\alpha\alpha} \\ & - 2\alpha_\varphi \frac{\zeta_\varphi + nh_\varphi}{(\zeta + nh)^2 n_a h}\partial_{a\alpha}\end{aligned} \tag{36}$$

The operator (36) is identical to the one derived in [1].

4. Derivation of Laplace operator using differential geometry

In differential geometry, the Laplace operator can be generalized to operate on functions defined on surfaces in Euclidean space. The more general operator is called the Laplace-Beltrami operator. This operator of a scalar function in any curvilinear co-ordinate system can be expressed using Einstein notation [6-8]:

$$\nabla_i \nabla^i V = \frac{1}{\sqrt{g}}\partial_i\left(\sqrt{g} g^{ij}\partial_j V\right) \tag{37}$$

g^{ij} in (37) is here the contravariant metric tensor of the second rank. Its general covariant form is [8]:

$$g_{ij} = \frac{\partial X^k}{\partial q^i}\frac{\partial X^k}{\partial q^j} \tag{38}$$

The k in (38) is the summation index. The position vector X is defined by (39):

$$X = \big(n(a)h(\varphi) + \zeta(\varphi)\big)\begin{pmatrix}\cos\varphi \\ \sin\varphi\end{pmatrix} \tag{39}$$

The variables of the position vector (39) are $q^1 = a$ and $q^2 = \alpha$. The derivatives of the position vector (39) are:

$$X_a = n_a h \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} \tag{40}$$

$$X_\alpha = \alpha_\varphi^{-1} \begin{pmatrix} (nh_\varphi + \zeta_\varphi)\cos\varphi - (nh+\zeta)\sin\varphi \\ (nh_\varphi + \zeta_\varphi)\sin\varphi + (nh+\zeta)\cos\varphi \end{pmatrix} \tag{41}$$

The use of (38) gives the components of the covariant metric tensors:

$$g_{11} = (n_a h)^2 \tag{42}$$

$$g_{12} = \frac{nh_\varphi + \zeta_\varphi}{\alpha_\varphi} n_a h \tag{43}$$

$$g_{21} = g_{12} \tag{44}$$

$$g_{22} = \frac{(nh_\varphi + \zeta_\varphi)^2 + (nh+\zeta)^2}{{\alpha_\varphi}^2} \tag{45}$$

The determinant of the metric tensors is equal to:

$$g = \frac{(n_a h)^2 (nh+\zeta)^2}{{\alpha_\varphi}^2} \tag{46}$$

The contravariant metric tensor is defined as [6]:

$$g^{ij} = g_{ij}^{-1} \tag{47}$$

The components of the metric tensor in the contravariant form are:

$$g^{11} = \frac{1}{(n_a h)^2} + \left(\frac{nh_\varphi + \zeta_\varphi}{n_a h (nh+\zeta)}\right)^2 \tag{48}$$

$$g^{12} = -\alpha_\varphi \frac{nh_\varphi + \zeta_\varphi}{n_a h (nh+\zeta)^2} \tag{49}$$

$$g^{21} = g^{12} \tag{50}$$

$$g^{22} = \frac{{\alpha_\varphi}^2}{(nh+\zeta)^2} \tag{51}$$

The Laplace operator simplifies in the considered case to:

$$\begin{aligned} \Delta = &\left(\partial_a(\sqrt{g}g^{11}) + \partial_\alpha(\sqrt{g}g^{21})\right)\frac{\partial_a}{\sqrt{g}} + \left(\partial_\alpha(\sqrt{g}g^{22}) + \partial_a(\sqrt{g}g^{12})\right)\frac{\partial_\alpha}{\sqrt{g}} \\ &+ g^{11}\partial_{aa} + g^{22}\partial_{\alpha\alpha} + 2g^{12}\partial_{a\alpha} \end{aligned} \tag{52}$$

The multiplier of $g^{-1/2}\partial_\alpha$ can be expressed as:

$$\partial_\alpha\left(\sqrt{g}g^{22}\right)+\partial_a\left(\sqrt{g}g^{12}\right)=\underbrace{\partial_\alpha\frac{n_a h\alpha_\varphi}{nh+\zeta}}_{A}-\underbrace{\partial_a\frac{nh_\varphi+\zeta_\varphi}{nh+\zeta}}_{B} \tag{53}$$

The first term of (53) is equal to:

$$A=\frac{n_a h_\varphi \varphi_\alpha \alpha_\varphi (nh+\zeta)+n_a h\alpha_{\varphi\varphi}\varphi_\alpha (nh+\zeta)-\left(nh_\varphi+\zeta_\varphi\right)\varphi_\alpha n_a h\alpha_\varphi}{(nh+\zeta)^2} \tag{54}$$

The second term of (53) is equal to:

$$B=\frac{n_a h_\varphi (nh+\zeta)-n_a h\left(nh_\varphi+\zeta_\varphi\right)}{(nh+\zeta)^2} \tag{55}$$

The subtraction of the terms (54) and (55) yields for (53):

$$\partial_\alpha\left(\sqrt{g}g^{22}\right)+\partial_a\left(\sqrt{g}g^{12}\right)=\frac{n_a h\alpha_{\varphi\varphi}}{\alpha_\varphi(nh+\zeta)} \tag{56}$$

The first part-multiplier of $g^{-1/2}\partial_a$ becomes:

$$\partial_a\left(\sqrt{g}g^{11}\right)=\underbrace{\partial_a\frac{nh+\zeta}{n_a h\alpha_\varphi}}_{C}+\underbrace{\partial_a\frac{\left(nh_\varphi+\zeta_\varphi\right)^2}{n_a h\alpha_\varphi(nh+\zeta)}}_{D} \tag{57}$$

The first term of (57) is equal to:

$$C=\frac{1}{\alpha_\varphi}-\frac{(nh+\zeta)n_{aa}}{h\alpha_\varphi {n_a}^2} \tag{58}$$

The second term of (57) is equal to:

$$D=2\frac{nh_\varphi+\zeta_\varphi}{h\alpha_\varphi(nh+\zeta)}h_\varphi-\frac{\left(nh_\varphi+\zeta_\varphi\right)^2 n_{aa}}{h\alpha_\varphi {n_a}^2(nh+\zeta)}-\frac{1}{\alpha_\varphi}\left(\frac{nh_\varphi+\zeta_\varphi}{nh+\zeta}\right)^2 \tag{59}$$

The second part-multiplier of $g^{-1/2}\partial_a$ becomes:

$$\partial_\alpha\left(\sqrt{g}g^{21}\right)=-\underbrace{\partial_\alpha\frac{nh_\varphi+\zeta_\varphi}{nh+\zeta}}_{E} \tag{60}$$

Differentiation of (60) gives:

$$E = \frac{1}{\alpha_\varphi}\frac{nh_{\varphi\varphi} + \zeta_{\varphi\varphi}}{nh + \zeta} - \frac{1}{\alpha_\varphi}\left(\frac{nh_\varphi + \zeta_\varphi}{nh + \zeta}\right)^2 \tag{61}$$

Addition of the terms C (58) and D (59) together with the subtraction of terms E (61) results in the multiplier of $g^{-1/2}\partial_a$. This multiplier equals:

$$\begin{aligned}\partial_a\left(\sqrt{g}g^{11}\right) + \partial_\alpha\left(\sqrt{g}g^{21}\right) &\\ = \frac{1}{\alpha_\varphi} - \frac{(nh+\zeta)n_{aa}}{h\alpha_\varphi {n_a}^2} &+ 2\frac{nh_\varphi + \zeta_\varphi}{h\alpha_\varphi(nh+\zeta)}h_\varphi \\ - \frac{\left(nh_\varphi + \zeta_\varphi\right)^2 n_{aa}}{h\alpha_\varphi {n_a}^2(nh+\zeta)} &- \frac{1}{\alpha_\varphi}\frac{nh_{\varphi\varphi} + \zeta_{\varphi\varphi}}{nh+\zeta}\end{aligned} \tag{62}$$

Finally, inserting of (56), (62) and (46)-(51) in (52) yields to the Laplace operator:

$$\begin{aligned}\Delta = &\frac{\alpha_{\varphi\varphi}}{(nh+\zeta)^2}\partial_\alpha \\ &+ \left(\frac{1}{(nh+\zeta)n_a h} - \frac{n_{aa}}{h^2 {n_a}^3} + 2\frac{nh_\varphi + \zeta_\varphi}{n_a h^2 (nh+\zeta)^2}h_\varphi \right.\\ &\left. - \frac{\left(nh_\varphi + \zeta_\varphi\right)^2 n_{aa}}{h^2 {n_a}^3 (nh+\zeta)^2} - \frac{nh_{\varphi\varphi} + \zeta_{\varphi\varphi}}{n_a h (nh+\zeta)^2}\right)\partial_a \\ &+ \left(\frac{1}{(n_a h)^2} + \left(\frac{nh_\varphi + \zeta_\varphi}{n_a h (nh+\zeta)}\right)^2\right)\partial_{aa} + \frac{{\alpha_\varphi}^2}{(nh+\zeta)^2}\partial_{\alpha\alpha} \\ &- 2\alpha_\varphi \frac{nh_\varphi + \zeta_\varphi}{n_a h (nh+\zeta)^2}\partial_{a\alpha}\end{aligned} \tag{63}$$

The operator (63) is identical to (36) and to the one derived in [1].

5. Gradient

The gradient of a scalar function V in any curvilinear co-ordinate system is a covariant vector defined as [7]

$$(\nabla V)_i = g^{ik}\frac{\partial X}{\partial q^i}\frac{\partial V}{\partial q^k} \tag{64}$$

The derivatives of the position vector (39) are given by (40) and (41). With the use of the new basis with unit vectors:

$$e^{a} = e^{r} \tag{65}$$

$$e^{\alpha} = \left(r^2 + \left(nh_\varphi + \zeta_\varphi\right)^2\right)^{-1/2} \left(\left(nh_\varphi + \zeta_\varphi\right)e^{r} + re^{\varphi}\right)\alpha_\varphi \tag{66}$$

and the use of (1) one can write these derivatives in more compact manner:

$$X_a = r_a e^{a} \tag{67}$$

$$X_\alpha = \left(r^2 + \left(nh_\varphi + \zeta_\varphi\right)^2\right)^{1/2} e^{\alpha} \tag{68}$$

The gradient (64) can be expressed by its components as follows:

$$(\nabla)_1 = g^{11} x_a \partial_a + g^{12} x_a \partial_\alpha \tag{69}$$

$$(\nabla)_2 = g^{21} x_\alpha \partial_a + g^{22} x_\alpha \partial_\alpha \tag{70}$$

The elements of the contravariant metric tensor (48) to (51) can also be written in a shorter way:

$$g^{11} = \frac{1}{r_a^2} + \left(\frac{\left(nh_\varphi + \zeta_\varphi\right)}{r_a r}\right)^2 \tag{71}$$

$$g^{12} = -\alpha_\varphi \frac{\left(nh_\varphi + \zeta_\varphi\right)}{r_a r^2} \tag{72}$$

$$g^{21} = g^{12} \tag{73}$$

$$g^{22} = \frac{{\alpha_\varphi}^2}{r^2} \tag{74}$$

The sum of (69) and (70) together with (67), (68) and with relations (71)-(74) yields:

$$\begin{aligned} \nabla = \frac{e^{a}}{r^2}&\left(\frac{r^2 + \left(nh_\varphi + \zeta_\varphi\right)^2}{r_a}\partial_a - \alpha_\varphi\left(nh_\varphi + \zeta_\varphi\right)\partial_\alpha\right) \\ &+ \frac{e^{\alpha}}{r^2}\sqrt{r^2 + \left(nh_\varphi + \zeta_\varphi\right)^2}\left(\alpha_\varphi \partial_\alpha - \frac{\left(nh_\varphi + \zeta_\varphi\right)}{r_a}\partial_a\right) \end{aligned} \tag{75}$$

The derivatives of (1) in a direction are:

$$r_a = n_a h \tag{76}$$

Setting of (1) and (76) in (75) results in the nabla operator in a, α co-ordinate system.

$$\begin{aligned}\nabla = \frac{e^a}{(nh+\zeta)^2}\left(\frac{(nh+\zeta)^2+\left(nh_\varphi+\zeta_\varphi\right)^2}{n_a h}\partial_a - \alpha_\varphi\left(nh_\varphi+\zeta_\varphi\right)\partial_\alpha\right) \\ + \frac{e^\alpha}{(nh+\zeta)^2}\sqrt{(nh+\zeta)^2+\left(nh_\varphi+\zeta_\varphi\right)^2}\left(\alpha_\varphi\partial_\alpha \right. \\ \left. - \frac{\left(nh_\varphi+\zeta_\varphi\right)}{n_a h}\partial_a\right)\end{aligned} \tag{77}$$

The operator (75) can also be expressed by means of the unit vectors of the r, φ basis:

$$\nabla = e^r\frac{\partial_a}{r_a} + \frac{e^\varphi}{r}\left(\alpha_\varphi\partial_\alpha - \frac{\left(nh_\varphi+\zeta_\varphi\right)}{r_a}\partial_a\right) \tag{78}$$

Now the operator (78) can be inspected by means of the operator (79) in polar co-ordinate system [5]:

$$\nabla = e^r\partial_r + r^{-1}e^\varphi\partial_\varphi \tag{79}$$

For the operator ∂_φ can be written by means of rules of partial differentiation:

$$\partial_\varphi = a_\varphi\partial_a + \alpha_\varphi\partial_\alpha \tag{80}$$

Substitution of a_φ from (28) yields:

$$\partial_\varphi = -\frac{\zeta_\varphi + nh_\varphi}{n_a h}\partial_a + \alpha_\varphi\partial_\alpha \tag{81}$$

After the setting of (19) and (24) in (9) one obtains the operator ∂_r:

$$\partial_r = \frac{\partial_a}{n_a h} \tag{82}$$

Finally, the usage of (81), (82) and (21) in (79) yields to

$$\nabla = \frac{e^r}{n_a h}\partial_a + \frac{e^\varphi}{\zeta + nh}\left(\alpha_\varphi\partial_\alpha - \frac{\zeta_\varphi + nh_\varphi}{n_a h}\partial_a\right) \tag{83}$$

Under the use of (21), (76) one can see that the operators (83) and (78) are identical to each other.

6. Vector surface element of the armature side surface

For the parameterization of the armature side surface the position vector (39) must be extended in the axial direction:

$$X = re^r + ze^z \tag{84}$$

The vector surface element can be obtained from the cross product of partial derivatives of (84) in the α and z direction:

$$dA = (X_\alpha \times X_z) d\alpha dz \tag{85}$$

The derivatives of (84) are:

$$X_\alpha = \alpha_\varphi^{-1}\left((nh_\varphi + \zeta_\varphi)e^r + re^\varphi\right) \tag{86}$$

$$X_z = e^z \tag{87}$$

Setting (86) and (87) in (85) yields the vector surface element of the armature side surface

$$dA = \alpha_\varphi^{-1}\left((\zeta + nh)e^r - (\zeta_\varphi + nh_\varphi)e^\varphi\right)_{a=0} d\alpha dz \tag{88}$$

7. Conclusions

The Laplace operator in the curvilinear co-ordinate system used for the numerical computation of electromagnetic force acting on the armature of high-speed solenoid valves was derived in three different ways: transformation shoal in [1], differential operators and differential geometry. All these three Laplace operators are identical to each other.

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