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ORTHGONALITY OF HERMITE POLYNOMIALS SYSTEM

Abstract

Introduction and aim: The paper presents some Hermite polynomials, orthogonality condition for Hermite polynomials, recurrence formula and differential equation for Hermite polynomials. The aim of the discussion was to give some proof of orthogonality of Hermite polynomial system.
Material and methods: Selected material based on some knowledge about Hermite polynomials which has been obtained from the right literature. The proof of the theorem describing the orthogonality of Hermite polynomials has been elaborated using a deduction method.
Results: Has been shown some proof of the theorem describing the orthogonality of Hermite polynomials. It has been shown an example of orthogonality testing a pair of two arbitrary Hermite polynomials.
Conclusions: In the paper has been shown the proof for theorem: The system of Hermite polynomials is orthogonal in the interval $(-\infty,+\infty)$ with the weighting function $p(z) = \exp(-z^2)$.

Keywords: The system of Hermite polynomials, theorem of Hermite polynomials orthogonality, proof.

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ORTOGONALNOŚĆ UKŁADU WIELOMIANÓW HERMITE'A

Streszczenie

Wstęp i cel: *W pracy przedstawiono wielomiany Hermite'a, warunek ortogonalności dla układu tych wielomianów, funkcję tworzącą oraz równanie różniczkowe dla wielomianów Hermite'a. Celem rozważań było przeprowadzenie dowodu twierdzenia o ortogonalności układów wielomianów Hermite'a.*
Materiał i metody: *Materiał stanowiły wybrane wiadomości o wielomianach Hermite'a uzyskane z literatury przedmiotu. W przeprowadzonym dowodzie zastosowano metodę dedukcji.*
Wyniki: *Pokazano dowód twierdzenia o ortogonalności układów wielomianów Hermite'a. Podano przykład badania ortogonalności pary dwóch dowolnych wielomianów Hermite'a.*
Wniosek: *W pracy przeprowadzono dowód twierdzenia: Układ wielomianów Hermite'a jest ortogonalny w przedziale $(-\infty,+\infty)$ z wagą $p(z) = \exp(-z^2)$.*

Słowa kluczowe: *Układ wielomianów Hermite'a, twierdzenie o ortogonalności układu wielomianów Hermite'a, dowód.*

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1. Analytical functions and orthogonal systems

Definition 1.

The function f(z) of the complex variable specified in a certain region D is called an analytic function in this area, where each point of the domain D has the first derivative of function f (z) [2].

Definition 2.

Two functions f (z) and g(z) defined in the interval ⟨a, b⟩ are called orthogonal functions in this interval with the weight function p(z) where the integral of the product of three functions p(z), f(z) and g(z) is equal to zero [2]:

$$\int_a^b p(z) f(z) g(z)\, dz = 0. \tag{1}$$

Consider the a system of functions $\{f_n(z)\}$ specified in the interval ⟨a, b⟩ and integrable in it, then as we know are also integrable some products of these functions taken together with the weight function p(z) as a third factor.

Definition 3.

If the functions of given system are pair wise orthogonal with weight function p(z), then the system is called orthogonal system of functions [2].

Definition 4.

The system functions $\{f_n(z)\}$, where n = 0, 1, 2, ... is called an orthogonal system with weight function p(z) in the range ⟨a, b⟩, if for each $m \neq n$ is true the following equality [1],[2]:

$$\int_a^b p(z) f_m(z) g_n(z)\, dz = 0 \tag{2}$$

where p(z) is a predetermined non-negative function that is independent from the indicators m and n known as a weight function.

2. Hermite polynomials

Definition 5.

Hermite polynomials $H_n(z)$ for the value of a variable z are determined by the formula [1]:

$$H_n(z) = (-1)^n \exp(z^2) \frac{d^n \exp(-z^2)}{dz^n} \quad \text{for} \quad n = 0, 1, 2, \ldots. \tag{3}$$

Let us compute a several Hermite polynomials and write down the general formula for the polynomial $H_{n-1}(z)$. A few Hermite polynomials calculated directly from the definition (3) is as follows:

$$H_0(z) = (-1)^0 \exp(z^2) \frac{d^0 \exp(-z^2)}{dz^0} = \exp(z^2)\exp(-z^2) = 1, \tag{4}$$

$$H_1(z) = (-1)^1 \exp(z^2) \frac{d^1 \exp(-z^2)}{dz^1} = -\exp(z^2)(-2z)\exp(-z^2) = 2z\,, \tag{5}$$

$$\begin{aligned} H_2(z) &= (-1)^2 \exp(z^2) \frac{d^2 \exp(-z^2)}{dz^2} = \exp(z^2) \frac{d}{dz}[(-2z)\exp(-z^2)] = \\ &= \exp(z^2)[(-2)\exp(-z^2) + (-2z)(-2z)\exp(-z^2)] = 4z^2 - 2\,, \end{aligned} \tag{6}$$

$$\begin{aligned} H_3(z) &= (-1)^3 \exp(z^2) \frac{d^3 \exp(-z^2)}{dz^3} = (-1)\exp(z^2) \frac{d^2}{dz^2}[(-2z)\exp(-z^2)] = \\ &= (-1)\exp(z^2) \frac{d}{dz}[(-2)\exp(-z^2) + (-2z)(-2z)\exp(-z^2)] = \\ &= 2\exp(z^2) \frac{d}{dz}[\exp(-z^2) + (-2z^2)\exp(-z^2)] = \\ &= 2\exp(z^2)[(-2z)\exp(-z^2) + (-4z)\exp(-z^2) + (-2z^2)(-2z)\exp(-z^2)] = \\ &= 2\exp(z^2)[(-6z)\exp(-z^2) + 4z^3\exp(-z^2)] = 8z^3 - 12z, \end{aligned} \tag{7}$$

$$\begin{aligned} H_4(z) &= (-1)^4 \exp(z^2) \frac{d^4 \exp(-z^2)}{dz^4} = \exp(z^2) \frac{d^3}{dz^3}[(-2z)\exp(-z^2)] = \\ &= \exp(z^2) \frac{d^2}{dz^2}[(-2)\exp(-z^2) + (-2z)(-2z)\exp(-z^2)] = \\ &= (-2)\exp(z^2) \frac{d^2}{dz^2}[\exp(-z^2) + (-2z^2)\exp(-z^2)] = \\ &= (-2)\exp(z^2) \frac{d}{dz}[(-2z)\exp(-z^2) + (-4z)\exp(-z^2) + (-2z^2)(-2z)\exp(-z^2)] = \\ &= (-2)\exp(z^2) \frac{d}{dz}[(-6z)\exp(-z^2) + 4z^3\exp(-z^2)] = \\ &= (-2)\exp(z^2) \frac{d}{dz}[(-6)\exp(-z^2) + (-6z)(-2z)\exp(-z^2) + 12z^2\exp(-z^2) + 4z^3(-2z)\exp(-z^2)] = \\ &= (-2)\exp(z^2) \frac{d}{dz}[(-6)\exp(-z^2) + 24z^2\exp(-z^2) + (-8z^4)\exp(-z^2)] = 12 - 48z^2 + 16z^4, \end{aligned} \tag{8}$$

....

$$H_{n-1}(z) = (-1)^{n-1} \exp(z^2) \frac{d^{n-1}}{dz^{n-1}} \exp(-z^2)\,, \tag{9}$$

....

and so on.

<u>Theorem 1</u>. *(About the generating function)*

Function

$$w(z,t) = \exp(2zt - t^2) \tag{10}$$

is the generating function for Hermite polynomials, i.e. for small values of $|t|$ there is the following series expansion [1]:

$$w(z,t) = \exp(2zt - t^2) \equiv \sum_{n=0}^{\infty} \frac{H_n(z)}{n!}\, t^n\,. \tag{11}$$

Theorem 2. *(The second order differential equation for Hermite polynomials)*

If $H_n(z)$ are Hermite polynomials, then [1]:

$$\frac{d^2m}{dz^2}+(2n+1-z^2)m(z)=0 \tag{12}$$

where

$$m(z)=\exp\left(-\frac{z^2}{2}\right)\cdot H_n(z) \tag{13}$$

for n = 0, 1, 2, … .

3. Orthogonality of Hermite polynomials system

One of the properties of Hermite polynomials is their orthogonality.

Theorem 3. *(Orthogonality of Hermite polynomials system) [1]*

The system of Hermite polynomials is orthogonal in the interval $(-\infty,+\infty)$ with weight function $p(z) = \exp(-z^2)$.

Proof:

To demonstrate the orthogonality of the Hermite polynomials system should be show in accordance with the formula (2) that:

$$\int_{-\infty}^{+\infty}\exp(-z^2)H_m(z)H_n(z)\,dz=0 \tag{14}$$

where $m \neq n$.

We use for proof equation (12) taking into account the following equality:

$$m_n(z)=\exp\left(-\frac{z^2}{2}\right)\cdot H_n(z). \tag{15}$$

The equation of m-th polynomial is multiplied by $m_n(z)$ and we get:

$$m_n(z)\frac{d^2m_m}{dz^2}+(2m+1-z^2)m_m(z)m_n(z)=0. \tag{16}$$

The equation of n-th polynomial is multiplied by $m_m(z)$ and we get:

$$m_m(z)\frac{d^2m_n}{dz^2}+(2n+1-z^2)m_n(z)m_m(z)=0. \tag{17}$$

So we subtract the obtained equations by sides, i.e. the second from the first. Therefore:

$$m_m(z)\frac{d^2m_n}{dz^2}+(2n+1-z^2)m_n(z)m_m(z)-m_n(z)\frac{d^2m_m}{dz^2}-(2m+1-z^2)m_m(z)m_n(z)=0. \tag{18}$$

From where:

$$m_m(z)\frac{d^2m_n}{dz^2}-m_n(z)\frac{d^2m_m}{dz^2}+2(n-m)m_m(z)m_n(z)=0. \tag{19}$$

To the left side of the equation (19) we add and subtract the expression $\frac{dm_n}{dz}\frac{dm_m}{dz}$ and apply the connectivity and commutative property of addition.

We therefore:

$$\left[m_m(z)\frac{d^2m_n}{dz^2}+\frac{dm_n}{dz}\frac{dm_m}{dz}\right]-\left[m_n(z)\frac{d^2m_m}{dz^2}+\frac{dm_n}{dz}\frac{dm_m}{dz}\right]+2(n-m)m_m(z)m_n(z)=0. \tag{20}$$

Hence:

$$\frac{d}{dz}\left[m_m(z)\frac{dm_n}{dz}\right]-\frac{d}{dz}\left[m_n(z)\frac{dm_m}{dz}\right]+2(n-m)m_m(z)m_n(z)=0. \tag{21}$$

From which it follows that:

$$\frac{d}{dz}\left[m_m(z)\frac{dm_n}{dz}-m_n(z)\frac{dm_m}{dz}\right]+2(n-m)m_m(z)m_n(z)=0. \tag{22}$$

Equation (22) we integrate in the range from –∞ to + ∞ and we obtain:

$$\int_{-\infty}^{+\infty}\frac{d}{dz}\left[m_m(z)\frac{dm_n}{dz}-m_n(z)\frac{dm_m}{dz}\right]dz+2(n-m)\int_{-\infty}^{+\infty}m_m(z)m_n(z)\,dz=\int_{-\infty}^{+\infty}0\,dz. \tag{23}$$

Now we calculate the value of the expression beneath the sign of the first derivative of the integral equation (23). We use it with the function (13).

Thus we have:

$$\begin{aligned}
&m_m(z)\frac{dm_n}{dz}-m_n(z)\frac{dm_m}{dz}=\\
&=\left[-z\exp\left(-\frac{z^2}{2}\right)H_n(z)+\exp\left(-\frac{z^2}{2}\right)\frac{dH_n(z)}{dz}\right]\exp\left(-\frac{z^2}{2}\right)H_m(z)+\\
&-\left[-z\exp\left(-\frac{z^2}{2}\right)H_m(z)+\exp\left(-\frac{z^2}{2}\right)\frac{dH_m(z)}{dz}\right]\exp\left(-\frac{z^2}{2}\right)H_n(z)=\\
&=-z\exp(-z^2)H_n(z)H_m(z)+\exp(-z^2)H_m(z)\frac{dH_n(z)}{dz}+\\
&+z\exp(-z^2)H_m(z)H_n(z)-\exp(-z^2)H_n(z)\frac{dH_m(z)}{dz}=\\
&=\left[H_m(z)\frac{dH_n(z)}{dz}-H_n(z)\frac{dH_m(z)}{dz}\right]\exp(-z^2).
\end{aligned} \tag{24}$$

Therefore the first integral in the equation (23) has the following form:

$$\begin{aligned}
&\int_{-\infty}^{+\infty}\frac{d}{dz}\left\{\left[H_m(z)\frac{dH_n(z)}{dz}-H_n(z)\frac{dH_m(z)}{dz}\right]\exp(-z^2)\right\}dz=\\
&=\lim_{a\to-\infty}\int_a^0\frac{d}{dz}\left\{\left[H_m(z)\frac{dH_n(z)}{dz}-H_n(z)\frac{dH_m(z)}{dz}\right]\exp(-z^2)\right\}dz+\\
&+\lim_{b\to+\infty}\int_0^b\frac{d}{dz}\left\{\left[H_m(z)\frac{dH_n(z)}{dz}-H_n(z)\frac{dH_m(z)}{dz}\right]\exp(-z^2)\right\}dz.
\end{aligned} \tag{25}$$

From which it follows that:

$$\int_{-\infty}^{+\infty}\frac{d}{dz}\left\{\left[H_m(z)\frac{dH_n(z)}{dz}-H_n(z)\frac{dH_m(z)}{dz}\right]\exp(-z^2)\right\}dz=$$
$$=\lim_{a\to-\infty}\left[H_m(z)\frac{dH_n(z)}{dz}-H_n(z)\frac{dH_m(z)}{dz}\right]\exp(-z^2)\Big|_a^0+$$
$$+\lim_{b\to+\infty}\left[H_m(z)\frac{dH_n(z)}{dz}-H_n(z)\frac{dH_m(z)}{dz}\right]\exp(-z^2)\Big|_0^b= \tag{26}$$
$$=\lim_{a\to-\infty}\left\{\left[H_m(0)\frac{dH_n(0)}{dz}-H_n(0)\frac{dH_m(0)}{dz}\right]-\left[H_m(a)\frac{dH_n(a)}{dz}-H_n(a)\frac{dH_m(a)}{dz}\right]\exp(-a^2)\right\}+$$
$$+\lim_{b\to+\infty}\left\{\left[H_m(b)\frac{dH_n(b)}{dz}-H_n(b)\frac{dH_m(b)}{dz}\right]\exp(-b^2)-\left[H_m(0)\frac{dH_n(0)}{dz}-H_n(0)\frac{dH_m(0)}{dz}\right]\right\}$$

Using the properties of the exponential function exp(z), we have that, when $a\to-\infty$ and $b\to+\infty$ are the adequate factors tend to zero. Then we have:

$$\int_{-\infty}^{+\infty}\frac{d}{dz}\left\{\left[H_m(z)\frac{dH_n(z)}{dz}-H_n(z)\frac{dH_m(z)}{dz}\right]\exp(-z^2)\right\}dz=$$
$$=\lim_{a\to-\infty}\left\{\left[H_m(0)\frac{dH_n(0)}{dz}-H_n(0)\frac{dH_m(0)}{dz}\right]-0\right\}+\lim_{b\to+\infty}\left\{0-\left[H_m(0)\frac{dH_n(0)}{dz}-H_n(0)\frac{dH_m(0)}{dz}\right]\right\}= \tag{27}$$
$$=H_m(0)\frac{dH_n(0)}{dz}-H_n(0)\frac{dH_m(0)}{dz}-H_m(0)\frac{dH_n(0)}{dz}+H_n(0)\frac{dH_m(0)}{dz}=0.$$

Considering the above results, and taking into account equality (23) we get finally:

$$2(n-m)\int_{-\infty}^{+\infty}m_m(z)m_n(z)\,dz=0 \quad \text{for} \quad m\neq n. \tag{28}$$

Dividing both sides of the above equality (28) by the expression $2(n-m)\neq 0$ we have that:

$$\int_{-\infty}^{+\infty}m_m(z)m_n(z)\,dz=0 \quad \text{for} \quad m\neq n. \tag{29}$$

Taking into account the function (13) we totally obtain:

$$\int_{-\infty}^{+\infty}\exp(-z^2)H_m(z)H_n(z)\,dz=0 \quad \text{for} \quad m\neq n. \tag{30}$$

That fact completes the proof of the theorem (3). ■

5. Conclusion

The system of Hermite polynomials is orthogonal in the interval $(-\infty,+\infty)$ with the weight function $p(z) = \exp(-z^2)$.

Literature

[1] Лебедев Н.Н.: *Специальные функци и их приложения,* Государственное Издательсто Физико-Математчеcкой Литературы, Москва-Ленинград 1963, издание второе.

[2] Leja F.: *Funkcje zespolone,* Biblioteka Matematyczna Tom 29, PWN Warszawa 1973, w. 3.