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Remarks on stability of magneto-elastic shocks

WŁODZIMIERZ DOMAŃSKI

Military University of Technology, Faculty of Cybernetics,
Institute of Mathematics and Cryptology,
2 Kaliskiego Str. 00-908 Warsaw, Poland
domanski.wlodek@gmail.com

Abstract. The problem of stability of plane shock waves for a model of perfect magneto-elasticity is investigated. Important mathematical properties, like loss of strict hyperbolicity and loss of genuine nonlinearity, and their consequences for the stability of magneto-elastic shocks are discussed. It is shown that some of these shocks do not satisfy classical Lax stability conditions. Both compressible and incompressible models of magneto-elasticity are discussed.
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1. Introduction

Magneto-elasticity is an example of coupled fields theory. It was developed many years ago (see e.g. classical book [4], and also later books [28] and [14]). One of the pioneers in this field were Polish scientists Sylwester Kaliski and Witold Nowacki. In a series of papers (see e.g. [20]–[24]) they analyzed different aspects of magneto-elastic interactions including the influence of thermal fields. The renewed interest in magneto-elastic interactions has been noticed with the possibility of new applications of magneto-sensitive elastomers, materials that change their mechanical behavior in response to the application of magnetic fields. More recently new constitutive formulation of magneto-elasticity based on a total energy density function has been developed in [11]. This was followed by a number of articles analyzing different aspects of magneto-elastic interactions (see e.g. [12], [6]) and the recent book ([13]).

We are interested in the problem of stability of shock waves for the equations of perfect magneto-elasticity. The modeling equations consist of Maxwell's system together with the equations of motion for an elastic medium, enhanced by the presence of the magnetic stresses. All equations are written in the material coordinates. Both geometrical and physical elastic nonlinearities are taken into account in the model. We analyze a compressible as well as an incompressible model. A perfect magneto-elastic medium is by definition an infinitely conducting, electrically neutral, non-magnetizable, homogeneous and isotropic elastic solid interacting with a magnetic field. We discuss some important mathematical properties, like loss of *strict hyperbolicity* and loss of *genuine nonlinearity*, and their consequences for the stability of magneto-elastic shocks. We show that some of these shocks do not satisfy classical Lax stability conditions [25].

In the mathematical literature (see e.g. [27]), shock waves are defined as special piecewise smooth discontinuous solutions of the first order hyperbolic systems of conservation laws. According to this definition, a shock wave is determined by a triple: a smooth singular surface $\mathcal{S}$ across which a conserved quantity $\boldsymbol{u}$ suffers a jump, and two functions $\boldsymbol{u}^+$ and $\boldsymbol{u}^-$ defined in respective domains Ω^+ and Ω^- on either side of this surface. The functions $\boldsymbol{u}^+$ and $\boldsymbol{u}^-$ are smooth solutions of the quasi-linear form of conservation laws in Ω^+ and Ω^-, respectively. Moreover, they satisfy *Rankine-Hugoniot* jump conditions [27] which relate to the values of the jumps in the conserved quantity and in the corresponding fluxes, to the shock speeds. Besides, shocks should satisfy certain stability requirements which usually follow from the second law of thermodynamics and are connected with the growth of the entropy across the shock front.

First mathematical papers devoted to shock stability were concerned with gas and fluid dynamics. Based on the applications in these fields, Lax [25] formulated general stability conditions for *plane* waves in *strictly hyperbolic* and *genuinely nonlinear* $m \times m$ systems:

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{A}(\boldsymbol{u}) \frac{\partial \boldsymbol{u}}{\partial x} = \boldsymbol{0} \tag{1.1}$$

where $\boldsymbol{A}(\boldsymbol{u})$ is an $m \times m$ matrix. *Strict hyperbolicity* means distinctiveness of the wave speeds (eigenvalues of matrix $\boldsymbol{A}(\boldsymbol{u})$), that is: $\lambda_j(\boldsymbol{u}) \neq \lambda_k(\boldsymbol{u})$ for $j \neq k$. *Genuine nonlinearity* of a k – wave is defined as follows:

$$\nabla_{\boldsymbol{u}} \lambda_k(\boldsymbol{u}) \cdot \boldsymbol{r}_k(\boldsymbol{u}) \neq 0 \tag{1.2}$$

where $\boldsymbol{r}_k(\boldsymbol{u})$ is an eigenvector of matrix $\boldsymbol{A}(\boldsymbol{u})$ corresponding to the eigenvalue $\lambda_k(\boldsymbol{u})$. Lax conditions generalize the *supersonic* ahead and *subsonic* behind the shock front, stability requirements known for gases. In gas dynamics they are equivalent to the physical condition of entropy growth under passage throw the discontinuity, hence even for abstract hyperbolic systems they are often called Lax *entropy* conditions. These conditions specify inequalities which every shock speed σ must satisfy to define a stable shock front, namely, the shock speed σ should be such that:

$$\lambda_k^+ < \sigma < \lambda_k^- \quad \wedge \quad \lambda_{k-1}^- < \sigma < \lambda_{k+1}^+ \tag{1.3}$$

where $\lambda_k^\pm$ are linearized wave speeds(eigenvalues of matrix $\boldsymbol{A}(\boldsymbol{u})$ of a k-wave evaluated at the appropriate side of the discontinuity). Equivalently, using Lax words [25], these requirements may be formulated as follows: *"There is an index k such that the shock speed lies between the* $(k-1)$ *and k-th characteristic speeds with respect to the state on the left of the shock, and between the k-th and* $(k+1)$ *characteristic speeds on the right."* Hence Lax shock inequalities determine the precise number of incoming and outgoing characteristics from the left and right hand side of the shock front.

Lax conditions can single out all stable shocks but only for strictly hyperbolic and genuinely nonlinear systems. However, these conditions do not give satisfactory stability results in many physical problems like e.g. multiphase flows, phase transitions or even purely elastic waves, so in particular for models with non-convex fluxes which arise quite naturally in many physical and mechanical problems where strict hyperbolicity and genuinely nonlinearity assumptions are typically violated. In such systems non-classical shocks may appear, that is e.g. shocks with too many incoming characteristics (*overcompressive shocks*) or too few characteristics (*undercompressive shocks*).

In this paper we first analyze the problem of loss of strict hyperbolicity and loss of genuine nonlinearity for the models of a perfect magneto-elasticity. Next we study the violation of Lax conditions for these models and the problem of existence of non-classical magneto-elastic shocks. We show that both kinds of these non-classical shock waves are possible in magneto-elasticity. We propose an asymptotic approach to derive simplified models for studying a subtle problem of stability of intermediate magneto-elastic shocks.

2. Compressible Perfect Magneto-elasticity

Dynamical equations of perfect magneto-elasticity were investigated at the beginning of the seventies by Bazer and some of his co-workers [1], [2] and [3] (see also Maugin's book [28]). The model of magneto-elasticity is not only challenging from the mathematical point of view, but it may have potential applications, apart from magneto-sensive elastomers mentioned in the introduction, also e.g. in astrophysics. The study of heavy neutron stars – pulsars, revealed (see e.g. [32] or [15]) that the outer crust of them could be treated as a solid. These small but very dense objects rotate very quickly producing sudden pulses of light. There are some speculations [32] that the model of magneto-elasticity may be helpful in explaining intriguing and strange phenomena connected with these peculiar objects i.e. pulsars.

The equations of perfect magneto-elasticity written in spatial (Eulerian) coordinates, marked here by a "prime", look as follows [2]:

$$\begin{aligned}
\mathrm{div}'\boldsymbol{B}' &= 0 \\
\frac{\partial \boldsymbol{B}'}{\partial t} + \mathrm{rot}'(\boldsymbol{B}' \times \boldsymbol{v}') &= \mathbf{0} \\
\rho'(\frac{\partial \boldsymbol{v}'}{\partial t} + \boldsymbol{v}' \cdot \mathrm{grad}'\boldsymbol{v}') - \mathrm{div}'\boldsymbol{T}' - \mathrm{rot}'\boldsymbol{B}' \times \boldsymbol{B}' &= 0 \\
\frac{\partial \rho'}{\partial t} + \mathrm{div}'(\rho'\boldsymbol{v}') &= 0 \\
\frac{\partial S'}{\partial t} + \boldsymbol{v}' \cdot \mathrm{grad}'S' &= 0
\end{aligned} \tag{2.1}$$

with $\boldsymbol{B}'$ – magnetic flux density, q' – density of electric charge ρ' – density, $\boldsymbol{v}'$ – velocity, $\boldsymbol{T}'$ – Cauchy stress tensor, S' – entropy.

Next we transform from spatial to material (Lagrangian) coordinates and restrict ourselves to the one-space dimensional case. This procedure is described in details in the Appendix I of [2]. The resulting equations look as follows:

$$\begin{aligned}
B_1 &= \mathrm{const.} \\
\frac{\partial (1+m_1)\boldsymbol{B}}{\partial t} - \frac{\partial (B_1 \mathbf{v})}{\partial x} &= \mathbf{0} \\
\frac{\partial \boldsymbol{v}}{\partial t} - \frac{\partial \boldsymbol{T}}{\partial x} - \frac{\partial \boldsymbol{M}}{\partial x} &= \mathbf{0} \\
\frac{\partial \boldsymbol{m}}{\partial t} - \frac{\partial \boldsymbol{v}}{\partial x} &= \mathbf{0} \\
\frac{\partial S}{\partial t} &= 0
\end{aligned} \tag{2.2}$$

here $\boldsymbol{B} = (B_1, B_2, B_3)^T$[1] is a vector of magnetic induction, $\boldsymbol{m} = (m_1, m_2, m_3)^T$ – displacement gradient vector, $\boldsymbol{v} = (v_1, v_2, v_3)^T$ – velocity, $\boldsymbol{T}$ – elastic stress, $\boldsymbol{M}$ – magnetic stress, S – entropy, x is a direction of wave propagation in Lagrangian coordinates. We assume from now on that the density ρ in material coordinates is normalized to 1.

2.1. Conservative and Quasilinear Forms

Writing system (2.2) in a standard form we have:

$$\frac{\partial \boldsymbol{F}_0(\boldsymbol{u})}{\partial t} + \frac{\partial \boldsymbol{F}_1(\boldsymbol{u})}{\partial x} = \boldsymbol{0}$$

with

$$\boldsymbol{u} = (B_2, B_3, v_1, v_2, v_3, m_1, m_2, m_3, S)^T = (\boldsymbol{B}_\perp, v_\parallel, \boldsymbol{v}_\perp, m_\parallel, \boldsymbol{m}_\perp, S)^T,$$

$$\boldsymbol{F}_0(\boldsymbol{u}) = \left((1+m_1)B_2,\ (1+m_1)B_3,\ v_1,\ v_2,\ v_3,\ m_1,\ m_2,\ m_3,\ S \right)^T,$$

$$\boldsymbol{F}_1(\boldsymbol{u}) = \left(B_1 v_2,\ B_1 v_3,\ T_{11}+M_{11},\ T_{12}+M_{12},\ T_{13}+M_{13},\ v_1,\ v_2,\ v_3,\ 0 \right)^T.$$

Specifying the components of magnetic stress tensor

$$M_{11} = \frac{1}{2\mu}(B_1^2 - B_2^2 - B_3^2), \quad M_{12} = \frac{1}{\mu} B_1 B_2, \quad M_{13} = \frac{1}{\mu} B_1 B_3,$$

and assuming that the medium is hyperelastic:

$$T_{1j} = \frac{\partial W}{\partial m_j} \equiv W'_j, \quad j = 1, 2, 3$$

with $W = W(m_1, m_2, m_3, S)$ — energy density, we write the system of perfect magneto-elasticity in the following *quasilinear form:*

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{A}(\boldsymbol{u}) \frac{\partial \boldsymbol{u}}{\partial x} = \boldsymbol{0} \tag{2.3}$$

with

$$\boldsymbol{A}(\boldsymbol{u}) = -\begin{pmatrix} 0 & 0 & -\frac{B_2}{1+m_1} & \frac{B_1}{1+m_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{B_3}{1+m_1} & 0 & \frac{B_1}{1+m_1} & 0 & 0 & 0 & 0 \\ -\tilde{B}_2 & -\tilde{B}_3 & 0 & 0 & 0 & W'_{11} & W'_{12} & W'_{13} & W'_{14} \\ \tilde{B}_1 & 0 & 0 & 0 & 0 & W'_{12} & W'_{22} & W'_{23} & W'_{24} \\ 0 & \tilde{B}_1 & 0 & 0 & 0 & W'_{13} & W'_{32} & W'_{33} & W'_{34} \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

[1] Superscript T denotes a transpose, so $(B_1, B_2, B_3)^T$ means a column vector.

where $\tilde{B}_j \equiv B_j/\mu$. It is possible to calculate (with the help of symbolic computations) all the eigenvalues of $\boldsymbol{A}(\boldsymbol{u})$ explicitly, however the formulas are extremely lengthy and we will not present them here. Instead we will analyze the linearized equations.

2.2. Linearized System

Let us first look at matrix $\boldsymbol{A}(\boldsymbol{u})$, linearized at the zero constant state $\boldsymbol{u}_0 = \mathbf{0}$. For simplicity, using the isotropy assumption, we may represent the energy function as $W = \Phi(m_1, N, S)$ with $N = m_2^2 + m_3^2$. Matrix $\boldsymbol{A}(\mathbf{0})$ looks then as follows:

$$\boldsymbol{A}(\mathbf{0}) = -\begin{pmatrix} 0 & 0 & 0 & B_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & B_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \Phi_{\prime 11} & 0 & 0 & \Phi_{\prime 1S} \\ \tilde{B}_1 & 0 & 0 & 0 & 0 & 0 & 2\Phi_{\prime N} & 0 & 0 \\ 0 & \tilde{B}_1 & 0 & 0 & 0 & 0 & 0 & 2\Phi_{\prime N} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

here $\Phi_{\prime 1} = \frac{\partial \Phi}{\partial m_1}$, $\Phi_{\prime N} = \frac{\partial \Phi}{\partial N}$, $\Phi_{\prime S} = \frac{\partial \Phi}{\partial S}$. The eigenvalues of $\boldsymbol{A}(\mathbf{0})$ are:

$$\begin{aligned} &\lambda_1 = -\sqrt{B_1\tilde{B}_1 + 2\Phi_{\prime N}} = \lambda_2 = -\lambda_3 = -\lambda_4, \\ &\lambda_5 = -\sqrt{\Phi_{\prime 11}} = -\lambda_6, \\ &\lambda_7 = \lambda_8 = \lambda_9 = 0. \end{aligned}$$

The corresponding right eigenvectors have the form:

$$\begin{aligned} \boldsymbol{r}_1 &= (0,\, B_1,\, 0,\, 0,\, -\lambda_1,\, 0,\, 0,\, 1,\, 0) && \text{(transverse)}, \\ \boldsymbol{r}_2 &= (B_1,\, 0,\, 0,\, -\lambda_2,\, 0,\, 0,\, 1,\, 0,\, 0) && \text{(transverse)}, \\ \boldsymbol{r}_3 &= (0,\, B_1,\, 0,\, 0,\, -\lambda_3,\, 0,\, 0,\, 1,\, 0) && \text{(transverse)}, \\ \boldsymbol{r}_4 &= (B_1,\, 0,\, 0,\, -\lambda_4,\, 0,\, 0,\, 1,\, 0,\, 0) && \text{(transverse)}, \\ \boldsymbol{r}_5 &= (0,\, 0,\, -\lambda_5,\, 0,\, 0,\, 1,\, 0,\, 0,\, 0) && \text{(longitudinal)}, \\ \boldsymbol{r}_6 &= (0,\, 0,\, -\lambda_6,\, 0,\, 0,\, 1,\, 0,\, 0,\, 0) && \text{(longitudinal)}, \\ \boldsymbol{r}_7 &= (0,\, -2\frac{\Phi_{\prime N}}{\tilde{B}_1},\, 0,\, 0,\, 0,\, 0,\, 0,\, 1,\, 0) && \text{(transverse)}, \end{aligned}$$

$$\boldsymbol{r}_8 = (-2\frac{\Phi_{,N}}{\tilde{B}_1},\ 0,\ 0,\ 0,\ 0,\ 0,\ 1,\ 0,\ 0\,) \qquad \text{(transverse)},$$

$$\boldsymbol{r}_9 = (\ 0,\ 0,\ 0,\ 0,\ 0,\ -\frac{\Phi_{,1S}}{\Phi_{,11}},\ 0,\ 0,\ 1\,) \qquad \text{(longitudinal)}.$$

At the zero constant state, we have three pairs of waves propagating in the mutually opposite directions in each pair. There are two pairs of doubled, coupled magneto–elastic shear waves, and one pair of longitudinal pure elastic waves. Hence even for the linearized equations of *the compressible perfect magneto-elasticity*, we have to deal with a *loss of strict hyperbolicity.*

2.3. Eigenvalues of $\boldsymbol{A}(\boldsymbol{u})$

As we have already mentioned, the formulas for the eigenvalues of matrix $\boldsymbol{A}(\boldsymbol{u})$ are cumbersome but when we drop out the third components of the magnetic field – B_3 and the gradient of displacement – m_3, these formulas simplify substantially. Namely let us assume that $\boldsymbol{B}_\perp = (\,B_2,\,0)$, $\boldsymbol{m}_\perp = (\,m_2,\,0)$. Then the squares of the nonzero eigenvalues of $\boldsymbol{A}(\boldsymbol{u})$ (the squares of the speeds of the intermediate, slow and fast waves) take the forms:

$$\begin{aligned} c_I^2 &= 2\Phi_{,N} + b_1^2, \\ c_S^2 &= \frac{1}{2}\left(\Phi_{,11} + 2\Phi_{,N} + 4\Phi_{,NN}\, m_2^2 + b_1^2 + b_2^2 - \sqrt{\Delta}\right), \\ c_F^2 &= \frac{1}{2}\left(\Phi_{,11} + 2\Phi_{,N} + 4\Phi_{,NN}\, m_2^2 + b_1^2 + b_2^2 + \sqrt{\Delta}\right), \end{aligned} \tag{2.4}$$

with $b_j^2 \equiv \frac{B_j^2}{\mu\ (1+m_1)}$ for $j = 1, 2$, and

$$\Delta = (\Phi_{,11} + b_2^2 - 2\Phi_{,N} - 4\Phi_{,NN}\, m_2^2 - b_1^2)^2 + 4(2\Phi_{,NN}\, m_2 - b_1 b_2)^2.$$

System (2.3) is hyperbolic under some additional assumptions on the energy potential Φ. We do not discuss this here but we will come back to this issue when we specify the energy potential Φ explicitly by the formula (2.7) for a special Hookean material.

2.4. Rankine-Hugoniot Jump Conditions

A shock wave in *solids* is related with the discontinuity in velocity, deformation gradient or gradient of displacement but not with displacement itself. Let us define the jump in quantity ϕ as

$$[\phi] = \phi^+ - \phi^-$$

where ϕ^+ and ϕ^- denote the values of ϕ ahesd and behind the shock front, appriopriately. Shocks in a perfect magneto-elastic compressible medium satisfy the following Rankine-Hugoniot (R-H) jump relations at the discontinuity

$$\begin{aligned} [B_1] &= 0 \\ \sigma\, [(1+m_1)\boldsymbol{B}_\perp] + [B_1\, \boldsymbol{v}_\perp] &= \mathbf{0} \\ \sigma\, [v_1] + \left[\Phi_{,1} - \frac{|\boldsymbol{B}_\perp|^2}{2\mu}\right] &= 0 \\ \sigma\, [\boldsymbol{v}_\perp] + \left[2\boldsymbol{m}_\perp \Phi_{,N} + \frac{B_1}{\mu}\boldsymbol{B}_\perp\right] &= \mathbf{0} \\ \sigma\, [\boldsymbol{m}] + [\boldsymbol{v}] &= \mathbf{0}. \\ |\sigma|\, [S] &\geq 0 \end{aligned} \tag{2.5}$$

Using the relations for the jump of the energy density, one can also derive the magneto-elastic *Hugoniot relation* which in our notation simplifies to

$$[\,\Phi\,] - \langle \Phi_{\boldsymbol{m}} \rangle [\boldsymbol{v}] + \frac{[m_1]}{4\mu}\, [\boldsymbol{B}_\perp]^2 = 0 \tag{2.6}$$

where $\langle \Phi_{\boldsymbol{m}} \rangle \equiv \frac{1}{2}\left((\Phi_{\boldsymbol{m}})^+ + (\Phi_{\boldsymbol{m}})^-\right)$. We will return to this formula in the concluding remarks.

Magneto-elastic (*ME*) shocks closely resemble *MHD* shocks [31]. In both systems we may distinguish fast, slow, intermediate, switch-off and switch-on shocks. Across the intermediate *MHD* shock as well as the intermediate *ME* shock, the sign of the tangential component of the magnetic field changes. However, while studying *ME* shocks, we have to overcome additional complications connected e.g. with the presence of shear stresses. To simplify the analysis, in the next subsection we specify the material and consider the special Hookean medium. Although the elastic constitutive relations in that model are physically linear, the magneto-elastic model is still highly nonlinear due to the coupling with the magnetic field.

2.5. Special Hookean Medium

For a special Hookean medium for which

$$\Phi = T_0 S + \frac{1}{2}(c_L^2 m_1^2 + c_T^2 N), \tag{2.7}$$

the formulas for the eigenvalues (2.4) reduce to

$$c_I^2 = c_T^2 + b_1^2,$$

$$c_{S,F}^2 = \frac{1}{2}\left(c_L^2 + c_T^2 + b_1^2 + b_2^2 \mp \sqrt{\Delta}\right)$$

with

$$\Delta = (c_L^2 - c_T^2)^2 + 2(c_L^2 - c_T^2)(b_2^2 - b_1^2) + (b_1^2 + b_2^2)^2 = b^4(\delta^2 - 2\delta\cos 2\alpha + 1),$$

where $b^2 \equiv b_1^2 + b_2^2$, $\delta \equiv (c_L^2 - c_T^2)/b^2$, α is an angle between vector $\boldsymbol{B} = (B_1, B_2, 0)$ and the x - axis; c_L and c_T are the speeds of the longitudinal and shear elastic waves. We assume that $c_L^2 > 2c_T^2$. Typically, under reasonable assumptions in most cases, we have that $c_S < c_I < c_F$. However, it may happen that locally $c_S = c_I < c_F$ or $c_S < c_I = c_F$ or else $c_S = c_I = c_F$. In particular, for a special Hookean material (2.7), for the angle $\alpha = 0$, that is $\boldsymbol{B} = (B_1, 0, 0)$, we have $c_F^2 = c_L^2$, and $c_S^2 = c_I^2 = c_T^2 + b_1^2$. On the other hand for $\alpha = \pi/2$, that is $\boldsymbol{B} = (0, B_2, 0)$, we have $c_F^2 = c_L^2 + b_2^2$, and $c_S^2 = c_I^2 = c_T^2$.

Hence we see that the eigenvalues may collide and the *strict hyperbolicity* assumption is *violated*. Similarly one can prove that the *genuine nonliearity* assumption also *fails*. Since these two assumptions are the key ingredients of the classical theory based on Lax conditions, therefore we need to apply different than classical approach to shock waves. First, in the next section, we will describe what kind of magneto-elastic shock waves appear in the special Hookean media.

2.6. Shocks in a Special Hookean Medium

Magneto-elastic shocks in the special Hookean medium satisfy the *R-H* conditions (2.5). Having specified the energy (2.7), we can write explicitly in terms of wave speeds, those *R-H* jumps which contain the energy (while the other *R-H* conditions remain the same). Namely, for jumps corresponding to the conservation of momentum equation, we have

$$\sigma\,[v_1] + \left[c_L^2 m_1 - \frac{|\boldsymbol{B}_\perp|^2}{2\mu}\right] = \mathbf{0}$$

$$\sigma\,[\boldsymbol{v}_\perp] + \left[c_L^2 m_1 + \frac{B_1}{\mu}\boldsymbol{B}_\perp\right] = \mathbf{0}.$$

In the case of a special Hookean medium, it is possible to derive from the *R-H* conditions, the explicit formulas for the shock speeds and express them in terms of the quantities, e.g. ahead or behind of the shock front. Comparing the shock velocities with characteristic speeds in front and behind of the shock front we get the information about shock stability.

An interesting feature can be observed for *slow* magneto-elastic shocks waves in the special Hookean medium. Namely, assume that the transverse magnetic field is perpendicular to the x-axis, and denote the relative strength of the shock by

$$\beta = [B_2]/|\boldsymbol{B}^-|$$

where $[B_2] = B_2^+ - B_2^-$, and $|\boldsymbol{B}^-| = \sqrt{(B_1^-)^2 + (B_2^-)^2}$. Then we have noticed that for a certain range of a parameter β, *slow* magneto-elastic shock waves satisfy classical geometric Lax conditions, that is they are subslow ahead while beeing superslow and subintermediate behind the shock front:

$$c_S^+ < \sigma_S < c_S^-,$$

$$\sigma_S < c_I^+.$$

There are yet values of β for which *slow* magneto-elastic shocks are *overcompressive*, that is, they are subslow behind, and superintermediate and subfast ahead of the shock front. In this case there are too many incoming characteristics into the shock front:

$$c_I^+ < \sigma_S < c_F^+,$$

$$\sigma_S < c_S^-.$$

On the other hand *intermediate* magneto-elastic shocks may as well in some range become *overcompressive* but they may also be *undercompressive*, in which case the following inequalities hold:

$$c_I^+ < \sigma_{SI} < c_I^-,$$

$$c_S^+ < \sigma_I < c_F^-.$$

Both overcompressive and also undercompressive shocks belong to the family of nonclassical shocks. Based on the analysis of a pure hyperbolic perfect magneto-elasticity model, such shocks were considered unphysical and have been rejected as unstable [3].

In the remaining part of this work we will concentrate on the incompressible model of perfect magneto-elasticity. The additional assumption of incompressibility allows us to keep a general energy density function Φ in the equations without restriction to a special Hookean material.

3. Incompressible Perfect Magneto-elasticity

Under the incompressibility assumption, the equations of perfect magneto-elasticity take the following form [9]:

$$\begin{aligned} \frac{\partial \boldsymbol{B}_\perp}{\partial t} - \frac{\partial (B_1\, \boldsymbol{v}_\perp)}{\partial x} &= \boldsymbol{0} \\ \frac{\partial \boldsymbol{v}_\perp}{\partial t} - \frac{\partial (2\Phi_{\prime N}\boldsymbol{m}_\perp + \frac{B_1}{\mu}\boldsymbol{B}_\perp)}{\partial x} &= \boldsymbol{0} \\ \frac{\partial \boldsymbol{m}_\perp}{\partial t} - \frac{\partial \boldsymbol{v}_\perp}{\partial x} &= \boldsymbol{0} \\ \frac{\partial S}{\partial t} &= 0 \end{aligned} \tag{3.1}$$

where now $\Phi = \Phi(N, S)$, $\boldsymbol{B}_\perp, \boldsymbol{v}_\perp$, and $\boldsymbol{m}_\perp$ are the transverse components of the appropriate fields, $B_1 = \text{const.}$, $v_1 = 0$ and $w_1 = 0$, the density ρ as before is normalized to 1.

3.1. Quasilinear Form

Let us denote the vector of dependent variables which consists of *transverse* components of the magnetic field, velocity and strain vectors as $\boldsymbol{u} = (\boldsymbol{B}_\perp, \boldsymbol{v}_\perp, \boldsymbol{m}_\perp, S)^T$. Writing the above system (3.1) in a quasilinear form (2.3), we can calculate the eigenvalues of the appropriate matrix:

$$\boldsymbol{A}(\boldsymbol{u}) = -\begin{pmatrix} 0 & 0 & B_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_1 & 0 & 0 & 0 \\ -\widetilde{B_1} & 0 & 0 & 0 & P_2 & Q & R_2 \\ 0 & \widetilde{B_1} & 0 & 0 & Q & P_3 & R_3 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.2}$$

here $\tilde{B_1} \equiv \frac{B_1}{\mu}, Q \equiv 4m_2m_3\Phi_{\prime NN}$, $P_j \equiv 2\Phi_{\prime N} + 4m_j^2\Phi_{\prime NN}$, and $R_j \equiv$ $\equiv 2m_j\Phi_{\prime NS}$ for j = 2, 3. Let $c_A^2 \equiv B_1^2/\mu$ be the speed of Alfven wave.

Under the following assumptions: $\Phi_{\prime NN} \geq 0$ and $\Phi_{\prime N} \geq -\frac{1}{2}c_A^2$, the quasilinear system with $\boldsymbol{A}(\boldsymbol{u})$ from (3.2) is *hyperbolic*. We have a pair of fast waves with speeds $\mp c_F$ such that $c_F^2 = c_A^2 + 2\Phi_{\prime N} + 4N\Phi_{\prime NN}$, and a pair of slow waves $\mp c_S$: $c_S^2 = c_A^2 + 2\Phi_{\prime N}$. When $\Phi_{\prime NN} = 0$, the speeds of fast and slow waves coincide and become the speeds of a pair of double intermediate waves $\pm c_I$: $c_I^2 = c_A^2 + 2\Phi_{\prime N}$.

3.2. Rankine-Hugoniot Jump Conditions

Incompressible magneto-elastic shocks satisfy the following Rankine-Hugoniot jump conditions at the surface of discontinuity

$$\sigma\,[\boldsymbol{B}_\perp] + [B_1\,\boldsymbol{v}_\perp] = \boldsymbol{0}$$
$$\sigma\,[\boldsymbol{v}_\perp] + \left[2N\Phi_{\prime N} + \frac{B_1}{\mu}\boldsymbol{B}_\perp\right] = \boldsymbol{0}$$
$$\sigma\,[\boldsymbol{m}_\perp] + [\boldsymbol{v}_\perp] = \boldsymbol{0}$$
$$|\sigma|\,[S] \geq 0.$$

In the intermediate incompressible magneto-elastic shocks, $|\boldsymbol{m}_\perp|^2$ does not change across the shock front, however, $\boldsymbol{m}_\perp$ may rotate across the shock. Since

$$[\Phi_{\prime N}]\ ([\boldsymbol{m}_\perp] \times <\boldsymbol{m}_\perp>) = \boldsymbol{0}$$

where $<\boldsymbol{m}_\perp> = \frac{1}{2}(\boldsymbol{m}_\perp^+ + \boldsymbol{m}_\perp^-)$, so we can classify the incompressible magneto-elastic shocks as *linearly* polarized (fast, slow, switch-on, switch-off), satisfying

$$[\boldsymbol{m}_\perp] \times <\boldsymbol{m}_\perp> = \boldsymbol{0},$$

and *circularly* polarized shocks (intermediate), satisfying

$$[\Phi_{\prime N}] = 0.$$

The speed of an intermediate shock wave σ_I, calculated from the R-H condition is equal to $\sigma_I^2 = c_A^2 + 2\Phi_{\prime N}{}^- = c_A^2 + 2\Phi_{\prime N}{}^+$.

3.3. Remarks on Stability of Intermediate Shocks

The stability of intermediate shock waves is a very controversial issue. Even for a simpler than magneto-elasticity but similar model of magnetohydrodynamics, there was a long lasting debate whether *MHD intermediate shocks* are stable or not. There was a strong belief in the scientific community (see e.g. [19]) that some properties of intermediate shocks were inconsistent with the mathematical theory of discontinues solutions of hyperbolic conservation laws. Therefore intermediate shocks should be disregarded and treated as unstable. However, later numerical simulations (see e.g. [5]) revealed that intermediate *MHD* shocks are stable at least for a range of material parameters.

The problem is that idealized mathematical theory that we have presented here, is not a suitable tool to investigate intermediate and other nonclassical shocks stability. A pure hyperbolic model in which a shock wave is treated as a discontinuity is not enough. One has to analyze the structure of a shock front. Therefore one needs to enhance the hyperbolic model by viscous, thermal and other effects like it was done e.g. for the equations of elasticity in [16], [17] or [18]. Then using e.g. a viscous profile criterion one can decide whether a given shock profile is stable. This criterion is based on studying travelling waves solutions to the conservation laws enhanced by the viscous and strain gradient terms. The problem of the stability of shocks is reduced to the existence of trajectories (viscous profiles) joining the critical points of the obtained dynamical system. Undercompressive shocks are very difficult to investigate by this method because they correspond to saddle - saddle connections which are very subtle to study. Moreover, the fact that the full system of magneto-elasticity is large and complicated makes the analysis even harder. Therefore it is very helpful to use the method of weakly nonlinear asymptotics and reduce the original equations to some canonical simplified models for small amplitude waves. We have derived asymptotic models (see e.g [7], [8], [9]) which are new in the context of magneto-elasticity, and which as we hope will be useful in the analysis of nonclassical magneto-elastic shocks. The obtained asymptotic models for small wave amplitudes are of the type of *mKdV-Burgers* equations, *complex Burgers* equations, etc. The analysis of viscous profiles for such equations is much easier then for original system. Hence, one can derive stability results for intermediate magneto-elastic shock waves on the basis of the study of these simplified models. The detailed analysis is yet to be done.

4. Concluding Remarks

We have presented a mathematical theory of stability of plane magneto-elastic shock waves based on Lax conditions. We have shown that these conditions are not satisfied for some of the magneto-elastic shocks and so these shocks were found to be nonclassical. On the example of magneto-elasticity we can see that classical mathematical theory of shock waves which disregards dissipative, dispersive and other physical effects, and which treats shocks as discontinues solutions of hyperbolic conservation laws is not satisfactory. This theory fails when we have to deal with nonclassical shocks. We propose to use the asymptotic method of multiple scale to derive simplified canonical models which are tractable as far as the travelling waves analysis is concerned. This method allows to draw conclusion about the stability of nonclassical shocks.

One should also emphasize that thermodynamics must be included in any consideration of shock stability, since shock stability is closely related to the growth of entropy. Maugin [30] showed that the entropy production rate is related to the power expanded by the driving force which acts on the material manifold. This approach has direct applications to shock stability since such a force is involved in a *kinetic relation*. This relation is an additional constraint which some of the nonclassical shock waves must satisfy in order to be stable [26]. In particular Maugin [29] has given an explicit form of the kinetic relation in magneto-elasticity based on the Hugoniot relation (2.6).

Part of this work is based on the conference proceedings paper [10].

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W. DOMAŃSKI

Uwagi na temat magneto-sprężystych fal uderzeniowych

Streszczenie. Rozważono problem stabilności fal uderzeniowych dla modelu doskonałej magneto-sprężystości. Przedyskutowano ważne matematyczne własności takie jak utrata ścisłej hiperboliczności i istotnej nieliniowości oraz ich konsekwencje dla stabilności magneto-sprężystych fal uderzeniowych. Pokazano, że pewne z tych fal nie spełniają klasycznego warunku stabilności Laxa. Zanalizowano zarówno ściśliwe jak i nieściśliwe modele magneto-sprężystości.
Słowa kluczowe: doskonała magneto-sprężystość, fale uderzeniowe, warunki stabilności
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