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The selected problems of controllability of discrete-time switched linear systems with constrained switching rule

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Abstract. In this paper the controllability problem for discrete-time linear switched systems is considered. The main goal is to find a control signal that steers any initial state to a given final state independently of the switching signal. In the paper, it is assumed that there are some constraints posed on the switching signal. Moreover, we present a necessary and sufficient conditions of some kinds of controllability. Three types of controllability, namely: from zero initial state to any final state, from any initial state to zero final state and from any initial state to any final state are considered. Finally, three illustrative examples are shown.

Key words: controllability, switched systems, discrete-time linear systems, switching rule, hybrid systems.

1. Introduction

Hybrid systems are the kind of the dynamical systems that connect simultaneously several kinds of dynamical behaviours in different parts of the system (e.g. discrete-time, continuous-time, jump phenomena and others). These systems are one of the most popular in the last decade (see e.g. [1–8]). Examples of such systems include constrained robotic systems [9], sampled-data systems [10], discrete event systems [11], inteligent vehicles/ highway systems [12] and many other types of systems [5, 6].

Switched systems are hybrid systems that consist of several subsystems. The switchings between these subsystems are dependent on the switching policy. A special class of switched systems are linear systems. They provide a framework that connects the linear systems and the complex and/or uncertain systems. Moreover, the study of switched linear systems provides additional insights into some long-standing and sophisticated problems, such as intelligent control, adaptive control and robust analysis.

Theoretical examination of switched linear systems is difficult due to their complex dynamics. Switching makes these systems much more complicated than standard-time invariant or even time-varying systems. Many intricated behaviours/dynamics and fundamentally new properties, which standard systems do not have, have been demonstrated on switched linear systems. From the control system design point of view, switching brings an additional degree of freedom in control system design. Switching rules, in addition to control laws, may be utilized to manipulate switched systems to achieve a better performance of a system. This can be seen as an added advantage for control design to attain certain control purposes like stabilizability or controllability.

The controllability problem is one of the most important concept in mathematical control theory [13–17]. In general, the controllability means, that it is possible to control dynamical system from an arbitrary initial state to an arbitrary final state using the set of admissible controls.

Systematic study of such a problem was started in the 1960's, when the theory of controllability based on the state space description for both time-invariant and time-varying linear control systems. The concept of controllability was first presented by Kalman and referred to linear dynamical systems. Because the most of practical dynamical systems are nonlinear, that is why, in recent years various controllability problems for different types of nonlinear or semilinear dynamical systems have been considered in many publications and monographs. Simultaneously there are more and more tools used to investigate controllability problems, for example, fixed point theorems and measures of noncompactness for function spaces [18]. The study of controllability for linear systems has spanned a great number of research directions (e.g. fractional and/or positive [19–23]). Testing degrees of controllability, and their numerical analysis aspects, are still a subject of intensive research. In the literature there are many different definitions of controllability, e.g. complete controllability [24], approximate controllability [25], exact controllability [26], trajectory-controllability [27]. Research of the controllability problems for different types of dynamical systems requires the application of numerous mathematical concepts and methods taken directly from differential geometry, functional analysis, topology, matrix analysis and theory of ordinary and partial differential equations and theory of difference equations [28].

In general case, for controllability analysis of switched linear control systems, a much more difficult situation arises since both the control input and the switching rule are designed variables to be determined. Thus, the interaction between them is very important from controllability point of view. For switched linear discrete-time control system, in gen-

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eral case the controllable set is not a subspace but a countable union of subspaces. In continuous-time case, the controllable set is an uncountable union of subspaces.

This paper is devoted to controllability of discrete-time linear switched systems (see [5] for definition and motivation). Such a system can be seen as a collection of discrete stationary linear systems between which is followed by the switching signal. The phenomenon over which we have control (i.e. change of regulator parameters, gear ratio) or uncontrolled events (failures, changing of the operating point) may be modelled by the switching rule. In the literature (e.g. [29–35]) the controllability of hybrid systems is related to the first case and then, the controllability problem is formulated as a problem of finding control and switching signal which steers an initial condition to a given final state. In this case the switching signal plays a role of additional control. Our paper is devoted to the second case. Hence, we are looking for a control, that regardless of the switching signal, steers an initial condition to a given final state. This situation is similar to a problem of controllability of jump linear systems ([36,37]) but in our framework we do not have a probabilistic model of the switching signal. Moreover, a new contribution of the paper is that we take into account the situation that usually occurs in engineering practice, i.e. in which certain switching sequences are not possible.

2. Notation and definitions

In this section we consider a class of linear switched systems described by difference state equation [33–35]:

$$x(k+1) = A(r(k))x(k) + B(r(k))u(k) \tag{1}$$

for $k \geq 0$, where $x(k) \in \mathbb{R}^n$ denotes the state vector; $r(k) \in \{1, 2, ..., s\} =: S$ is the switching signal; $u(k) \in \mathbb{R}^m$ is the control input, $k = 0, 1, ...$

Moreover, for $r(k) = i$, $A_i := A(i)$ and $B_i := B(i)$ are constant matrices of appropriate dimensions. Let us denote by $x(k, x_0, i_0, u)$ the solution of (1) at time k, under the control u with initial condition x_0 at time $k = 0$ and switching signal satisfying $r(0) = i_0$. The control

$$u = (u(0), u(1), ...)$$

is supposed to be such that

$$u(k) = f_k\left(r(0), r(1), ..., r(k)\right).$$

It means that the control $u(k)$ at time k depends only on the variables $r(0), r(1), ..., r(k)$. Such control will be called an admissible control.

Now, let us introduce the following notation:

$$\begin{aligned} F(k,k) &= I_{n\times n}, \\ F(k,l,i_{k-1},...,i_l) &= A(i_{k-1})A(i_{k-2})...A(i_l), \\ F_r(k,l) &= A\left(r(k-1)\right)A\left(r(k-2)\right)...A\left(r(l)\right) \end{aligned} \tag{2}$$

for $k > l \geq 0$ and $i_{k-1}, i_{k-2}, ..., i_l \in S$.

Let us recall, that $n \times n$ dimensional matrix $F_r(k,l)$ is the so-called state transition matrix. Using the above-established notation we can express the solution of difference Eq. (1) in the following form

$$x(k, x_0, i_0, u) = F_r(k,0)x_0 + \sum_{t=0}^{k-1} F_r(k, t+1)B\left(r(t)\right)u(t) \tag{3}$$

or equivalently

$$x(k, x_0, i_0, u) = F(k, 0, r(k-1), ..., r(0))x_0 + \sum_{t=0}^{k-1} F\left(k, t+1, r(k-1), ..., r(t+1)\right)B\left(r(t)\right)u(t) \tag{4}$$

for $k \geq 1$.

In addition, we use the following notation

$$S_{i_0}^{(N)} = \left\{(i_0, i_1, ..., i_{N-1}) : i_0, i_1, ..., i_{N-1} \in S\right\}. \tag{5}$$

In further considerations, it will be convenient to have the elements of $S_{i_0}^{(N)}$ ordered as a sequence. To do it, let us order the elements of $S_{i_0}^{(N)}$ in lexicographical order, i.e. they are ordered in the following way

$$\begin{gathered} (i_0,1,1,...,1,1), (i_0,1,1,...,1,2), ..., (i_0,1,1,...,1,s), \\ (i_0,1,1,...,2,1), (i_0,1,1,...,2,2), ..., (i_0,1,1,...,2,s), ... \\ (i_0,s,s,...,s,1), (i_0,s,s,...,s,2), ..., (i_0,s,s,...,s,s). \end{gathered}$$

In many engineering problems some switches are impossible, i.e. we have certain set Λ of pairs $(i,j) \in S \times S$ such that it is impossible that

$$r(k) = i, \quad r(k+1) = j \quad \text{for a} \quad k = 0, 1, ...$$

Withdraw from $S_{i_0}^{(N)}$ all the elements $(i_0, i_1, ..., i_{N-1})$ such that

$$(i_l, i_{l+1}) \in \Lambda \quad \text{for certain} \quad l = 0, 1, ..., N-1.$$

and denote by $\overline{S}_{i_0}^{(N)}$ the set obtained in this way. In this notation $\overline{S}_{i_0}^{(N)}$ is a sequence of all possible switching paths of the length N. By $\overline{\overline{s}}_{i_0}^{(N)}$ we will denote the number of elements of $\overline{S}_{i_0}^{(N)}$.

Now, let us fix a number $N > 0$ and a sequence

$$(i_0, i_1, ..., i_{N-1})$$

of elements of S. Consider a matrix column blocks, which are numbered successively by sequences:

$$i_0, \overline{S}_{i_0}^{(2)}, ..., \overline{S}_{i_0}^{(N)}$$

and the block

$$(i_0, i_1, ..., i_k)$$

for $k = 0, 1, .., N-1$ is given by

$$F(N, k+1, i_{N-1}, ..., i_{k+1})B_{i_k}$$

and the others are equal to 0. Denote the matrix obtained in this way by

$$C(i_0, i_1, ..., i_{N-1})$$

and by $G(i_0)$ – the matrix consisting of all

$$C(i_0, i_1, ..., i_{N-1})$$

(as row blocks numbered by $\overline{S}_{i_0}^{(N)}$) for

$$(i_0, i_1, ..., i_{N-1}) \in \overline{S}_{i_0}^{(N)}.$$

Moreover, by

$$H(i_0) \in R^{n\overline{\overline{s}}_{i_0}^{(N)} \times m}$$

let us denote a matrix row blocks, of which are numbered by the sequence $\overline{S}_{i_0}^{(N)}$. The block with number

$$(i_0, i_1, ..., i_{N-1})$$

is given by

$$F(N, 0, i_{N-1}, i_{N-2}, ..., i_0).$$

In order to illustrate the introduced notation, let us discuss the following example. For $S = \{1, 2, 3\}$, $N = 3$, $i_0 = 1$ and

$$\Lambda = \{(1, 2), (3, 2)\}$$

we have

$$G(i_0) = G(1) = \begin{bmatrix} C(1,1,1) \\ C(1,1,3) \\ C(1,3,1) \\ C(1,3,3) \end{bmatrix} =$$

$$= \left[\begin{array}{ccc} (1) & (1,1) & (1,3) \\ A^2(1)B(1) & A(1)B(1) & 0 \\ A(3)A(1)B(1) & A(3)B(1) & 0 \\ A(1)A(3)B(1) & 0 & A(1)B(3) \\ A^2(3)B(1) & 0 & A(3)B(3) \end{array} \right.$$

$$\left. \begin{array}{cccc} (1,1,1) & (1,1,3) & (1,3,1) & (1,3,3) \\ B(1) & 0 & 0 & 0 \\ 0 & B(3) & 0 & 0 \\ 0 & 0 & B(1) & 0 \\ 0 & 0 & 0 & B(3) \end{array} \right]$$

and

$$H(i_0) = H(1) = \begin{bmatrix} A^3(1) \\ A(3)A^2(1) \\ A(1)A(3)A(1) \\ A^2(3)A(1) \end{bmatrix}.$$

Moreover, let us denote by

$$f_1^{(k)}, f_2^{(k)}, ..., f_n^{(k)} \in \mathbb{R}^{nk}$$

the vectors defined by

$$f_l^{(k)} = \left. \begin{bmatrix} e_l \\ e_l \\ \vdots \\ e_l \end{bmatrix} \right\} k \text{ times } e_l, \quad l = 1, 2, ..., n$$

where $\{e_1, e_2, ..., e_n\}$ is the standard basis of $\mathbb{R}^n$.

In the next part of the paper, we shall use the following definition:

Definition 1. The system (1) is i_0- controllable at time N if, for all x_0, $x_1 \in \mathbb{R}^n$ there exists an admissible control u such that

$$x(N, x_0, i_0, u) = x_1. \tag{6}$$

Analogously, we say that system (1) is i_0 – controllable at time N to zero (from zero) if, for all $x_0 \in \mathbb{R}^n$ ($x_1 \in \mathbb{R}^n$) there exists a control u such that

$$x(N, x_0, i_0, u) = 0 \qquad (x(N, 0, i_0, u) = x_1). \tag{7}$$

If the system (1) is i_0 – controllable at time N (i_0 – controllable at time N to zero, i_0 – controllable at time N from zero) for all $i_0 \in S$ then we say that (1) is controllable at time N (controllable at time N to zero, controllable at time N from zero).

Let us observe that the controllability of each time-varying system corresponding to switching paths of the length N is only the necessary, but not the sufficient condition for controllability at time N of the system (1).

3. Main results

In this section, using notation given in Sec. 2, we prove main results of the paper.

Theorem 1 contains the necessary and sufficient conditions for i_0 – controllability at time N as well as i_0 – controllability at time N from zero and to zero.

Theorem 1. The system (1) is i_0 – controllable at time N from zero if and only if

$$rankG(i_0) = rank \left[G(i_0) \vdots f_l^{\left(\overline{\overline{s}}_{i_0}^{(N)}\right)} \right] \tag{8}$$

for all $l = 1, 2, ..., n$.

The system (1) is i_0 – controllable at time N to zero if and only if

$$ImH(i_0) \subset ImG(i_0) \tag{9}$$

and it is i_0 – controllable at time N if and only if

$$rankG(i_0) = rank \left[G(i_0) \vdots f_l^{\left(\overline{\overline{s}}_{i_0}^{(N)}\right)} \right] \tag{10}$$

for all $l = 1, 2, ..., n$, and

$$ImH(i_0) \subset ImG(i_0). \tag{11}$$

Before beginning the formal proof let us briefly discuss the main idea. Since the set $\overline{S}_{i_0}^{(N)}$ of all possible switching paths is finite, therefore, the question about i_0 – controllability can be reformulated, similarly as for classical time-varying systems, as a question about existence of a solution of a finite set of linear algebraic equations. However, now we must take into account the constraint that control $u(k)$ at time k may depend only on the variables

$$r(0), r(1), ..., r(k)$$

and should be independent of variables

$$r(k+1), r(k+2), ..., r(N).$$

The proper definition of matrices $G(i_0)$ and $H(i_0)$ plays an essential role in the proof of the above-mentioned constraints.

Proof. Suppose that switched the system (1) is i_0 – controllable at time N from zero. Then for any $y \in \mathbb{R}^n$ there exists an admissible control sequence

$$u(0), u(1), ..., u(N-1)$$

such that

$$u(k) = g_k\left(i_0, r(1), r(2), ..., r(k)\right)$$

for $k = 0, 1, ..., N-1$ and

$$x(N, 0, i_0, u) = y, \tag{12}$$

where g_k is a function from $\overline{S}_{i_0}^{(k)}$ to $\mathbb{R}^m$ for $k = 0, 1, ..., N-1$. It means that for any

$$(i_0, i_1, ..., i_{N-1}) \in \overline{S}_{i_0}^{(N)}$$

the following equality

$$\sum_{t=0}^{N-1} F(N, t+1, i_{N-1}, ..., i_{t+1}) B(i_t) g_t(i_0, ..., i_t t) = y$$

holds.

This implies that the system of algebraic equations

$$G(i_0)v = z,$$

where

$$z = \left.\begin{bmatrix} t \\ t \\ \vdots \\ t \end{bmatrix}\right\} \overline{\overline{s}}_{i_0}^{(N)} \quad \text{times}$$

has a solution for each $t \in \mathbb{R}^n$. Since vectors

$$f_l^{\left(\overline{\overline{s}}_{i_0}^{(N)}\right)}$$

for $l = 1, 2, ..., n$ form a basis in the space

$$\left\{ \begin{bmatrix} t \\ t \\ \vdots \\ t \end{bmatrix} \in R^{n\overline{\overline{s}}_{i_0}^{(N)}} : t \in R^n \right\},$$

the Kronecker-Capelli theorem (see e.g. [38]) implies that equality (8) is valid.

Assume now that the equality (8) holds. Using once again the Kronecker-Capelli theorem we obtain, that the set of algebraic equations

$$G(i_0)v = z,$$

where

$$z = \left.\begin{bmatrix} y \\ y \\ \vdots \\ y \end{bmatrix}\right\} \overline{\overline{s}}_{i_0}^{(N)} \quad \text{times}$$

has a solution for any $y \in \mathbb{R}^n$.

This clearly forces that for any $y \in \mathbb{R}^n$ and each

$$(i_0, i_1, ..., i_{N-1}) \in \overline{S}_{i_0}^{(N)}$$

there exists a sequence

$$g_k(i_0, i_1, ..., i_k),$$

for $k = 0, 1, ..., N-1$ such that

$$\sum_{t=0}^{N-1} F(N, t+1, i_{N-1}, i_{N-2}, ..., i_{t+1}) B(i_t) g_t(i_0, ..., i_t) = y.$$

For the control defined as

$$u(k) = g_k(i_0, r(1), r(2), ..., r(k))$$

we obtain

$$x(N, 0, i_0, u) = y$$

which implies that the system (1) is i_0 – controllable at time N from zero.

Now, let us assume that the system (1) is i_0 – controllable at time N to zero. Then for any $y \in \mathbb{R}^n$ there exists an admissible control sequence

$$u(0), u(1), ..., u(N-1)$$

such that

$$u(k) = g_k\left(i_0, r(1), r(2), ..., r(k)\right)$$

and

$$x(N, y, i_0, u) = 0, \tag{13}$$

where g_k is a function from $\overline{S}_{i_0}^{(k)}$ to $\mathbb{R}^m$ for $k = 0, 1, ..., N-1$. From the Eq. (13), we get

$$\sum_{t=0}^{N-1} F(N, t+1, i_{N-1}, ..., i_{t+1}) B(i_t) g_t(i_0, i_1, ..., i_t)$$
$$= -F(N, 0, i_{N-1}, i_{N-2}, ..., i_0) y.$$

and therefore,

$$-H(i_0)y \in ImG(i_0)$$

which implies that inclusion (9) holds. Now, let us suppose that the inclusion (9) is valid. It means that for any $x_0 \in \mathbb{R}^n$ there exists v such that

$$-H_\Lambda(i_0)x_0 = G_\Lambda(i_0)v$$

which implies that for any

$$(i_0, i_1 ..., i_{N-1}) \in \overline{S}_{i_0}^{(N)}$$

there exists a sequence

$$g_k(i_0, i_1, ..., i_k)$$

for $k = 0, 1, ..., N-1$ such that

$$\sum_{t=0}^{N-1} F(N, t+1, i_{N-1}, ..., i_{t+1}) B(i_t) g_t(i_0, i_1, ..., i_t)$$
$$= -F(N, 0, i_{N-1}, i_{N-2}, ..., i_0) x_0.$$

Defining now the control

$$u(k) = g_k\left(i_0, r(1), r(2), ..., r(k)\right)$$

we have

$$x(N, x_0, i_0, u) = 0$$

and the switched system (1) is i_0 – controllable at time N to zero. By analogy with the above-mentioned idea we can prove the part concerning i_0 – controllability at time N.

4. Examples

Considering the linear discrete time-varying system it is well known [13] that the controllability from zero implies the controllability to zero and the inverse implication is not true. The next example shows that for the linear switched system the controllability from zero generally does not imply the controllability to zero.

Example 1. Consider the switched system (1) with $S = \{1,2\}$, $N = 2$, $\Lambda = \emptyset$ and any i_0.

$$A(1) = \begin{bmatrix} 4 & 8 \\ 12 & 4 \end{bmatrix}, \quad A(2) = \begin{bmatrix} -4 & 8 \\ 4 & -4 \end{bmatrix},$$

$$B(1) = \begin{bmatrix} 0 \\ 8 \end{bmatrix}, \quad B(2) = \begin{bmatrix} 0 \\ 4 \end{bmatrix}.$$

We check controllability at time 2. According to the notation we have

$$S_1^{(2)} = \{(1,1),(1,2)\},$$

$$S_2^{(2)} = \{(2,1),(2,2)\},$$

$$\overset{=}{s}{}_1^{(2)} = \overset{=}{s}{}_2^{(2)} = 2,$$

$$G(1) = \begin{bmatrix} C(1,1) \\ C(1,2) \end{bmatrix} = \begin{bmatrix} A(1)B(1) & B(1) & 0 \\ A(2)B(1) & 0 & B(2) \end{bmatrix}$$

$$= \begin{bmatrix} 64 & 0 & 0 \\ 32 & 8 & 0 \\ 64 & 0 & 0 \\ -32 & 0 & 4 \end{bmatrix},$$

$$G(2) = \begin{bmatrix} C(2,1) \\ C(2,2) \end{bmatrix} = \begin{bmatrix} A(1)B(2) & B(1) & 0 \\ A(2)B(2) & 0 & B(2) \end{bmatrix}$$

$$= \begin{bmatrix} 32 & 0 & 0 \\ 16 & 8 & 0 \\ 32 & 0 & 0 \\ -16 & 0 & 4 \end{bmatrix}$$

and

$$f_1^{(2)} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad f_2^{(2)} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

Because

$$rankG(1) = rank\left[G(1) \vdots f_1^{(2)}\right] = 3$$

and

$$rankG(2) = rank\left[G(2) \vdots f_2^{(2)}\right] = 3$$

then the condition (8) is satisfied. The control that steers the zero initial condition to final state

$$\begin{bmatrix} x_1^{(0)} \\ x_2^{(0)} \end{bmatrix}$$

at time $N = 2$ is given by

$$u(0) = \begin{cases} \frac{1}{64}x_1^{(0)} & \text{if} \quad r(0) = 1 \\ \frac{1}{32}x_1^{(0)} & \text{if} \quad r(0) = 2 \end{cases},$$

$$u(1) = \begin{cases} -\frac{1}{16}x_1^{(0)} + \frac{1}{8}x_2^{(0)} & \text{if} \quad r(0) = 1,\ r(1) = 1 \\ \frac{1}{8}x_1^{(0)} + \frac{1}{4}x_2^{(0)} & \text{if} \quad r(0) = 1,\ r(1) = 2 \\ \frac{1}{8}x_1^{(0)} + \frac{1}{4}x_2^{(0)} & \text{if} \quad r(0) = 2,\ r(1) = 2 \\ -\frac{1}{16}x_1^{(0)} + \frac{1}{8}x_2^{(0)} & \text{if} \quad r(0) = 2,\ r(1) = 1 \end{cases}.$$

From the other hand the system is not controllable to zero at time 2. In fact we have

$$H(1) = \begin{bmatrix} A^2(1) \\ A(2)A(1) \end{bmatrix} = \begin{bmatrix} 112 & 64 \\ 96 & 112 \\ 80 & 0 \\ -32 & 16 \end{bmatrix},$$

$$H(2) = \begin{bmatrix} A^2(2) \\ A(1)A(2) \end{bmatrix} = \begin{bmatrix} 48 & -64 \\ -32 & 48 \\ 16 & 0 \\ -32 & 80 \end{bmatrix}$$

and

$$\begin{bmatrix} 22 \\ 26 \\ 10 \\ -2 \end{bmatrix} \in ImH(1), \qquad \begin{bmatrix} 2 \\ -2 \\ -2 \\ -6 \end{bmatrix} \in ImH(2),$$

but

$$\begin{bmatrix} 22 \\ 26 \\ 10 \\ -2 \end{bmatrix} \notin ImG(1), \qquad \begin{bmatrix} 2 \\ -2 \\ -2 \\ -6 \end{bmatrix} \notin ImG(2).$$

According to the Theorem 1 the system (1) is not controllable to zero at time 2.

The second example relates to a human arm, which model has been described by switched system. The kinematic scheme is shown in Fig. 1. We decided to model the human arm by a switched linear system, because after scrutinizing human limb motion and taking into account results in [40–42] it is evident that muscles are changing their shape in the process of contraction, which influences the moments of inertia during limb motion.

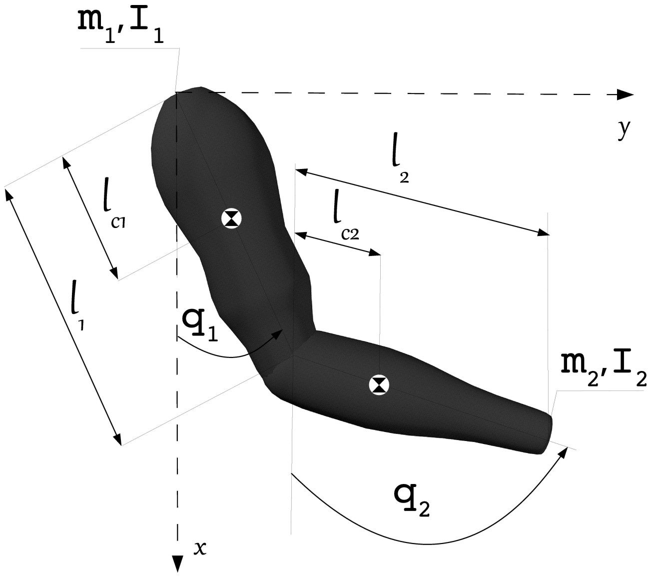

Fig. 1. The model of two-link human arm

As a result, we can assume that the matrix of inertia and the distance from the center of gravity of each joint, are changed. However, changes of these parameters are dependent on the configuration of the arm. On the other hand, the muscles impact is omitted. In this case, the motion equation is presented in the following nonlinear differential equation [40, 43–46]:

$$\frac{d}{dt}\begin{bmatrix} q \\ \dot{q} \end{bmatrix} = \begin{bmatrix} \dot{q} \\ M^{-1}(q)[u - C(q,\dot{q})\dot{q} - G(q) - W\dot{q}] \end{bmatrix}, \quad (14)$$

where

$$M(q) = \begin{bmatrix} m_1 l_{c1}^2 + m_2 l_1^2 + I_1 & m_2 l_1 l_{c2} \cos(q_1 - q_2) \\ m_2 l_1 l_{c2} \cos(q_1 - q_2) & m_2 l_{c2}^2 + I_2 \end{bmatrix},$$

$$C(q,\dot{q}) = $$
$$= \begin{bmatrix} 0 & m_2 l_1 l_{c2} \sin(q_1 - q_2)\dot{q}_2 \\ -m_2 l_1 l_{c2} \sin(q_1 - q_2)\dot{q}_1 & 0 \end{bmatrix},$$

$$G(q) = \begin{bmatrix} -(m_1 l_{c1} + m_2 l_1) g \sin q_1 \\ -m_2 l_{c2} g \sin q_2 \end{bmatrix},$$

$$W = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix}$$

and $M(q) \in \mathbb{R}^{2\times 2}$ – is a positive definite symmetric inertia matrix; $C(q,\dot{q}) \in \mathbb{R}^{2\times 2}$ – is a vector centripetal and Coriolis forces; $G(q) \in \mathbb{R}^2$ – is gravity forces vector; $W \in \mathbb{R}^{2\times 2}$ – is the joint friction matrix; $u = [u_1\, u_2]^T \in \mathbb{R}^2$ – is the joint torque; $q = [q_1\, q_2]^T \in \mathbb{R}^2$ – is the angular displacement; m_i – is the mass; l_i – is the link length; l_{ci} – is the distance from the joint to the center of mass; I_i – is the moment of inertia relative to a normal z-axis to the i-th link of frame attached at the center of mass of the i-th link; i – is the number of human link, $i = 1, 2$.

We assume that the state vector is expressed by

$$\begin{bmatrix} q \\ \dot{q} \end{bmatrix}.$$

In order to obtain an equivalent set of first-order state equations, we may use in the second-order differential state equations (14) the following state variables

$$x_1 = q_1, \quad x_2 = q_2, \quad x_3 = \dot{x}_1 = \dot{q}_1, \quad x_4 = \dot{x}_2 = \dot{q}_2,$$

$$x = \begin{bmatrix} q_1, & q_2, & \dot{q}_1, & \dot{q}_2 \end{bmatrix}^T.$$

As a result, we can write the two-link human arm system into a state space form, as vector first-order nonlinear differential equations:

$$\dot{x} = F(x) + G(x)u. \quad (15)$$

In (15), the vector functions $F(x)$, $G(x)$ are given by

$$F(x) = [F_1(x), F_2(x), F_3(x), F_4(x)]^T,$$

where

$$F_1(x) = x_3, \qquad F_2(x) = x_4,$$

$$F_3(x) = \frac{m_2^2 l_1^2 l_{c2}^2 \sin(x_1 - x_2)\cos(x_1 - x_2)}{D_M} x_3^2 +$$

$$-\frac{m_2^3 l_1^3 l_{c2}^3 \sin(x_1 - x_2)\cos^2(x_1 - x_2)}{(m_1 l_{c1}^2 + m_2 l_1^2 + I_1)} x_4^2 +$$

$$-\left(\frac{m_2 l_1 l_{c2} \sin(x_1 - x_2)}{m_1 l_{c1}^2 + m_2 l_1^2 + I_1}\right) x_4^2 +$$

$$+\left(\frac{m_2 l_1 l_{c2} \cos(x_1 - x_2) w_{21}}{D_M}\right) x_3 +$$

$$-\left(\frac{m_2^2 l_1^2 l_{c2}^2 \cos^2(x_1 - x_2) w_{11}}{D_M(m_1 l_{c1}^2 + m_2 l_1^2 + I_1)} + \frac{w_{11}}{m_1 l_{c1}^2 + m_2 l_1^2 + I_1}\right) x_3 +$$

$$-\left(\frac{m_2^2 l_1^2 l_{c2}^2 \cos^2(x_1 - x_2) w_{12}}{D_M(m_1 l_{c1}^2 + m_2 l_1^2 + I_1)}\right) x_4 +$$

$$+\left(\frac{m_2 l_1 l_{c2} \cos(x_1 - x_2) w_{22}}{D_M} - \frac{w_{12}}{m_1 l_{c1}^2 + m_2 l_1^2 + I_1}\right) x_4 +$$

$$+\left(\frac{m_2^2 l_1^2 l_{c2}^2 \cos^2(x_1 - x_2)(m_1 l_{c1} + m_2 l_1) g \sin(x_1)}{D_M(m_1 l_{c1}^2 + m_2 l_1^2 + I_1)}\right) +$$

$$+\left(\frac{(m_1 l_{c1} + m_2 l_1) g \sin(x_1)}{m_1 l_{c1}^2 + m_2 l_1^2 + I_1}\right) +$$

$$-\frac{m_2^2 l_1 l_{c2}^2 g \sin(x_2)\cos(x_1 - x_2)}{D_M},$$

$$F_4(x) = \frac{m_2^2 l_1^2 l_{c2}^2 \sin(x_1 - x_2)\cos(x_1 - x_2)}{D_M} x_4^2 +$$

$$+ \frac{m_2 l_1 l_{c2} \sin(x_1 - x_2)(m_1 l_{c1}^2 + m_2 l_1^2 + I_1)}{D_M} x_3^2 +$$

$$+ \frac{m_2 l_1 l_{c2} \cos(x_1 - x_2) w_{11}}{D_M} x_3 +$$

$$- \frac{(m_1 l_{c1}^2 + m_2 l_1^2 + I_1) w_{21}}{D_M} x_3 +$$

$$+ \frac{m_2 l_1 l_{c2} \cos(x_1 - x_2) w_{12}}{D_M} x_4 +$$

$$- \frac{(m_1 l_{c1}^2 + m_2 l_1^2 + I_1) w_{22}}{D_M} x_4 +$$

$$+ \frac{1}{D_M}\left((m_1 l_{c1}^2 + m_2 l_1^2 + I_1) m_2 l_{c2} g \sin(x_2)\right) +$$

$$+ \frac{1}{D_M}\left(m_2 l_1 l_{c2}(m_1 l_{c1} + m_2 l_1) g \sin(x_1)\cos(x_1 - x_2)\right)$$

and

$$G(x) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{D_M + m_2^2 l_1^2 l_{c2}^2 \cos(x_1 - x_2)}{D_M(m_1 l_{c1}^2 + m_2 l_1^2 + I_1)} & -\frac{m_2 l_1 l_{c2}\cos(x_1 - x_2)}{D_M} \\ -\frac{m_2 l_1 l_{c2}\cos(x_1 - x_2)}{D_M} & \frac{m_1 l_{c1}^2 + m_2 l_1^2 + I_1}{D_M} \end{bmatrix},$$

note that

$$D_M = m_2 l_{c2}^2 (m_1 l_{c1}^2 + m_2 l_1^2 + I_1 - m_2 l_1^2 \cos^2(x_1 - x_2)) + (m_1 l_{c1}^2 + m_2 l_1^2) I_2 + I_1 I_2.$$

is the determinant of matrix M.

Using series expansion linearization method, the nonlinear model was linearized at three operating points. The operating points are expressed as $[x^i, u^i]^T \in \mathbb{R}^6$ for $i = 1, 2, 3$, where:

$$x^{(1)} = [-0.523 \text{ rad}, 0 \text{ rad}, 0 \text{ rad/s}, 0 \text{ rad/s}]^T,$$
$$u^{(1)} = [0 \text{ Nm}, 0 \text{ Nm}]^T,$$

$$x^{(2)} = [0 \text{ rad}, 0.523 \text{ rad}, 0 \text{ rad/s}, 0 \text{ rad/s}]^T,$$
$$u^{(2)} = [0 \text{ Nm}, 0 \text{ Nm}]^T$$

and

$$x^{(3)} = [0.523 \text{ rad}, 0.523 \text{ rad}, 0 \text{ rad/s}, 0 \text{ rad/s}]^T,$$
$$u^{(3)} = [0 \text{ Nm}, 0 \text{ Nm}]^T.$$

All operating points were chosen arbitrarily. The parameters of two-link human arm, that were used during linearization, are shown in Table 1 [41] (the length l_2 is omitted, it does not appear in the final mathematical model of the two-link human arm).

Table 1
The parameters of human arm's model

	$m\,[kg]$	$l\,[m]$		
Link 1	1.4	0.3		
Link 2	1.1	–		
	$l_{c1}\,[m]$	$l_{c2}\,[m]$	$I_1\,[kgm^2]$	$I_2\,[kgm^2]$
Subsystem I	0.11	0.16	0.027	0.045
Subsystem II	0.1	0.14	0.018	0.04
Subsystem III	0.11	0.14	0.02	0.04

After linearization, the linear model was discretized using a zero-order hold discretization method with a sample time of T_s equal to 0.1 seconds.

At this point, we can consider the human arm as switched linear discrete-time system with state-dependent switching. In this case, the function $r(k)$ in (1) depends on $x(k)$ in the following way: the state space $\mathbb{R}^n$ is divided into a collection of disjoint regions

$$\Omega_1, \Omega_2, \Omega_3$$

with

$$\bigcup_{i=1}^{i=3} \Omega_i = \mathbb{R}^n$$

and then

$$x(k+1) = A(i)x(k) + B(i)u(k) \quad \text{for} \quad x \in \Omega_i \tag{16}$$

for $i = 1, 2, 3$.

Then, the switched discrete linear system (16) can be expressed in the following form:

$$x(k+1) = \begin{cases} A(1)x(k) + B(1)u(k) & \text{if} \quad x_1 < 0,\ x_2 \geq 0 \\ A(2)x(k) + B(2)u(k) & \text{if} \quad x_1 = 0,\ x_2 \geq 0 \\ A(3)x(k) + B(3)u(k) & \text{if} \quad x_1 > 0,\ x_2 > 0 \end{cases} \tag{17}$$

It should be noted that some switching are not allowed, namely: it is impossible to take $r(k) = 1$ and $r(k+1) = 3$ and reversely $r(k+1) = 3$ and $r(k) = 1$, therefore $\Lambda = \{(1,3),(3,1)\}$.

Example 2. Let us consider the system (17) where we have: $S = \{1, 2, 3\}, N = 3, i_0 = 1$ and the appropriate matrices:

$$A(1) = \begin{bmatrix} 0.893 & -0.0290 & 0.09567 & -0.0015 \\ -0.023 & 0.9782 & -0.00169 & 0.0981 \\ -2.088 & -0.5653 & 0.8789 & -0.0389 \\ -0.449 & -0.4281 & -0.0411 & 0.9544 \end{bmatrix},$$

$$B(1) = \begin{bmatrix} 0.03439 & -0.01166 \\ -0.01172 & 0.04997 \\ 0.6751 & -0.235 \\ -0.2374 & 0.9933 \end{bmatrix},$$

$$A(2) = \begin{bmatrix} 0.8123 & 0.0184 & 0.09304 & 0.00037 \\ 0.1112 & 0.8948 & 0.00223 & 0.09429 \\ -3.618 & 0.3509 & 0.8005 & 0.01341 \\ 2.113 & -2.045 & 0.08081 & 0.8526 \end{bmatrix},$$

$$B(2) = \begin{bmatrix} 0.04324 & -0.02754 \\ -0.0274 & 0.09487 \\ 0.8335 & -0.5271 \\ -0.5216 & 1.841 \end{bmatrix},$$

$$A(3) = \begin{bmatrix} 0.8084 & 0.04949 & 0.09294 & 0.00151 \\ 0.1362 & 0.8657 & 0.00334 & 0.09359 \\ -3.673 & 0.9324 & 0.7972 & 0.04588 \\ 2.565 & -2.586 & 0.112 & 0.8294 \end{bmatrix},$$

$$B(3) = \begin{bmatrix} 0.04658 & -0.03311 \\ -0.03311 & 0.0898 \\ 0.8931 & -0.6238 \\ -0.624 & 1.73 \end{bmatrix}.$$

Let us check i_0 – controllability at time N from zero for $i_0 = 1$ and $N = 3$. We have

$$\overline{S}_1^{(3)} = \{(1,1,1),(1,1,2),(1,2,1),(1,2,2),(1,2,3)\}$$

$$G(i_0) = G(1) = \begin{bmatrix} C(1,1,1) \\ C(1,1,2) \\ C(1,2,1) \\ C(1,2,2) \\ C(1,2,3) \end{bmatrix}$$

$$= \begin{array}{c} \begin{array}{cccc} (1) & (1,1) & (1,2) & (1,1,1) \end{array} \\ \left[\begin{array}{cccc} A^2(1)B(1) & A(1)B(1) & 0 & B(1) \\ A(2)A(1)B(1) & A(2)B(1) & 0 & 0 \\ A(1)A(2)B(1) & 0 & A(1)B(2) & 0 \\ A^2(2)B(1) & 0 & A(2)B(2) & 0 \\ A(3)A(2)B(1) & 0 & A(3)B(2) & 0 \end{array} \right. \end{array}$$

$$\begin{array}{c} \begin{array}{cccc} (1,1,2) & (1,2,1) & (1,2,2) & (1,2,3) \end{array} \\ \left. \begin{array}{cccc} 0 & 0 & 0 & 0 \\ B(2) & 0 & 0 & 0 \\ 0 & B(1) & 0 & 0 \\ 0 & 0 & B(2) & 0 \\ 0 & 0 & 0 & B(3) \end{array} \right] \end{array}$$

Matrix $G(i_0)$ is 20×16 – dimensional matrix. Moreover, using condition (8), we calculate that

$$rankG(1) = 16 \neq rank[G(1) \vdots f_1^{(5)}] = 17.$$

Therefore, the system is not i_0 – controllable at time N equal to 3 and $i_0 = 1$.

Example 3. Now, we focus on the case when $N = 6$ and $i_0 = 1$. Let us order the elements of $\overline{S}_{i_0}^{(6)}$, $i_0 = 1,2,3$ in lexicographical order by the following way:

$$\begin{aligned} \overline{S}_1^{(6)} = \{&(1,1,1,1,1,1),(1,1,1,1,1,2),(1,1,1,1,2,1),\\ &(1,1,1,1,2,2),(1,1,1,1,2,3),(1,1,1,2,1,1)\\ &\vdots\\ &(1,2,3,2,1,1),(1,2,3,2,1,2),(1,2,3,2,2,1)\\ &\vdots\\ &(1,2,3,3,2,3),(1,2,3,3,3,2),(1,2,3,3,3,3)\}, \end{aligned}$$

$$\begin{aligned} \overline{S}_2^{(6)} = \{&(2,1,1,1,1,1),(2,1,1,1,1,2),(2,1,1,1,2,1),\\ &(2,1,1,1,2,2),(2,1,1,1,2,3),(2,1,1,2,1,1)\\ &\vdots\\ &(2,2,3,2,1,1),(2,2,3,2,1,2),(2,2,3,2,2,1)\\ &\vdots\\ &(2,3,3,3,2,3),(2,3,3,3,3,2),(2,3,3,3,3,3)\} \end{aligned}$$

and

$$\begin{aligned} \overline{S}_3^{(6)} = \{&(3,2,1,1,1,1),(3,2,1,1,1,2),(3,2,1,1,2,1),\\ &(3,2,1,1,2,2),(3,2,1,1,2,3),(3,2,1,2,1,1)\\ &\vdots\\ &(3,2,3,3,2,1),(3,2,3,3,2,2),(3,2,3,3,2,3)\\ &\vdots\\ &(3,3,3,3,2,3),(3,3,3,3,3,2),(3,3,3,3,3,3)\}. \end{aligned}$$

The numbers $\overline{\overline{s}}_1^{(6)}$ and $\overline{\overline{s}}_3^{(6)}$ are equal to 70. The number $\overline{\overline{s}}_2^{(6)}$ is equal to 108. Finally, after the tedious and longer calculations, we may conclude, that the system (17) is i_0 – controllable for time N equal to 6 and all $i_0 = 1,2,3$.

5. Conclusions

In the paper we presented the necessary and sufficient conditions for controllability (controllability to zero and controllability from zero) for linear discrete-time switched linear systems. These conditions are given in terms of relations consisting of ranks and images of matrices constructed on the base of the system coefficients. The proposed controllability concept is appropriate to the situation when the switching signal models unpredictable events, for example systems failures. Additionally, a new contribution of the paper is that we took into account the situation in which certain switching sequences are not possible. This situation often occurs in engineering practice.

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