

AGGREGATION AND RELATED FUNCTIONS IN FUZZY SET THEORY: DUALITIES AND MONOTONICITY-BASED GENERALIZATIONS

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Abstract:

We discuss several extensions of binary Boolean functions acting on the domain $[0, 1]$. Formally, there are 16 disjoint classes of such functions, covering a majority of binary functions considered in fuzzy set theory. We introduce and discuss dualities in this framework, stressing the links between different subclasses of considered functions, e.g., the link between conjunctive and implication functions. Special classes of considered functions are characterized, among others, by particular kinds of monotonicity. Relaxing these constraints by considering monotonicity in one direction only, we generalize standard classes of aggregation functions, implications, semicopulas, etc., into larger classes called pre-aggregations, pre-implications, pre-semicopulas, etc. Note that the dualities discussed for the standard classes also relate the new extended classes of pre-functions.

Keywords: aggregation function, conjuctor, directional monotonicity, disjunctive, duality, implication function, pre-aggregation function, pre-implication function

1. Introduction

Fuzzy set theory introduced by Zadeh [11] was based on $[0, 1]$ -extensions of Boolean functions. We will consider the binary functions only, though the other arities could be considered in a similar way. Recall that there are exactly 16 Boolean functions $B_\beta: \{0, 1\}^2 \rightarrow \{0, 1\}$, $\beta \in \{0, 1, \dots, 15\}$, where

$$\beta = 8B_\beta(1, 1) + 4B_\beta(1, 0) + 2B_\beta(0, 1) + B_\beta(0, 0).$$

So, for example,

$$B_6(1, 1) = 0, B_6(1, 0) = 1, B_6(0, 1) = 1 \text{ and } B_6(0, 0) = 0.$$

Observe that, e.g., B_8 is the Boolean conjunction, B_{14} Boolean disjunction, B_{11} Boolean implication, and B_9 is the Boolean bi-implication.

Denote by \mathcal{B} the set of all Boolean binary functions, i.e., $\mathcal{B} = \{B_0, B_1, \dots, B_{15}\}$. A mapping $\varphi: \mathcal{B} \rightarrow \mathcal{B}$ is called a duality whenever it is involutive, i.e., if $\varphi \circ \varphi = Id$ is the identity on \mathcal{B} (Id is the trivial duality). It is not difficult to check that there are exactly 8 dualities on \mathcal{B} and when their composition is considered, they form an Abelian group isomorphic to $(\mathbb{Z}_2, +)^3$.

Zadeh in his introduction of fuzzy sets [11] has extended the Boolean range $\{0, 1\}$ into the real unit interval $[0, 1]$ and as the basic fuzzy connectives he has considered piece-wise linear extensions of the corresponding Boolean functions. Such an extension of the

Boolean conjunction B_8 has resulted in Min operator, $Min(x, y) = \min\{x, y\}$ and B_{14} has been extended to Max operator, $Max(x, y) = \max\{x, y\}$ (when linearity on simplices determined by vertices $(0, 0)$, $(0, 1)$, $(1, 1)$ and $(0, 0)$, $(1, 0)$, $(1, 1)$ was considered). Note that if the Boolean implication B_{11} is extended in this way, the Łukasiewicz implication $I_L: [0, 1]^2 \rightarrow [0, 1]$ given by $I_L(x, y) = \min\{1, 1 - x + y\}$ is obtained. Also observe that when linearity on simplices determined by vertices $(0, 1)$, $(0, 0)$, $(1, 0)$ and $(0, 1)$, $(1, 1)$, $(1, 0)$ is considered, then the Łukasiewicz connectives T_L and S_L , as well as the Kleene-Dienes implication I_{KL} are obtained. Recall that for each $(x, y) \in [0, 1]^2$ they are given by $T_L(x, y) = \max\{0, x + y - 1\}$, $S_L(x, y) = \min\{1, x + y\}$, and $I_{KL}(x, y) = \max\{1 - x, y\}$.

Considering a binary function $F: [0, 1]^2 \rightarrow [0, 1]$ (a grupoid $([0, 1], F)$), we can distinguish 81 disjoint classes of such functions, depending on their values at Boolean inputs $(0, 0)$, $(0, 1)$, $(1, 0)$ and $(1, 1)$. Note that the value of F at a Boolean input can be 0, 1 or can belong to $]0, 1[$ (i.e., not being Boolean). We restrict our considerations to functions F extending some of the Boolean functions from \mathcal{B} , i.e., we will only discuss 16 classes of such functions. All such extensions form the class \mathcal{EB} of extended Boolean functions, and $\mathcal{EB}_\beta, \beta \in \{0, 1, \dots, 15\}$ denotes the class of all extensions of B_β . It holds $F \in \mathcal{EB}_\beta$ if and only if $F: [0, 1]^2 \rightarrow [0, 1]$ satisfies $F|_{\{0, 1\}^2} = B_\beta$.

Note that a similar study can be done when considering all 81 classes mentioned above. We restrict our discussion to 16 classes forming \mathcal{EB} only for the sake of the transparency of our study. The dualities on \mathcal{EB} are discussed in the next section. There we also recall particular extended Boolean functions characterized by some additional properties, including monotonicity constraints. In Section 3, we recall recently introduced concepts of directional monotonicity [1] and pre-aggregation functions [9]. In Section 4, we generalize some of the discussed classes of extended Boolean functions by relaxing the monotonicity constraints and replacing them by some directional monotonicity. In this way, new classes of pre-conjunctors or pre-implications are introduced and exemplified, and moreover, their duality is shown to be inherited from the duality of conjunctors and implications considered in Section 2. Finally, some concluding remarks are added.

2. Basic Dualities in the Class of Extended Boolean Functions

Recall that there are just two unary non-constant Boolean functions, and both of them can be seen as dualities. Their linear extensions to the domain $[0, 1]$ are just the functions $id, n: [0, 1] \rightarrow [0, 1]$ given by $id(x) = x$ and $n(x) = 1 - x$. Recall that n was proposed by Zadeh [11] as a negation for building complements of fuzzy sets, and it is often called Zadeh's or the standard negation. The mentioned 8 dualities on the class \mathcal{B} can be straightforwardly extended to the dualities on the class \mathcal{EB} as follows:

For any $F \in \mathcal{EB}$ and all $(x, y) \in [0, 1]^2$ we put

$$\begin{aligned}\varphi_0(F)(x, y) &= F(x, y) \quad (\text{i.e., } \varphi_0 \text{ is the identity on } \mathcal{EB}); \\ \varphi_1(F)(x, y) &= F(x, 1 - y); \\ \varphi_2(F)(x, y) &= F(1 - x, y); \\ \varphi_3(F)(x, y) &= F(1 - x, 1 - y); \\ \varphi_4(F)(x, y) &= 1 - F(x, y); \\ \varphi_5(F)(x, y) &= 1 - F(x, 1 - y); \\ \varphi_6(F)(x, y) &= 1 - F(1 - x, y); \\ \varphi_7(F)(x, y) &= 1 - F(1 - x, 1 - y).\end{aligned}$$

Observe that $\varphi_7(F)$ is the dual to F in the sense of the standard duality of aggregation functions [7].

More about the group structure of dualities $\varphi_0, \dots, \varphi_7$ can be found in our recent paper [8]. Each of the introduced dualities preserves the partition of \mathcal{EB} into the classes $\mathcal{EB}_0, \dots, \mathcal{EB}_{15}$. So, for example, for any function $F \in \mathcal{EB}_8$ (i.e., extending the Boolean conjunction), $\varphi_5(F) \in \mathcal{EB}_{11}$ (i.e., $\varphi_5(F)$ extends the Boolean implication). Vice-versa, if $F \in \mathcal{EB}_{11}$ then $\varphi_5(F) \in \mathcal{EB}_8$. More detailed information can be found in [8]. Note that there are also other types of dualities on \mathcal{EB} , when instead of linear extensions of Boolean unary functions continuous involutive extensions are considered (i.e., the identity and strong negations). As the extension of our discussion considering dualities $\varphi_0, \dots, \varphi_7$ can be introduced to this more general setting in a direct way [8], we avoid this step to keep the transparency of our ideas.

We now recall a few of the distinguished subclasses of \mathcal{EB} that are often considered in fuzzy set theory:

- coordinate-wise monotone extensions, in particular conjunctors (subclass of \mathcal{EB}_8), disjunctors (subclass of \mathcal{EB}_{14}), implications (subclass of \mathcal{EB}_{11}), co-implications (subclass of \mathcal{EB}_2);
- semicopulas, i.e., conjunctors with neutral element $e = 1$ [5];
- quasi-copulas, i.e., 1-Lipschitz semicopulas [6];
- overlap functions, i.e., symmetric continuous conjunctors with annihilator 0 and no unit multipliers ($F(x, y) = 1$ only if $x = y = 1$) [2].

3. Directional Monotonicity

Directional monotonicity was introduced in [1] for n -ary functions. We now recall this notion for binary functions only.

Definition 1. Let $F: [0, 1]^2 \rightarrow [0, 1]$ be a function and $\bar{r} \in \mathbb{R}^2$ a vector with unit length, i.e., $\bar{r} = (r_1, r_2)$ and $r_1^2 + r_2^2 = 1$. F is said to be \bar{r} -increasing whenever for all $(x, y) \in [0, 1]^2$ and $c > 0$ such that also $(x + cr_1, y + cr_2) \in [0, 1]^2$ we have

$$F(x + cr_1, y + cr_2) \geq F(x, y).$$

If F is \bar{r} -increasing for some vector \bar{r} , F is called directionally monotone.

For any function $F: [0, 1]^2 \rightarrow [0, 1]$, we denote by R_F the set of all 2-dimensional vectors \bar{r} with unit length for which F is \bar{r} -increasing. Note that $R_F = \emptyset$, i.e., F is not directionally monotone, whenever F attains a strict local extreme on $]0, 1[^2$. If R_F is maximal (i.e., it contains all possible directions) then, and only then, F is a constant function. Consider the weighted Lehmer mean $L_{(w_1, w_2)}$ related to positive weights w_1, w_2 such that $w_1^2 + w_2^2 = 1$, given by

$$L_{(w_1, w_2)}(x, y) = \frac{w_1 x^2 + w_2 y^2}{w_1 x + w_2 y}$$

with convention $\frac{0}{0} = 0$.

Then $R_{L_{(w_1, w_2)}} = \{(w_2, w_1)\}$ is a singleton, i.e., $L_{(w_1, w_2)}$ is directionally monotone, but it is increasing only in direction $\bar{r} = (w_2, w_1)$.

For a class \mathcal{H} of some functions from \mathcal{EB} , we put

$$R_{\mathcal{H}} = \bigcap_{F \in \mathcal{H}} R_F,$$

i.e., $R_{\mathcal{H}}$ is the set of all directions \bar{r} for which any function $F \in \mathcal{H}$ is \bar{r} -increasing. Thus $R_{\mathcal{H}}$ is a monotonicity characterization of the class \mathcal{H} . For example, for the class \mathcal{C} of all conjunctors, the class \mathcal{S} of all semicopulas and \mathcal{Q} of all quasi-copulas, we have

$$R_{\mathcal{C}} = R_{\mathcal{S}} = R_{\mathcal{Q}} = \{\bar{r} \in \mathbb{R}^2 \mid r_1 \geq 0, r_2 \geq 0, r_1^2 + r_2^2 = 1\}.$$

It is not difficult to check that if $F \in \mathcal{EB}$ is directionally monotone, then, for any duality φ_i , $i = 0, 1, \dots, 7$, the function $\varphi_i(F)$ is also directionally monotone. In particular, we have the following result:

Theorem 1. Let $F \in \mathcal{EB}$ be \bar{r} -increasing for a direction $\bar{r} = (r_1, r_2)$. Then:

- $\varphi_0(F) = F$ is \bar{r} -increasing;
- $\varphi_1(F)$ is $(r_1, -r_2)$ -increasing;
- $\varphi_2(F)$ is $(-r_1, r_2)$ -increasing;
- $\varphi_3(F)$ is $-\bar{r}$ -increasing;
- $\varphi_4(F)$ is $-\bar{r}$ -increasing;
- $\varphi_5(F)$ is $(-r_1, r_2)$ -increasing;
- $\varphi_6(F)$ is $(r_1, -r_2)$ -increasing;
- $\varphi_7(F)$ is \bar{r} -increasing.

As already mentioned, all above introduced notions and results can be extended for arbitrary functions $F: [0, 1]^2 \rightarrow [0, 1]$. As a distinguished example, we recall aggregation functions [7] $A: [0, 1]^2 \rightarrow [0, 1]$ which

are characterized by the (Boolean) boundary conditions $A(0,0) = 0$ and $A(1,1) = 1$ (i.e., we consider functions belonging to the convex closure of the class $\mathcal{EB}_8 \cup \mathcal{EB}_{10} \cup \mathcal{EB}_{12} \cup \mathcal{EB}_{14}$) and by directional monotonicity with respect to the directions from the first quadrant (i.e., A is \bar{r} -increasing whenever $\bar{r} \in R_C$). Hence, for the class \mathcal{A} of all binary aggregation functions we have

$$R_{\mathcal{A}} = R_C = \{\bar{r} \in \mathbb{R}^2 \mid r_1 \geq 0, r_2 \geq 0, r_1^2 + r_2^2 = 1\}.$$

Recently, we have generalized the class \mathcal{A} , keeping the boundary conditions and directional monotonicity, but requiring the \bar{r} -increasingness for some $\bar{r} \in R_{\mathcal{A}}$ only, see [9].

Definition 2. Let $B: [0,1]^2 \rightarrow [0,1]$ be a function such that $B(0,0) = 0, B(1,1) = 1$ and let B be \bar{r} -increasing for some $\bar{r} \in R_{\mathcal{A}}$. Then B is called a pre-aggregation function.

As an example of a proper pre-aggregation function (i.e., a pre-aggregation function that is not an aggregation function) we recall the above discussed weighted Lehmer mean $L_{(w_1, w_2)}$.

4. Generalizations of Some Subclasses of Extended Boolean Functions

Each particular subclass of \mathcal{EB} recalled in Section 2 is characterized by some properties, including monotonicity, i.e., each subclass \mathcal{H} is characterized by the set $R_{\mathcal{H}}$ of all directions \bar{r} such that any $F \in \mathcal{H}$ is \bar{r} -increasing. Following the idea of pre-aggregation functions, see Definition 2, we propose the following generalization of a particular subclass $\mathcal{H} \subset \mathcal{EB}$.

Definition 3. Let $\mathcal{H} \subset \mathcal{EB}$ be completely characterized by $R_{\mathcal{H}}$ and some other properties forming a set $\mathcal{P}_{\mathcal{H}}$. Let the class $\mathcal{G}_{\mathcal{H}} \subset \mathcal{EB}$ consist of all functions $F \in \mathcal{EB}$ satisfying all properties in $\mathcal{P}_{\mathcal{H}}$ and being \bar{r} -increasing for some $\bar{r} \in R_{\mathcal{H}}$. If functions F in \mathcal{H} are called \mathcal{H} -functions, then functions $G \in \mathcal{G}_{\mathcal{H}}$ will be called pre- \mathcal{H} -functions.

Definition 3 allows to introduce pre-conjunctors (subclass of \mathcal{EB}_8), pre-disjunctors (subclass of \mathcal{EB}_{14}), pre-implications (subclass of \mathcal{EB}_{11}), pre-semicopulas, pre-overlap functions, etc. Note that the notion of pre-t-norms was already introduced and discussed in [4]. Also observe that in some cases our generalization is empty. In particular, there is no proper pre-copula as well as no proper continuous pre-t-norm either pre-t-conorm.

Dualities between particular subclasses of \mathcal{EB} are also valid between the corresponding generalized subclasses.

Theorem 2. Let $\mathcal{H} \subset \mathcal{EB}$ be given and let $\mathcal{H}_i = \{\varphi_i(F) \mid F \in \mathcal{H}\}$, $i \in \{0, 1, \dots, 7\}$. Then $\mathcal{G}_{\mathcal{H}_i} = \{\varphi_i(F) \mid F \in \mathcal{G}_{\mathcal{H}}\}$.

Example 1. For $k \in [0, \infty[$, let $F_k: [0,1]^2 \rightarrow \mathbb{R}$ be given by $F_k = (k+1)T_P - kT_L$, where $T_P(x,y) = xy$ and

$$T_L(x,y) = \max\{0, x+y-1\}, \text{ i.e.,}$$

$$\begin{aligned} F_k(x,y) &= \min\{(k+1)xy, (k+1)xy - kx - ky + k\} \\ &= \min\{(k+1)xy, xy + k(1-x)(1-y)\}. \end{aligned}$$

Then $F_k \in \mathcal{EB}$ if and only if $k \in [0, 3]$, and then $F_k \in \mathcal{EB}_8$.

F_k has a neutral element $e = 1$ and an annihilator $a = 0$. F_k is a conjunctor (and thus a semicopula) only if $k = 0$, and it is a proper pre-conjunctive (a proper pre-semicopula) if $k \in]0, 1[$. If $k \in]1, 3[$, then $R_{F_k} = \emptyset$. If $k \in]0, 1[$, then $R_{F_k} = \left\{ \frac{(1,t)}{\sqrt{1+t^2}} \mid t \in [k, \frac{1}{k}] \right\}$.

Note that F_k is also a proper pre-overlap function whenever $k \in]0, 1[$.

Example 2. Recall that for any conjunctor $C \in \mathcal{C}$, $\varphi_5(C) = I \in \mathcal{J}$ is an implication function. Thus, if we go back to Example 1 and put $I_k = \varphi_5(F_k)$ for $k \in [0, 3]$, then evidently $I_k \in \mathcal{EB}_{11}$. Then $I_k: [0,1]^2 \rightarrow [0,1]$ is given by

$$I_k(x,y) = 1 - F_k(x, 1-y) =$$

$$\max\{1 - (k+1)x(1-y), 1 - x(1-y) - k(1-x)y\},$$

and I_k is an implication function only if $k = 0$ (then $I_0 = I_R$ is the Reichenbach implication). For $k \in]0, 1[$, I_k is a pre-implication function and $R_{I_k} = \left\{ \frac{(-1,t)}{\sqrt{1+t^2}} \mid t \in [k, \frac{1}{k}] \right\}$. Note also that $I_k(x,0) = 1-x$ and $I_k(1,y) = y$ for all $x, y \in [0,1]$ and $k \in [0,3]$.

5. Concluding Remarks

We have introduced and discussed some new looks at fuzzy connectives and related functions, including their generalizations based on replacing their monotonicity constraints by a weaker requirement of directional monotonicity. We have also discussed dualities in the class of extended Boolean functions, which can also be applied in a more general setting of binary functions (operations) on $[0,1]$. Note that this look at fuzzy connectives allows to transform the results obtained in some particular subclass \mathcal{H} of extended Boolean function to the dual class \mathcal{H}_i obtained by the duality φ_i , $i \in \{0, 1, \dots, 7\}$. So, for example, all results known for conjunctors can be transformed into the corresponding results for disjunctors (when the duality φ_7 is considered) or into the corresponding results for implications (when the duality φ_5 is considered, see also [10]).

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