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REGULARITY, POINTWISE COMPLETENESS AND POINTWISE GENERACY OF DESCRIPTOR LINEAR ELECTRICAL CIRCUITS

The regularity, pointwise completeness and pointwise generacy of descriptor linear electrical circuits composed of resistances, capacitances, inductances and voltage (current) sources are addressed. It is shown that every descriptor electrical circuit is a linear system with regular pencil. Conditions for the pointwise completeness and pointwise generacy of the descriptor linear electrical circuits are established. The considerations are illustrated by examples of descriptor electrical circuits.

KEYWORDS: regularity, pointwise completeness, pointwise generacy, descriptor electrical circuits

1. INTRODUCTION

Descriptor (singular) linear systems have been considered in many papers and books [1, 2, 5-8, 11-16, 21-23, 28]. The eigenvalues and invariants assignment by state and output feedbacks have been investigated in [14, 22] and the minimum energy control of descriptor linear systems in [16]. In positive systems inputs, state variables and outputs take only non-negative values [9, 20]. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models having positive linear behavior can be found in engineering, management science, economics, social sciences, biology and medicine, etc. The positive fractional linear systems and some of selected problems in theory of fractional systems have been addressed in monograph [22].

Descriptor standard positive linear systems by the use of Drazin inverse has been addressed in [1, 2, 6, 11-13, 21, 23]. The shuffle algorithm has been applied to checking the positivity of descriptor linear systems in [12]. The stability of positive descriptor systems has been investigated in [28]. Reduction and decomposition of descriptor fractional discrete-time linear systems have been considered in [21]. A new class of descriptor fractional linear discrete-time system has been introduced in [23].

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The pointwise completeness and pointwise degeneracy for standard and fractional linear systems have been investigated in [3-5, 10, 17-19, 24-27]. The Drazin inverse of matrices has been applied to find the solutions of the state equations of the fractional descriptor continuous-time linear systems with regular pencils in [13].

In this paper the regularity, pointwise completeness and pointwise generacy of the descriptor electrical circuits are investigated.

The paper is organized as follows. In section 2 three methods of analysis of descriptor linear continuous-time systems are recalled. The regularity of the descriptor linear electrical circuits is addressed in section 3. It is shown that every descriptor electrical circuit is a linear system with regular pencil. The pointwise completeness of descriptor electrical circuits is addressed in section 4 and the pointwise generacy in section 5. Concluding remarks are given in section 6.

2. PRELIMINARIES

Consider the descriptor continuous-time linear system

$$E\dot{x} = Ax + Bu\,, \tag{2.1}$$

where $\dot{x} = \frac{dx}{dt}$, $x \in \Re^n$, $u \in \Re^m$ are the state and input vectors and $E, A \in \Re^{n \times n}$, $B \in \Re^{n \times m}$.

It is assumed that the matrix E is singular and the pencil of (E, A) is regular, i.e.

$$\det E = 0 \text{ and } \det[Es - A] \neq 0 \text{ for some } s \in C \tag{2.2}$$

where C is the field of complex numbers.

Method 1. (Weierstrass-Kronecker decomposition)

It is well-known [15] that if the condition (2.2) is satisfied then there exist nonsingular matrices $P, Q \in \Re^{n \times n}$ such that

$$PEQ = \begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix}, \quad PAQ = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n_2} \end{bmatrix}, \tag{2.3}$$

where $N \in \Re^{n_2 \times n_2}$ is a nilpotent matrix with nilpotency index μ ($N^{\mu-1} \neq 0$ and $N^{\mu} = 0$) and $A_1 \in \Re^{n_1 \times n_1}$, $n_1 + n_2 = n$.

The matrices P and Q can be found by the use of the following elementary row and column operations [15]:

1. multiplication of i-th row (column) by a nonzero scalar c. This operation will be denoted by $L[i \times c]$ ($R[i \times c]$),
2. addition to the i-th row (column) of the j-th row (column) multiplied by a polynomial $p(s)$. This operation will be denoted by $L[i + j \times p(s)]$ ($R[i + j \times p(s)]$),

3. interchange of the i-th and j-th rows (columns). This operation will be denoted by $L[i,j]$ ($R[i,j]$).

Lemma 2.1. The characteristic polynomial of the descriptor system (2.1) and the characteristic polynomial of the matrix A_1 are related by

$$\det[Es - A] = k\det[I_{n_1}s - A_1] \tag{2.4}$$

where $k = (-1)^{n_2}\det P^{-1}\det Q^{-1} = (-1)^{n_2}\det(PQ)^{-1}$.

Proof. From (2.2) and (2.3) we have

$$\begin{aligned}\det[Es - A] &= \det\left\{P^{-1}\begin{bmatrix} I_{n_1}s - A_1 & 0 \\ 0 & Ns - I_{n_2}\end{bmatrix}Q^{-1}\right\} \\ &= \det P^{-1}\det Q^{-1}\det\left[I_{n_1}s - A_1\right]\det\left[Ns - I_{n_2}\right] = k\det\left[I_{n_1}s - A_1\right],\end{aligned} \tag{2.5}$$

since $\det\left[Ns - I_{n_2}\right] = (-1)^{n_2}$. □

<u>Method 2.</u> (Shuffle algorithm)

Performing elementary row operations on the equation (2.1) or equivalently on the array

$$E \quad A \quad B \tag{2.6}$$

we obtain

$$\begin{matrix} \overline{E}_1 & \overline{A}_{11} & \overline{B}_{11} \\ 0 & \overline{A}_{12} & \overline{B}_{12} \end{matrix} \tag{2.7}$$

and

$$\overline{E}_1\dot{x} = \overline{A}_{11}x + \overline{B}_{11}u \tag{2.8a}$$

$$0 = \overline{A}_{12}x + \overline{B}_{12}u \tag{2.8b}$$

where $\overline{E}_1 \in \Re^{r\times n}$ has full row rank and $r = \text{rank}E$.

Differentiation with respect to time of (2.8b) yields

$$\overline{A}_{12}\dot{x} = -\overline{B}_{12}\dot{u}\,. \tag{2.9}$$

The equations (2.8a) and (2.9) can be written in the form

$$\begin{bmatrix} \overline{E}_1 \\ \overline{A}_{12} \end{bmatrix}\dot{x} = \begin{bmatrix} \overline{A}_{11} \\ 0 \end{bmatrix}x + \begin{bmatrix} \overline{B}_{11} \\ 0 \end{bmatrix}u + \begin{bmatrix} 0 \\ -\overline{B}_{12} \end{bmatrix}\dot{u}\,. \tag{2.10}$$

If $\det\begin{bmatrix} \overline{E}_1 \\ \overline{A}_{12} \end{bmatrix} \neq 0$ then from (2.10) we have

$$\dot{x} = \hat{A}_1x + \hat{B}_{10}u + \hat{B}_{11}\dot{u}\,, \tag{2.11a}$$

where

$$\hat{A}_1 = \begin{bmatrix} \bar{E}_1 \\ \bar{A}_{12} \end{bmatrix}^{-1} \begin{bmatrix} \bar{A}_{11} \\ 0 \end{bmatrix}, \; \hat{B}_{10} = \begin{bmatrix} \bar{E}_1 \\ \bar{A}_{12} \end{bmatrix}^{-1} \begin{bmatrix} \bar{B}_{11} \\ 0 \end{bmatrix}, \; \hat{B}_{11} = \begin{bmatrix} \bar{E}_1 \\ \bar{A}_{12} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -\bar{B}_{12} \end{bmatrix}. \tag{2.11b}$$

If $\det \begin{bmatrix} \bar{E}_1 \\ \bar{A}_{12} \end{bmatrix} = 0$ then performing elementary row operations on the array

$$\begin{matrix} \bar{E}_1 & \bar{A}_{11} & \bar{B}_{11} & 0 \\ \bar{A}_{12} & 0 & 0 & -\bar{B}_{12} \end{matrix} \tag{2.12}$$

we eliminate the linearly dependent rows in the matrix $\begin{bmatrix} \bar{E}_1 \\ \bar{A}_{12} \end{bmatrix}$ and we obtain

$$\begin{matrix} \bar{E}_2 & \bar{A}_{21} & \bar{B}_{20} & \bar{B}_{21} \\ 0 & \bar{A}_{22} & \bar{B}_{30} & \bar{B}_{31} \end{matrix} \tag{2.13}$$

and

$$\bar{E}_2 \dot{x} = \bar{A}_{21} x + \bar{B}_{20} u + \bar{B}_{21} \dot{u} \tag{2.14a}$$

$$0 = \bar{A}_{22} x + \bar{B}_{30} u + \bar{B}_{31} \dot{u} \tag{2.14b}$$

Differentiation with respect to time of (2.14b) yields

$$\bar{A}_{22} \dot{x} = -\bar{B}_{30} \dot{u} - \bar{B}_{31} \ddot{u}\,. \tag{2.15}$$

The equations (2.14a) and (2.15) can be written in the form

$$\begin{bmatrix} \bar{E}_2 \\ \bar{A}_{22} \end{bmatrix} \dot{x} = \begin{bmatrix} \bar{A}_{21} \\ 0 \end{bmatrix} x + \begin{bmatrix} \bar{B}_{20} \\ 0 \end{bmatrix} u + \begin{bmatrix} \bar{B}_{21} \\ -\bar{B}_{30} \end{bmatrix} \dot{u} + \begin{bmatrix} 0 \\ -\bar{B}_{31} \end{bmatrix} \ddot{u}\,. \tag{2.16}$$

If $\det \begin{bmatrix} \bar{E}_2 \\ \bar{A}_{22} \end{bmatrix} \neq 0$ then from (2.16) we have

$$\dot{x} = \hat{A}_2 x + \hat{B}_{20} u + \hat{B}_{21} \dot{u} + \hat{B}_{22} \ddot{u}\,, \tag{2.17a}$$

where

$$\hat{A}_2 = \begin{bmatrix} \bar{E}_2 \\ \bar{A}_{22} \end{bmatrix}^{-1} \begin{bmatrix} \bar{A}_{21} \\ 0 \end{bmatrix}, \qquad \hat{B}_{20} = \begin{bmatrix} \bar{E}_2 \\ \bar{A}_{22} \end{bmatrix}^{-1} \begin{bmatrix} \bar{B}_{20} \\ 0 \end{bmatrix}, \qquad \hat{B}_{21} = \begin{bmatrix} \bar{E}_2 \\ \bar{A}_{22} \end{bmatrix}^{-1} \begin{bmatrix} \bar{B}_{21} \\ -\bar{B}_{30} \end{bmatrix},$$

$$\hat{B}_{22} = \begin{bmatrix} \bar{E}_2 \\ \bar{A}_{22} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -\bar{B}_{31} \end{bmatrix} \tag{2.17b}$$

If $\det \begin{bmatrix} \bar{E}_2 \\ \bar{A}_{22} \end{bmatrix} = 0$ then we repeat the procedure. It is well-known [12, 15, 21] that if the condition (2.2) is met then after finite number of steps we obtain the standard system equivalent to the descriptor system (2.1).

3. REGULARITY OF DESCRIPTOR LINEAR ELECTRICAL CIRCUITS

3.1. Examples

We start with simple examples of descriptor linear circuits and next the considerations will be extended to general case.

Example 3.1. Consider the descriptor electrical circuit shown in Fig 3.1 with given resistance R_1, capacitances C_1, C_2, C_3 and source voltages e_1 and e_2.

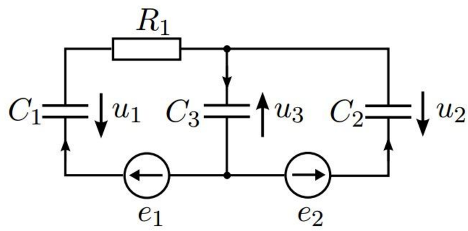

Fig. 3.1. Electrical circuit of Example 3.1

Using Kirchhoff's laws for the electrical circuit we obtain the equations

$$e_1 = R_1 C_1 \frac{\mathrm{d}u_1}{\mathrm{d}t} + u_1 + u_3 \tag{3.1a}$$

$$0 = C_1 \frac{\mathrm{d}u_1}{\mathrm{d}t} + C_2 \frac{\mathrm{d}u_2}{\mathrm{d}t} - C_3 \frac{\mathrm{d}u_2}{\mathrm{d}t} \tag{3.1b}$$

$$e_2 = u_2 + u_3 \tag{3.1c}$$

The equations (3.1) can be written in the form (2.1), where

$$x = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix},\ u = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix},\ E = \begin{bmatrix} R_1C_1 & 0 & 0 \\ C_1 & C_2 & -C_3 \\ 0 & 0 & 0 \end{bmatrix},\ A = \begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & -1 \end{bmatrix},\ B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}. \tag{3.2}$$

The assumption (2.2) for the electrical circuit is satisfied since

$$\det E = \begin{vmatrix} R_1C_1 & 0 & 0 \\ C_1 & C_2 & -C_3 \\ 0 & 0 & 0 \end{vmatrix} = 0 \tag{3.3a}$$

and

$$\det[Es - A] = \begin{vmatrix} R_1C_1s+1 & 0 & 1 \\ C_1s & C_2s & -C_3s \\ 0 & 1 & 1 \end{vmatrix} = s\left[R_1C_1\left(C_2+C_3\right)s + \left(C_1+C_2+C_3\right)\right] \tag{3.3b}$$

Therefore, the electrical circuit is a descriptor linear system with regular pencil.

Method 1.

Performing on the matrix

$$Es-A=\begin{bmatrix} R_1C_1s+1 & 0 & 1 \\ C_1s & C_2s & -C_3s \\ 0 & 1 & 1 \end{bmatrix} \quad (3.4)$$

the following elementary column operations $R\left[3+2\times\frac{C_3}{C_2}\right]$, $R\left[1+2\times\left(-\frac{C_1}{C_2}\right)\right]$, $R\left[1\times\frac{1}{R_1C_1}\right]$, $R\left[2\times\frac{1}{C_2}\right]$ we obtain the matrix

$$\begin{bmatrix} s+\frac{1}{R_1C_1} & 0 & 1 \\ 0 & s & 0 \\ -\frac{1}{R_1C_2} & \frac{1}{C_2} & \frac{C_2+C_3}{C_2} \end{bmatrix} \quad (3.5)$$

Performing on the matrix (3.5) the following elementary row operations $L\left[3\times\frac{C_2}{C_2+C_3}\right]$, $L[1+3\times(-1)]$ and the elementary column operations $R\left[1+3\times\frac{1}{R_1(C_2+C_3)}\right]$, $R\left[2+3\times\left(-\frac{1}{C_2+C_3}\right)\right]$ we obtain the desired matrix

$$\bar{E}s-\bar{A}=P[Es-A]Q=\begin{bmatrix} I_{n_1}s-A_1 & 0 \\ 0 & Ns-I_{n_2} \end{bmatrix}, \quad (3.6a)$$

where

$$A_1=\begin{bmatrix} -\frac{C_1+C_2+C_3}{R_1C_1(C_2+C_3)} & \frac{1}{C_2+C_3} \\ 0 & 0 \end{bmatrix}, \quad N=[0], \quad n_2=1 \quad (3.6b)$$

and

$$P=\begin{bmatrix} 1 & 0 & -\frac{C_2}{C_2+C_3} \\ 0 & 1 & 0 \\ 0 & 0 & \frac{C_2}{C_2+C_3} \end{bmatrix}, \quad Q=\begin{bmatrix} \frac{1}{R_1C_1} & 0 & 0 \\ -\frac{1}{R_1(C_2+C_3)} & \frac{1}{C_2+C_3} & \frac{C_3}{C_2} \\ \frac{1}{R_1(C_2+C_3)} & -\frac{1}{C_2+C_3} & 1 \end{bmatrix} \quad (3.6c)$$

The matrices (3.6c) can be found by performing the elementary row and column operations on the identity matrix I_3. Performing the elementary row operations $L\left[3\times\frac{C_2}{C_2+C_3}\right]$ and $L\left[1+3\times(-1)\right]$ on the matrix I_3 we obtain P and the elementary column operations $R\left[3+2\times\frac{C_3}{C_2}\right]$, $R\left[1+2\times\left(-\frac{C_1}{C_2}\right)\right]$, $R\left[1\times\frac{1}{R_1C_1}\right]$, $R\left[2\times\frac{1}{C_2}\right]$, $R\left[1+3\times\frac{1}{R_1(C_2+C_3)}\right]$ and $R\left[2+3\times\left(-\frac{1}{C_2+C_3}\right)\right]$ the matrix Q.

Method 2.

In this case the matrix E has already the desired form $\begin{bmatrix}\bar{E}_1\\0\end{bmatrix}$, where

$$\bar{E}_1=\begin{bmatrix}R_1C_1 & 0 & 0\\ C_1 & C_2 & -C_3\end{bmatrix} \tag{3.7}$$

and it has full row rank, i.e. $\operatorname{rank}\bar{E}_1=2$.
Taking into account that

$$\bar{A}_{12}=\begin{bmatrix}0 & -1 & -1\end{bmatrix} \tag{3.8a}$$

and

$$\det\begin{bmatrix}\bar{E}_1\\ \bar{A}_{12}\end{bmatrix}=\begin{vmatrix}R_1C_1 & 0 & 0\\ C_1 & C_2 & -C_3\\ 0 & -1 & -1\end{vmatrix}=-R_1C_1(C_2+C_3)\neq 0 \tag{3.8b}$$

from (2.10) we obtain

$$\begin{bmatrix}R_1C_1 & 0 & 0\\ C_1 & C_2 & -C_3\\ 0 & -1 & -1\end{bmatrix}\dot{x}=\begin{bmatrix}-1 & 0 & -1\\ 0 & 0 & 0\\ 0 & -1 & -1\end{bmatrix}x+\begin{bmatrix}1 & 0\\ 0 & 0\\ 0 & 0\end{bmatrix}u+\begin{bmatrix}0 & 0\\ 0 & 0\\ 0 & -1\end{bmatrix}\dot{u} \tag{3.9}$$

and

$$\dot{x}=\hat{A}_1x+\hat{B}_{10}u+\hat{B}_{11}\dot{u}\,, \tag{3.10a}$$

where

$$\hat{A}_1 = \begin{bmatrix} R_1C_1 & 0 & 0 \\ C_1 & C_2 & -C_3 \\ 0 & -1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{R_1C_1} & 0 & -\frac{1}{R_1C_1} \\ \frac{1}{R_1(C_2+C_3)} & 0 & \frac{1}{R_1(C_2+C_3)} \\ -\frac{1}{R_1(C_2+C_3)} & 0 & -\frac{1}{R_1(C_2+C_3)} \end{bmatrix}, \quad (3.10b)$$

$$\hat{B}_{10} = \begin{bmatrix} R_1C_1 & 0 & 0 \\ C_1 & C_2 & -C_3 \\ 0 & -1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{R_1C_1} & 0 \\ -\frac{1}{R_1(C_2+C_3)} & 0 \\ \frac{1}{R_1(C_2+C_3)} & 0 \end{bmatrix},$$

$$\hat{B}_{11} = \begin{bmatrix} R_1C_1 & 0 & 0 \\ C_1 & C_2 & -C_3 \\ 0 & -1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{C_3}{C_2+C_3} \\ 0 & -\frac{C_2}{C_2+C_3} \end{bmatrix}.$$

Method 3. (Elimination method)

From (3.1c) we have

$$u_3 = e_2 - u_2 . \quad (3.11)$$

Substituting (3.11) into (3.1a) and (3.1b) we obtain

$$\begin{bmatrix} R_1C_1 & 0 \\ C_1 & C_2+C_3 \end{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & C_3 \end{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \quad (3.12)$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = A \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + B_0 \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} + B_1 \frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, \quad (3.13a)$$

where

$$A=\begin{bmatrix} R_1C_1 & 0 \\ C_1 & C_2+C_3 \end{bmatrix}^{-1}\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}=\begin{bmatrix} -\frac{1}{R_1C_1} & \frac{1}{R_1C_1} \\ \frac{1}{R_1(C_2+C_3)} & -\frac{1}{R_1(C_2+C_3)} \end{bmatrix},$$

$$B_0=\begin{bmatrix} R_1C_1 & 0 \\ C_1 & C_2+C_3 \end{bmatrix}^{-1}\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}=\begin{bmatrix} \frac{1}{R_1C_1} & -\frac{1}{R_1C_1} \\ -\frac{1}{R_1(C_2+C_3)} & \frac{1}{R_1(C_2+C_3)} \end{bmatrix}, \quad (3.13b)$$

$$B_1=\begin{bmatrix} R_1C_1 & 0 \\ C_1 & C_2+C_3 \end{bmatrix}^{-1}\begin{bmatrix} 0 & 0 \\ 0 & C_3 \end{bmatrix}=\begin{bmatrix} 0 & 0 \\ 0 & \frac{C_3}{C_2+C_3} \end{bmatrix}.$$

The characteristic polynomial of the matrix A has the form

$$\det[I_2s-A]=\begin{vmatrix} s+\frac{1}{R_1C_1} & -\frac{1}{R_1C_1} \\ -\frac{1}{R_1(C_2+C_3)} & s+\frac{1}{R_1(C_2+C_3)} \end{vmatrix}=s\left[s+\frac{C_1+C_2+C_3}{R_1C_1(C_2+C_3)}\right], \quad (3.14)$$

of the matrix A_1 (given by (3.6b))

$$\det[I_2s-A_1]=\begin{vmatrix} s+\frac{C_1+C_2+C_3}{R_1C_1(C_2+C_3)} & -\frac{1}{C_2+C_3} \\ 0 & s \end{vmatrix}=s\left[s+\frac{C_1+C_2+C_3}{R_1C_1(C_2+C_3)}\right] \quad (3.15)$$

and of the matrix $\hat{A}_1$ (given by (3.10b))

$$\det\left[I_3s-\hat{A}_1\right]=\begin{vmatrix} s+\frac{1}{R_1C_1} & 0 & \frac{1}{R_1C_1} \\ -\frac{1}{R_1(C_2+C_3)} & s & -\frac{1}{R_1(C_2+C_3)} \\ \frac{1}{R_1(C_2+C_3)} & 0 & s+\frac{1}{R_1(C_2+C_3)} \end{vmatrix}$$

$$=s\begin{vmatrix} s+\frac{1}{R_1C_1} & \frac{1}{R_1C_1} \\ \frac{1}{R_1(C_2+C_3)} & s+\frac{1}{R_1(C_2+C_3)} \end{vmatrix}=s^2\left[s+\frac{C_1+C_2+C_3}{R_1C_1(C_2+C_3)}\right] \quad (3.16)$$

Note that the additional eigenvalue $s = 0$ has been introduced in Method 2 by the differentiation with respect to time of the equation (2.8b).

From (3.3b), (3.14), (3.15) and (3.16) it follows that the spectrum of the electrical circuit is the same for the three different methods and it is equal to

$$\sigma = \left\{ s_1 = 0, \quad s_2 = -\frac{C_1 + C_2 + C_3}{R_1 C_1 (C_2 + C_3)} \right\}.$$

Example 3.2. Consider the descriptor electrical circuit shown in Fig. 3.2 with given resistances R_1, R_2, R_3; inductances L_1, L_2, L_3 and source voltages e_1 and e_2.

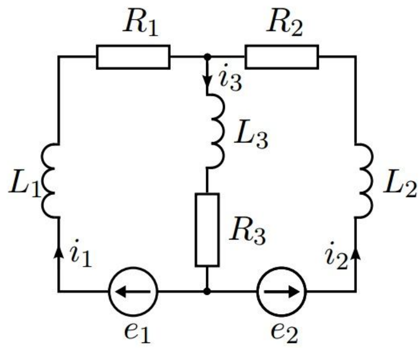

Fig. 3.2. Electrical circuit of Example 3.2

Using Kirchhoff's laws we can write the equations

$$e_1 = R_1 i_1 + L_1 \frac{di_1}{dt} + R_3 i_3 + L_3 \frac{di_3}{dt} \tag{3.17a}$$

$$e_2 = R_2 i_2 + L_2 \frac{di_2}{dt} + R_3 i_3 + L_3 \frac{di_3}{dt} \tag{3.17b}$$

$$0 = i_1 + i_2 - i_3 \tag{3.17c}$$

The equations (3.17) can be written in the form (2.1), where

$$x = \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix}, \ u = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, \ E = \begin{bmatrix} L_1 & 0 & L_3 \\ 0 & L_2 & L_3 \\ 0 & 0 & 0 \end{bmatrix}, \ A = \begin{bmatrix} -R_1 & 0 & -R_3 \\ 0 & -R_2 & -R_3 \\ 1 & 1 & -1 \end{bmatrix}, \ B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}. \tag{3.18}$$

The assumption (2.2) for the electrical circuit is satisfied, since

$$\det E = \begin{vmatrix} L_1 & 0 & L_3 \\ 0 & L_2 & L_3 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

and

$$\det[Es-A]=\begin{vmatrix} L_1s+R_1 & 0 & L_3s+R_3 \\ 0 & L_2s+R_2 & L_3s+R_3 \\ -1 & -1 & 1 \end{vmatrix}$$

$$=\left[L_1(L_2+L_3)+L_2L_3\right]s^2+\left[R_1(L_2+L_3)+R_2(L_1+L_3)+R_3(L_1+L_2)\right]s \quad (3.19)$$
$$+R_1(R_2+R_3)+R_2R_3$$

Therefore, the electrical circuit is a descriptor system with regular pencil.

Method 1.

Performing on the matrix

$$Es-A=\begin{bmatrix} L_1s+R_1 & 0 & L_3s+R_3 \\ 0 & L_2s+R_2 & L_3s+R_3 \\ -1 & -1 & 1 \end{bmatrix} \quad (3.20)$$

the elementary column operations $R\left[3+1\times\left(-\frac{L_3}{L_1}\right)\right]$, $R\left[3+2\times\left(-\frac{L_3}{L_2}\right)\right]$, $R\left[1\times\frac{1}{L_1}\right]$, $R\left[2\times\frac{1}{L_2}\right]$ we obtain

$$\begin{bmatrix} s+\frac{R_1}{L_1} & 0 & \frac{R_3L_1-R_1L_3}{L_1} \\ 0 & s+\frac{R_2}{L_2} & \frac{R_3L_2-R_2L_3}{L_2} \\ -\frac{1}{L_1} & -\frac{1}{L_2} & \Delta \end{bmatrix}, \quad \Delta=\frac{L_2(L_1+L_3)+L_1L_3}{L_1L_2}. \quad (3.21)$$

Next, performing on the matrix (3.21) the elementary operations $L\left[1+3\times\frac{R_1L_3-R_3L_1}{\Delta L_1}\right]$, $L\left[2+3\times\frac{R_2L_3-R_3L_2}{\Delta L_2}\right]$ we obtain

$$\begin{bmatrix} s+\frac{\Delta R_1L_1+R_3L_1-R_1L_3}{\Delta L_1^2} & \frac{R_3L_1-R_1L_3}{\Delta L_1L_2} & 0 \\ \frac{R_3L_2-R_2L_3}{\Delta L_1L_2} & s+\frac{\Delta R_2L_2+R_3L_2-R_2L_3}{\Delta L_2^2} & 0 \\ -\frac{1}{L_1} & -\frac{1}{L_2} & \Delta \end{bmatrix} \quad (3.22)$$

and finally $R\left[1+3\times\frac{1}{\Delta L_1}\right]$, $R\left[2+3\times\frac{1}{\Delta L_2}\right]$, $R\left[3\times\frac{1}{\Delta}\right]$ the desired form

$$\bar{E}s-\bar{A}=P\left[Es-A\right]Q=\begin{bmatrix} I_{n_1}s-A_1 & 0 \\ 0 & Ns-I_{n_2} \end{bmatrix}, \tag{3.23a}$$

where

$$A_1=\begin{bmatrix} -\frac{\Delta R_1L_1+R_3L_1-R_1L_3}{\Delta L_1^2} & \frac{R_1L_3-R_3L_1}{\Delta L_1L_2} \\ \frac{R_2L_3-R_3L_2}{\Delta L_1L_2} & -\frac{\Delta R_2L_2+R_3L_2-R_2L_3}{\Delta L_2^2} \end{bmatrix},\quad N=\left[0\right],\quad n_2=1 \tag{3.23b}$$

and

$$P=\begin{bmatrix} 1 & 0 & \frac{R_1L_3-R_3L_1}{\Delta L_1} \\ 0 & 1 & \frac{R_2L_3-R_3L_2}{\Delta L_2} \\ 0 & 0 & 1 \end{bmatrix},\quad Q=\begin{bmatrix} \frac{\Delta L_1-L_3}{\Delta L_1^2} & -\frac{L_3}{\Delta L_1L_2} & -\frac{L_3}{\Delta L_1} \\ -\frac{L_3}{\Delta L_1L_2} & \frac{\Delta L_2-L_3}{\Delta L_2^2} & -\frac{L_3}{\Delta L_2} \\ \frac{1}{\Delta L_1} & \frac{1}{\Delta L_2} & \frac{1}{\Delta} \end{bmatrix} \tag{3.23c}$$

Performing the elementary row operations on I_3 we obtain the matrix P and the elementary column operations the matrix Q.

Method 2.

The matrix E given by (3.18) has already the desired form $\begin{bmatrix} \bar{E}_1 \\ 0 \end{bmatrix}$, where

$$\bar{E}_1=\begin{bmatrix} L_1 & 0 & L_3 \\ 0 & L_2 & L_3 \end{bmatrix} \tag{3.24}$$

and $\operatorname{rank}\bar{E}_1=2$.

Taking into account that

$$\bar{A}_{12}=\left[1 \quad 1 \quad -1\right] \tag{3.25}$$

and

$$\det\begin{bmatrix} \bar{E}_1 \\ \bar{A}_{12} \end{bmatrix}=\begin{vmatrix} L_1 & 0 & L_3 \\ 0 & L_2 & L_3 \\ 1 & 1 & -1 \end{vmatrix}=-\left[L_1\left(L_2+L_3\right)+L_2L_3\right]\neq 0 \tag{3.26}$$

from (2.10) we obtain

$$\begin{bmatrix} L_1 & 0 & L_3 \\ 0 & L_2 & L_3 \\ 1 & 1 & -1 \end{bmatrix} \dot{x} = \begin{bmatrix} -R_1 & 0 & -R_3 \\ 0 & -R_2 & -R_3 \\ 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} u \tag{3.27}$$

and

$$\dot{x} = \hat{A}_1 x + \hat{B}_{10} u, \tag{3.28a}$$

where

$$\begin{aligned} \hat{A}_1 &= \begin{bmatrix} L_1 & 0 & L_3 \\ 0 & L_2 & L_3 \\ 1 & 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} -R_1 & 0 & -R_3 \\ 0 & -R_2 & -R_3 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \frac{1}{\Delta_1} \begin{bmatrix} -R_1(L_2+L_3) & -R_2 L_3 & -R_3 L_2 \\ R_1 L_3 & -R_2(L_1+L_3) & -R_3 L_1 \\ -R_1 L_2 & -R_2 L_1 & -R_3(L_1+L_2) \end{bmatrix} \\ \hat{B}_{10} &= \begin{bmatrix} L_1 & 0 & L_3 \\ 0 & L_2 & L_3 \\ 1 & 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \frac{1}{\Delta_1} \begin{bmatrix} L_1+L_3 & -L_3 \\ -L_3 & L_1+L_3 \\ L_2 & L_1 \end{bmatrix}, \\ \Delta_1 &= L_1(L_2+L_3) + L_2 L_3. \end{aligned} \tag{3.28b}$$

<u>Method 3.</u>

From (3.17c) we have

$$i_3 = i_1 + i_2. \tag{3.29}$$

Substituting (3.29) into (3.17a) and (3.17b) we obtain

$$\begin{bmatrix} L_1+L_3 & L_3 \\ L_3 & L_2+L_3 \end{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} -(R_1+R_3) & -R_3 \\ -R_3 & -(R_2+R_3) \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = A \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + B_0 \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, \tag{3.30a}$$

where

$$\begin{aligned} A &= \begin{bmatrix} L_1+L_3 & L_3 \\ L_3 & L_2+L_3 \end{bmatrix}^{-1} \begin{bmatrix} -(R_1+R_3) & -R_3 \\ -R_3 & -(R_2+R_3) \end{bmatrix} \\ &= \frac{1}{\Delta_1} \begin{bmatrix} -R_1(L_2+L_3) - R_3 L_2 & R_2 L_3 - R_3 L_2 \\ R_1 L_3 - R_3 L_1 & -R_2(L_1+L_3) - R_3 L_1 \end{bmatrix}, \end{aligned} \tag{3.30b}$$

$$B=\begin{bmatrix} L_1+L_3 & L_3 \\ L_3 & L_2+L_3 \end{bmatrix}^{-1}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}=\frac{1}{\Delta_1}\begin{bmatrix} L_2+L_3 & -L_3 \\ -L_3 & L_1+L_3 \end{bmatrix}, \qquad (3.30c)$$

$$\Delta_1=L_1\left(L_2+L_3\right)+L_2L_3.$$

The characteristic polynomial of the matrix (3.30b) has the form

$$\det\left[I_2s-A\right]=\frac{1}{\Delta_1^2}\begin{vmatrix} \Delta_1 s+R_1\left(L_2+L_3\right)+R_3L_2 & R_3L_2-R_2L_3 \\ R_3L_1-R_1L_3 & \Delta_1 s+R_2\left(L_1+L_3\right)+R_3L_1 \end{vmatrix}$$

$$=s^2+\frac{1}{\Delta_1}\left[R_1\left(L_2+L_3\right)+R_2\left(L_1+L_3\right)+R_3\left(L_1+L_2\right)\right]s+\frac{1}{\Delta_1}\left[R_1\left(R_2+R_3\right)+R_2R_3\right], \qquad (3.31)$$

of the matrix A_1 (given by (3.23b))

$$\det\left[I_2s-A_1\right]=\begin{vmatrix} s+\frac{\Delta R_1L_1+R_3L_1-R_1L_3}{\Delta L_1^2} & -\frac{R_1L_3-R_3L_1}{\Delta L_1L_2} \\ -\frac{R_2L_3-R_3L_2}{\Delta L_1L_2} & s+\frac{\Delta R_2L_2+R_3L_2-R_2L_3}{\Delta L_2^2} \end{vmatrix}$$

$$=s^2+\frac{1}{\Delta_1}\left[R_1\left(L_2+L_3\right)+R_2\left(L_1+L_3\right)+R_3\left(L_1+L_2\right)\right]s+\frac{1}{\Delta_1}\left[R_1\left(R_2+R_3\right)+R_2R_3\right], \qquad (3.32)$$

and of the matrix $\hat{A}_1$ (given by (3.10b))

$$\det\left[I_3s-\hat{A}_1\right]=\begin{bmatrix} s+\frac{R_1\left(L_2+L_3\right)}{\Delta_1} & \frac{R_2L_3}{\Delta_1} & \frac{R_3L_2}{\Delta_1} \\ \frac{R_1L_3}{\Delta_1} & s+\frac{R_2\left(L_1+L_3\right)}{\Delta_1} & \frac{R_3L_1}{\Delta_1} \\ \frac{R_1L_2}{\Delta_1} & \frac{R_2L_1}{\Delta_1} & s+\frac{R_3\left(L_1+L_2\right)}{\Delta_1} \end{bmatrix}$$

$$=s\left\{s^2+\frac{1}{\Delta_1}\left[R_1\left(L_2+L_3\right)+R_2\left(L_1+L_3\right)+R_3\left(L_1+L_2\right)\right]s+\frac{1}{\Delta_1}\left[R_1\left(R_2+R_3\right)+R_2R_3\right]\right\} \qquad (3.33)$$

Note that in (3.33) the additional eigenvalue $s=0$ has been introduced in Method 2 by the differentiation with respect to time of the equation (2.8b).

From (3.19), (3.31), (3.32) and (3.33) it follows that the spectrum of the electrical circuit is the same for the three different methods and it is determined by the zeros of the polynomial (3.31).

3.2. General case

Note that the electrical circuit shown in Fig. 3.1 contains one mesh consisting of branches with only ideal capacitors and voltage sources and the one shown in Fig. 3.2 contains one node with branches with coils. The equations (3.1) and (3.17) are two differential equations and one algebraic equation.

In general case we have the following theorem.

Theorem 3.1. Every electrical circuit is a descriptor system if it contains at least one mesh consisting with only ideal capacitors and voltage sources or at least one node with branches with coils.

Proof. If the electrical circuit contains at least one mesh consisting of branches with ideal capacitors and voltage sources then the rows of the matrix E corresponding to the meshes are zero rows and the matrix E is singular. If the electrical circuit contains at least one node with branches with coils then the equations written on Kirchhoff's current law for these nodes are algebraic ones and the corresponding rows of E are zero rows and it is singular. □

Theorem 3.2. Every descriptor electrical circuit is a linear system with regular pencil.

Proof. It is well-known [15, 20, 22] that for a descriptor electrical circuit with n branches and q nodes using current Kirchhoff's law we can write $q-1$ algebraic equations and the voltage Kirchhoff's law $n-q+1$ differential equations. The equalities are linearly independent and can be written in the form (2.1). From linear independence of the equations it follows that the condition (2.2) is satisfied and the pencil of the electrical circuit is regular.

Remark 3.1. The spectrum of descriptor electrical circuits is independent of the method used of their analysis.

4. POINTWISE COMPLETENESS OF DESCRIPTOR ELECTRICAL CIRCUITS

Consider the descriptor electrical circuit described by the equation (2.1) for $u(t)=0$, $t \ge 0$.

Defining the new state vector

$$\overline{x} = \begin{bmatrix} \overline{x}_1 \\ \overline{x}_2 \end{bmatrix} = Q^{-1}x, \quad \overline{x}_1 \in \Re^{n_1}, \quad \overline{x}_2 \in \Re^{n_2}, \quad n = n_1 + n_2 \tag{4.1}$$

and using (2.1) for $u(t)=0$, $t \ge 0$ and (2.3) we obtain

$$\dot{\overline{x}}_1(t) = A_1\overline{x}_1(t) \tag{4.2}$$

and

$$\overline{x}_1(t) = e^{A_1 t}\overline{x}_{10}, \quad t \ge 0, \tag{4.3}$$

where

$$\bar{x}_{10} = \bar{Q}_1 x_0, \quad (4.4a)$$

$$\begin{bmatrix} \bar{x}_{10} \\ \bar{x}_{20} \end{bmatrix} = Q^{-1} x_0 = \begin{bmatrix} \bar{Q}_1 \\ \bar{Q}_2 \end{bmatrix} x_0, \quad \bar{Q}_1 \in \Re^{n_1 \times n}, \quad \bar{Q}_2 \in \Re^{n_2 \times n}. \quad (4.4b)$$

Note that [15]

$$\bar{x}_2(t) = -\sum_{k=0}^{q-2} \delta^{(k)} N^{k+1} \bar{x}_2(0) = 0, \quad t > 0 \quad (4.5)$$

where $\delta^{(k)}$ is the k-th derivative of the Dirac impulse. From (4.1) for $\bar{x}_2(t_f) = 0$ we have

$$x_f = x(t_f) = Q_1 \bar{x}_1(t_f), \quad (4.6)$$

where $Q = [Q_1 \quad Q_2]$, $Q_1 \in \Re^{n \times n_1}$, $Q_2 \in \Re^{n \times n_2}$.

Definition 4.1. The descriptor electrical circuit (2.1) is called pointwise complete for $t = t_f$ if for every final state $x_f \in \Re^n$ there exist initial conditions $x_0 \in \Re^n$ satisfying (4.4a) such that $x_f = x(t_f) \in \operatorname{Im} Q_1$.

Theorem 4.1. The descriptor electrical circuit is pointwise complete for any $t = t_f$ and every $x_f \in \Re^n$ satisfying the condition

$$x_f \in \operatorname{Im} Q_1 \quad (4.7)$$

Proof. Taking into account that for any A_1 $\det\left[e^{A_1 t}\right] \neq 0$ and $\left[e^{A_1 t}\right]^{-1} = e^{-A_1 t}$ from (4.3) and (4.5) for $t = t_f$ we obtain

$$\bar{x}_{10} = e^{-A_1 t} \bar{x}_1(t_f) \text{ and } \bar{x}_2(t_f) = 0. \quad (4.8)$$

Therefore, from (4.6) it follows that there exist initial conditions $x_0 \in \Re^n$ such that $x_f = x(t_f)$ if (4.7) holds. □

Example 4.1. (Continuation of Example 3.1)
In this case from (3.6c) we have

$$Q = [Q_1 \quad Q_2], \quad Q_1 = \begin{bmatrix} \frac{1}{R_1 C_1} & 0 \\ -\frac{1}{R_1(C_2 + C_3)} & \frac{1}{C_2 + C_3} \\ \frac{1}{R_1(C_2 + C_3)} & -\frac{1}{C_2 + C_3} \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 0 \\ \frac{C_3}{C_2} \\ 1 \end{bmatrix} \quad (4.9)$$

and

$$x_f \in \operatorname{Im} Q_1 = \begin{bmatrix} \frac{1}{R_1C_1}a \\ -\frac{1}{R_1(C_2+C_3)}a + \frac{1}{C_2+C_3}b \\ \frac{1}{R_1(C_2+C_3)}a - \frac{1}{C_2+C_3}b \end{bmatrix} \tag{4.10}$$

for arbitrary *a* and *b*.

The eigenvalues of the matrix A_1 (given by (3.6b)) are $s_1 = 0$, $s_2 = -\frac{C_1+C_2+C_3}{R_1C_1(C_2+C_3)}$ and using the Sylvester formula [15] we obtain

$$e^{A_1 t} = Z_1 + Z_2 e^{s_2 t} = \begin{bmatrix} e^{s_2 t} & \frac{R_1C_1}{C_1+C_2+C_3}\left(1-e^{s_2 t}\right) \\ 0 & 1 \end{bmatrix}, \tag{4.11a}$$

since

$$\begin{aligned} Z_1 &= \frac{A_1 - I_2 s_2}{s_1 - s_2} = I_2 - \frac{1}{s_2}A_1 = \begin{bmatrix} 0 & \frac{R_1C_1}{C_1+C_2+C_3} \\ 0 & 1 \end{bmatrix} \\ Z_2 &= \frac{A_1 - I_2 s_1}{s_2 - s_1} = \frac{1}{s_2}A_1 = \begin{bmatrix} 1 & -\frac{R_1C_1}{C_1+C_2+C_3} \\ 0 & 0 \end{bmatrix} \end{aligned} \tag{4.11b}$$

Therefore, the descriptor electrical circuit shown in Fig. 3.1 is pointwise complete for any $t = t_f$ and every x_f satisfying (4.10).

5. POINTWISE GENERACY OF DESCRIPTOR ELECTRICAL CIRCUIT

Consider the descriptor electrical circuit described by equation (2.1) for $u(t) = 0$, $t \geq 0$.

Definition 5.1. The descriptor electrical circuit (2.1) is called pointwise degenerated in the direction $v \in \Re^n$ for $t = t_f$ if there exists nonzero vector *v* such that for all initial conditions $x_0 \in \operatorname{Im} Q_1$ the solution of (2.1) satisfies the condition

$$v^T x_f = 0 \quad (T \text{ denotes transpose}) \tag{5.1}$$

Theorem 5.1. The descriptor electrical circuit (2.1) is pointwise degenerated in the direction v defined by

$$v^T Q_1 = 0 \tag{5.2}$$

for any $t_f > 0$ and all initial conditions $\bar{x}_{10} \in \operatorname{Im} \overline{Q}_1$, where Q_1 and $\overline{Q}_1$ are determined by (4.6) and (4.4), respectively.

Proof. Substitution of (4.3) into (4.6) yields

$$x_f = Q_1 e^{A_1 t_f} \bar{x}_{10} \tag{5.3}$$

and

$$v^T x_f = v^T Q_1 e^{A_1 t_f} \bar{x}_{10} = 0 \tag{5.4}$$

since (5.2) holds for all $\bar{x}_{10} = \overline{Q}_1 x_0 \in \operatorname{Im} \overline{Q}_1$. □

Example 5.1. (Continuation of Example 3.1 and 4.1)

From (5.2) and (4.9) we have

$$v^T = [0 \quad 1 \quad 1] \tag{5.5}$$

since

$$v^T Q_1 = [0 \quad 1 \quad 1] \begin{bmatrix} \frac{1}{R_1 C_1} & 0 \\ -\frac{1}{R_1(C_2 + C_3)} & \frac{1}{C_2 + C_3} \\ \frac{1}{R_1(C_2 + C_3)} & -\frac{1}{C_2 + C_3} \end{bmatrix} = [0 \quad 0]. \tag{5.6}$$

Therefore, the descriptor electrical circuit shown in Fig. 3.1 in pointwise generated in the direction defined by (5.5) for any $t_f > 0$ and any values of the resistance R_1 and capacitances C_1, C_2, C_3.

6. CONCLUDING REMARKS

The regularity, pointwise completeness and pointwise generacy of descriptor linear electrical circuits composed of resistances, capacitances, inductances and voltage (current) sources have been investigated. Three basic methods of the analysis of the descriptor linear systems with regular pencils have been presented. It has been shown that every descriptor electrical circuit is a linear descriptor system with regular pencil. Conditions for the pointwise completeness and pointwise generacy of the descriptor electrical circuits have been established and illustrated by simple electrical circuits. The considerations can be extended to the fractional descriptor electrical circuits [22].

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