Scientific Issues
Jan Długosz University
in Częstochowa
Mathematics XIX (2014)
29–36

SOME REMARKS ON FLUID FLOW IN HOURGLASSES

ARKADIUSZ BRYLL, ROBERT SOCHACKI

Abstract

In the paper the authors analyse different shapes of an hourglass for the linearity of their graduation. We also assume that any hourglass (more precisely, each of the two congruent parts) has the shape of a solid of revolution and any cross section at height h of this hourglass depends on the base radius r, i.e. $h = f(r)$.

1. Introduction

In the paper [2] the author analyses three different shapes of an hourglass for the linearity of their graduation. We think (and this is the purpose of this paper) that it is worthwhile expanding and generalizing the results contained in the above mentioned work. Accordingly, we will use the simplest knowledge of higher mathematics (see [1]). More specifically, we will make use of the solution of a differential equation with separated variables in the form:

$$F(h)dh = \alpha dt,$$

where $F(h)$ is a given function and $\alpha = const$. We also assume that an hourglass (more precisely, each of the two congruent parts) has the shape of a solid of revolution and any cross section at height h of this hourglass depends on the base radius r, i.e., $h = f(r)$. The flow of liquid from the upper part to the lower part depends on the shape of an hourglass, and the hole cross section of the flow.

• *Arkadiusz Bryll* — e-mail: tex@op.pl
Częstochowa University of Technology.
• *Robert Sochacki* — e-mail: rsochacki@uni.opole.pl
University of Opole.

2. Main results

We will consider hourglasses for which the upper part is described by the equations of curves: $r = r_0$, $h = ar^b$, $\frac{r^2}{a^2} + \frac{h^2}{b^2} = 1$, $\frac{r^2}{a^2} - \frac{h^2}{b^2} = 1$, $(a > 0, b > 0)$. We rotate these curves around the axis Oh in the co-ordinate system Orh. Moreover, we will analyse an hourglass in which any change of the liquid column depending on the time has a linear dependence. Rotating the mentioned above curves, we obtain the following solids of revolution: a cylindrical surface, a conical surface (when $h = ar$, $a > 0$), a paraboloid of revolution, and a hyperboloid of one sheet (see Fig. 1).

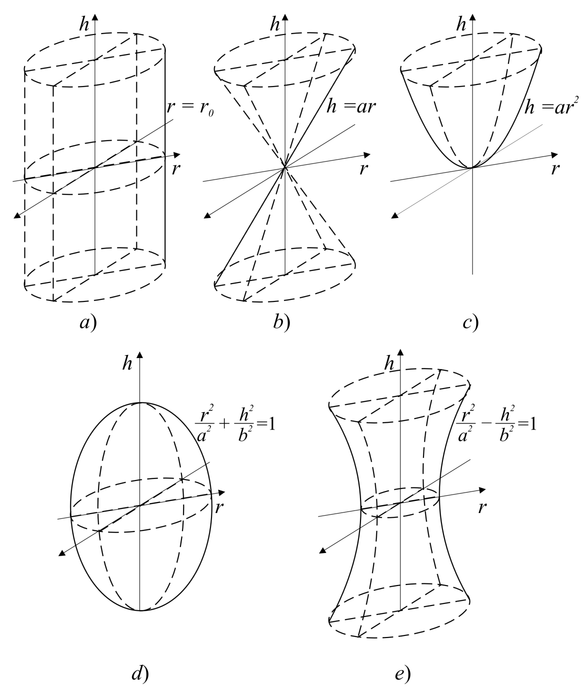

Figure 1

Let $r(t)$ be the base radius of the cross section of an hourglass at height $h(t)$ in the time t (see Fig.2).

The volume of the liquid passing through the cross-sectional area of S at the time t has the form:

$$V(t) = v(t) \cdot S \cdot dt, \tag{1}$$

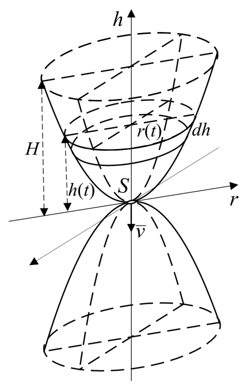

FIGURE 2

where $v(t)$ is the flow velocity of the liquid through the cross-sectional area of S. By the Bernoulli law, the velocity is:

$$v(t) = \sqrt{2g \cdot h(t)},$$

where g is the gravitational acceleration. On the other hand, we have the following formula:

$$V(t) = -\pi r^2(t)dh, \qquad (dh < 0). \tag{2}$$

Using (1) and (2), we obtain:

$$-\pi r^2(t)dh = v(t)Sdt,$$

i.e.

$$-\pi r^2(t)dh = \sqrt{2g \cdot h(t)}Sdt.$$

Hence:

$$\frac{r^2(t)dh}{\sqrt{h(t)}} = -kdt, \tag{3}$$

where $k = \frac{\sqrt{2g}S}{\pi}$. Thus, we obtain the differential equation with separated variables. The solution of this equation depends on the choice of the radius $r(t)$. We will consider four cases.

I. $r(t) = r_0, \quad r_0 = const$

If $r(t) = r_0$, then the solid of revolution has the shape of a cylinder (see Fig. 1a). Then, the formula (3) has the form:

$$\frac{dh}{\sqrt{h(t)}} = -\frac{k}{r_0^2}dt.$$

Thus, after the integration we obtain:

$$2\sqrt{h} = -\frac{k}{r_0^2}t + C.$$

Based on the assumption that $h(0) = H$, the integration constant C equals $2\sqrt{H}$. Thus, we have:

$$2\sqrt{h} = -\frac{k}{r_0^2}t + 2\sqrt{H},$$

$$h(t) = \left(-\frac{k}{2r_0^2}t + \sqrt{H}\right)^2,$$

hence:

$$h(t) = \left(-\frac{\sqrt{2g}S}{2\pi r_0^2}t + \sqrt{H}\right)^2. \tag{4}$$

If $h(t) = 0$, then the upper vessel is empty and all the liquid is in the lower vessel. Then, the total time of the flow of the liquid from the vessel top to the bottom one is:

$$t = \frac{2\pi r_0^2\sqrt{H}}{\sqrt{2g}S} = \sqrt{\frac{2H}{g}} \cdot \frac{\pi r_0^2}{S}.$$

Thus, the lower vessel of the hourglass can be scaled by the relationship (see Fig. 3):

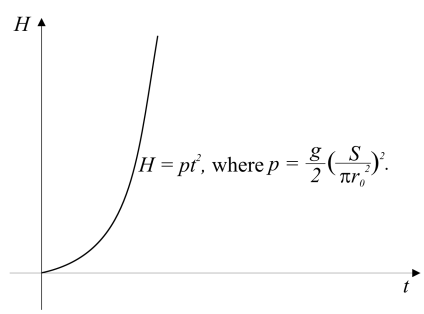

FIGURE 3

$$H = \frac{g}{2}\left(\frac{S}{\pi r_0^2}\right)^2 t^2. \tag{5}$$

II. $h = ar^b, \quad (a > 0, \quad b > 0)$

If $b = 1$, then the solid of revolution has the shape of a conic (see Fig. 1b). In the case of $b = 2$, the solid of revolution has the shape of a paraboloid (see Fig. 1c). We will solve the equation (3) for any a and b $(a > 0, b > 0)$. From the assumption $h = ar^b$ it follows that $r = \left(\frac{h}{a}\right)^{\frac{1}{b}}$ and the equation (3) has the form:

$$\frac{\left(\frac{h}{a}\right)^{\frac{2}{b}} dh}{\sqrt{h}} = -kt,$$

that is:

$$h^{\frac{4-b}{2b}} dh = -ka^{\frac{2}{b}} dt.$$

Thus, after the integration we obtain:

$$\frac{2b}{4+b} h^{\frac{4+b}{2b}} = -ka^{\frac{2}{b}} t + C_1.$$

From the condition $h(0) = H$ we have:

$$C_1 = \frac{2b}{4+b} H^{\frac{4+b}{2b}}.$$

Thus, finally:

$$\frac{2b}{4+b} h^{\frac{4+b}{2b}} = -ka^{\frac{2}{b}} t + \frac{2b}{4+b} H^{\frac{4+b}{2b}}.$$

Hence we can find the formula of the function h:

$$h(t) = \left(-\frac{4+b}{2b} ka^{\frac{2}{b}} t + H^{\frac{4+b}{2b}}\right)^{\frac{2b}{4+b}}.$$

Taking into account that $k = \frac{\sqrt{2g}S}{\pi}$, we have:

$$h(t) = \left(-\frac{4+b}{2b}\frac{\sqrt{2g}}{\pi} Sa^{\frac{2}{b}} t + H^{\frac{4+b}{2b}}\right)^{\frac{2b}{4+b}}. \tag{6}$$

In the case of the conic $(r = \frac{h}{a},\ b = 1)$ we obtain:

$$h(t) = \left(-\frac{5}{2}\frac{\sqrt{2g}}{\pi} Sa^2 t + H^{\frac{5}{2}}\right)^{\frac{2}{5}}, \tag{7}$$

while, in the case of the paraboloid $(r = \left(\frac{h}{a}\right)^{\frac{1}{2}},\ b = 2)$ we obtain:

$$h(t) = \left(-\frac{3}{2}\frac{\sqrt{2g}}{\pi} Sat + H^{\frac{3}{2}}\right)^{\frac{2}{3}}. \tag{8}$$

On the basis of (6) we can determine the total time to empty the upper vessel. Taking $h(t) = 0$, we obtain:

$$t = \frac{2b\pi a^{-\frac{2}{b}}}{(4+b)\sqrt{2g}S} \cdot H^{\frac{4+b}{2b}}. \tag{9}$$

For the conic ($b = 1$), the total time to empty the upper vessel is:

$$t = \frac{2\pi a^{-2}}{5\sqrt{2g}S} \cdot H^{\frac{5}{2}}.$$

Since $a = \frac{h}{r} = \frac{H}{R}$, hence:

$$t = \frac{1}{5}\sqrt{\frac{2H}{g}} \cdot \frac{\pi R^2}{S}.$$

For the paraboloid of revolution ($b = 2$), the total time to empty the vessel is:

$$t = \frac{2}{3}\frac{\pi a^{-1}}{\sqrt{2g}S} \cdot H^{\frac{3}{2}}.$$

In the case of $a = \frac{h}{r^2} = \frac{H}{R^2}$ we obtain:

$$t = \frac{1}{3}\sqrt{\frac{2H}{g}} \cdot \frac{\pi R^2}{S}.$$

The lower vessel of the hourglass can be scaled by the formula (see (9)):

$$H = \left[\frac{4+b}{2b} \cdot \frac{\sqrt{2g}}{\pi} S a^{\frac{2}{b}}\right]^{\frac{2b}{4+b}} \cdot t^{\frac{2b}{4+b}}. \tag{10}$$

In the case of the conic ($b = 1$) we have:

$$H = \left[\frac{5}{2} \cdot \frac{\sqrt{2g}}{\pi} S a^2\right]^{\frac{2}{5}} \cdot t^{\frac{2}{5}}.$$

Moreover, if we consider the paraboloid ($b = 2$), then:

$$H = \left[\frac{3}{2} \cdot \frac{\sqrt{2g}}{\pi} S a\right]^{\frac{2}{3}} \cdot t^{\frac{2}{3}}.$$

III. $\frac{r^2}{a^2} \pm \frac{h^2}{b^2} = 1$

In this case the solid of revolution has the shape of ellipsoid (Fig.1d) or the shape of hyperboloid of one sheet (Fig.1e). Since $r^2 = \left(\frac{a}{b}\right)^2 (b^2 \mp h^2)$, thus the equation (3) has the form:

$$\frac{\left(\frac{a}{b}\right)^2 \left(b^2 \mp h^2\right)}{\sqrt{h}} dh = -k dt,$$

that is

$$\left(\frac{a}{b}\right)^2\left(b^2h^{-\frac{1}{2}}\mp h^{\frac{3}{2}}\right)dh=-kdt.$$

After the integration we obtain:

$$\left(\frac{a}{b}\right)^2\left(2b^2h^{\frac{1}{2}}\mp\frac{2}{5}h^{\frac{5}{2}}\right)=-kt+C_2. \tag{11}$$

From the condition $h(0)=H$ we can determine the constant C_2:

$$C_2=\left(\frac{a}{b}\right)^2\sqrt{H}\left(2b^2\mp\frac{2}{5}H^2\right).$$

Thus, finally, the equation (11) has the form:

$$\left(\frac{a}{b}\right)^2\left[\left(2b^2h^{\frac{1}{2}}\mp\frac{2}{5}h^{\frac{5}{2}}\right)-\left(2b^2H^{\frac{1}{2}}\mp\frac{2}{5}H^{\frac{5}{2}}\right)\right]=-kt, \tag{12}$$

where $k=\frac{\sqrt{2g}S}{\pi}$.

The upper vessel will be empty when $h(t)=0$. Then:

$$t=\frac{1}{k}\left(\frac{a}{b}\right)^2\sqrt{H}\left(2b^2\mp\frac{2}{5}H^2\right).$$

Note that the "$-$" sign refers to the ellipsoid, while the "$+$" sign refers to the hyperboloid of one sheet.

IV. $h(t)=-wt+H$

This case concerns the situation in which the change of the column of the liquid in the hourglass, depends linear on the time. Indeed, if $\frac{dh}{dt}=-w$ and $w=const$, then $h(t)=-wt+C_3$, wherein $h(0)=H$. So, we have $h(t)=-wt+H$ (see Fig. 4).

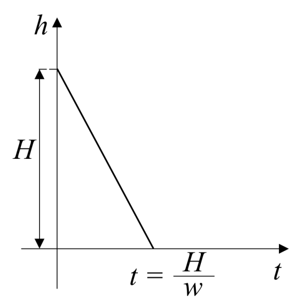

FIGURE 4

By the assumption $\frac{dh}{dt}=-w$, the equation (3) has the form:

$$\frac{r^2(t)}{\sqrt{h(t)}}(-w)=-k,$$

that is

$$\sqrt{h(t)} = \frac{r^2 w}{k},$$

hence

$$h = \left(\frac{w}{k}\right)^2 \cdot r^4. \tag{13}$$

This formula allows to determine the shape of the curve $h = f(r)$, if the height of the liquid column decreases linearly with the time and $f(r)$ is a bi-quadratic function.

3. Final remarks

As the above analysis shows, shaping hourglasses as typical solids of revolution does not guarantee the linearity of their graduation. This can be seen in the following equations: (4), (7), (8) and (12). If we still do want to obtain the linearity for the lower part of the hourglass, we must shape it another way. The shape is defined by the formula (13).

References

[1] F. Leja, *Differential and Integral Calculus* (in Polish), PWN, Warszawa 2012.
[2] K. Omiljanowski, *Make yourself an hourglass*, (in Polish), Magazyn Miłośników Matematyki. **1**, (2010), 15-17.

Received: May 2014

Arkadiusz Bryll
Częstochowa University of Technology
Faculty of Management
ul. Armii Krajowej 19B, Częstochowa
E-mail address: `tex@op.pl`

Robert Sochacki
University of Opole
Institute of Philosophy, Department of Logic and Epistemology
ul. Katowicka 89, Opole
E-mail address: `rsochacki@uni.opole.pl`