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STABILITY BY KRASNOSELSKII'S THEOREM IN TOTALLY NONLINEAR NEUTRAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper we use fixed point methods to prove asymptotic stability results of the zero solution of a class of totally nonlinear neutral differential equations with functional delay. The study concerns

$$x'(t) = -a(t)x^3(t) + c(t)x'(t - r(t)) + b(t)x^3(t - r(t)).$$

The equation has proved very challenging in the theory of Liapunov's direct method. The stability results are obtained by means of Krasnoselskii-Burton's theorem and they improve on the work of T.A. Burton (see Theorem 4 in [*Liapunov functionals, fixed points, and stability by Krasnoselskii's theorem*, Nonlinear Studies **9** (2001), 181–190]) in which he takes $c = 0$ in the above equation.

Keywords: fixed point, stability, nonlinear neutral equation, Krasnoselskii-Burton theorem.

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1. INTRODUCTION

Without doubt, Liapunov's direct method has been, for more than 100 years, the main tool for investigating the stability properties of a wide variety of ordinary, functional, partial differential and Volterra integral equations. Nevertheless, the application of this method to problems of stability in differential and Volterra integral equations with delay has encountered serious obstacles if the delay is unbounded or if the equation has unbounded terms (see [2, 6–8, 10–14, 19, 23] and references therein) and it does seem that other ways need to be investigated. In recent years, several investigators have tried stability by using a new technique. Particularly, Burton, Furumochi, Zhang and others began a study in which they noticed that some of these difficulties vanish or might be overcome by means of fixed point theory (see [1, 2, 6–17, 20, 21] and [23]).

The most striking object is that the fixed point method does not only solve the problem on stability but has a significant advantage over Liapunov's direct method. The conditions of the former are often averages but those of the latter are usually pointwise (see [2]). While it remains an art to construct a Liapunov's functional when it exists, a fixed point method, in one step, yields existence, uniqueness and stability. All we need, to use the fixed point method, is a complete metric space, a suitable fixed point theorem and an elementary variation of parameters formula to solve problems that have frustrated investigators for decades.

Below, we present a study which concerns a totally nonlinear neutral differential equation with functional delay. In our situation it is necessary to invert the differential equation to obtain a mapping equation suitable for the fixed point theory. Unfortunately, our equation does not contains a linear term, the variation of parameters cannot be used. So, we resort to the old method of adding and substracting a linear term for the mapping.

Our equation is a totally nonlinear neutral differential equation with functional delay expressed as follows

$$x'(t) = -a(t)x^3(t) + c(t)x'(t - r(t)) + b(t)x^3(t - r(t)), \quad t \geq 0, \tag{1.1}$$

with an assumed initial function

$$x(t) = \psi(t), \quad t \in [m_0, 0],$$

with $\psi \in C([m_0, 0], \mathbb{R})$, $[m_0, 0] = \{u \leq 0 \mid\ u = t - r(t),\ t \geq 0\}$. Throughout this paper we assume that $a, b \in C(\mathbb{R}_+, \mathbb{R})$ with $a(t) \geq 0$, $c \in C^1(\mathbb{R}_+, \mathbb{R})$ and $r \in C^2(\mathbb{R}_+, \mathbb{R})$ such that

$$r'(t) \neq 1, \quad t \in \mathbb{R}_+. \tag{1.2}$$

Special cases of equation (1.1) have been recently considered and studied under various conditions and with several methods. Particularly, when $c = 0$ we obtain the following delay equation

$$x'(t) = -a(t)x^3(t) + b(t)x^3(t - r(t)), \quad t \geq 0. \tag{1.3}$$

A stability result which concerns (1.3), when both a and b are bounded, may be found in [18, p. 117].

In his book [2], Burton has devoted a large space to this challenging equation (1.3) and has discussed it under a variety of assumptions. He has studied (1.3) using both techniques, fixed point theory and Liapunov's. He clarified the difficulties encountered in studying this problem via liapunov's direct method if no restrictions are given on a, b and r. He successfully, demonstrated, by means of a new version of Krasnoseskii fixed point theorem, developed by him, how to overcome those difficulties. Burton has been able, in his work, to avoid the derivative $r'(t)$ of the delay which provoke serious difficulties and proved that the solutions are bounded and converge to zero at infinity. Such a convergence of solutions has not been seen before the publication of his paper [6]. More precisely, he established and proved the following theorem (see [6] or [2], Theorem 2.8.1, p. 196).

Theorem 1.1 (Burton)**.** *Suppose the following conditions are true:*

$$\int_0^\infty a(u)du = \infty, \quad J\,|b(t)| \le a(t) \quad with \quad J > 1,$$

$$\frac{b(t)}{a(t)} \to \infty \quad as \quad t \to \infty.$$

If $L = \sqrt{3}/3$ *and if* ψ *is continuous satisfying*

$$\|\psi\| + \frac{2\sqrt{3}}{9} + \frac{\sqrt{3}}{9J} \le L,$$

then there exists a solution of (1.3) *with* $|x(t,0,\psi)| < L$ *for* $t \ge 0$ *and* $x(t,0,\psi) \to 0$ *as* $t \to \infty$.

Motivated by the work of Burton we will try, here, to give a study of boundedness and stability of the zero solution which concerns the neutral type of totally nonlinear differential equation (1.1).

2. THE INVERSION AND THE FIXED POINT THEOREM

We have to invert (1.1) and during the process an integration by parts will have to be performed on the neutral term $x'(t - r(t))$. Unfortunately, when doing this, a derivative $r'(t)$ of the delay will appear on the way and so we have to support it.

Lemma 2.1. *Suppose* (1.2) *holds.* $x(t)$ *is a solution of equation* (1.1) *if and only if*

$$\begin{aligned} x(t) = &\left[\psi(0) - \frac{c(0)}{1-r'(0)}\psi(-r(0))\right] e^{-\int_0^t a(u)du} + \\ &+ \frac{c(t)}{1-r'(t)} x(t-r(t)) - \int_0^t \mu(s) x(s-r(s)) e^{-\int_s^t a(u)du} ds + \\ &+ \int_0^t b(s) x^3(s-r(s)) e^{-\int_s^t a(u)du} ds + \int_0^t a(s)\left[x(s) - x^3(s)\right] e^{-\int_s^t a(u)du} ds, \end{aligned} \tag{2.1}$$

where

$$\mu(t) = \frac{(c'(t) + a(t)c(t))(1-r'(t)) + c(t)r''(t)}{(1-r'(t))^2}. \tag{2.2}$$

Proof. Let $x(t)$ be a solution of equation (1.1). Rewrite (1.1) as

$$x'(t) + a(t)x(t) = a(t)x(t) - a(t)x^3(t) + c(t)x'(t-r(t)) + b(t)x^3(t-r(t)).$$

Multiply both sides of the above equation by $e^{\int_0^t a(u)du}$ and then integrate from 0 to t to obtain

$$
\begin{aligned}
x(t) = {} & \psi(0)\, e^{-\int_0^t a(u)du} + \\
& + \int_0^t a(s)\left[x(s) - x^3(s)\right] e^{-\int_s^t a(u)du} ds + \int_0^t c(s) x'(s - r(s))\, e^{-\int_s^t a(u)du} ds + \\
& + \int_0^t b(s) x^3(s - r(s))\, e^{-\int_s^t a(u)du} ds.
\end{aligned} \tag{2.3}
$$

Letting

$$
\int_0^t c(s) x'(s - r(s))\, e^{-\int_s^t a(u)du} ds = \int_0^t \frac{c(s)}{(1 - r'(s))} (1 - r'(s))\, x'(s - r(s))\, e^{-\int_s^t a(u)du} ds.
$$

By performing an integration by parts, we obtain

$$
\begin{aligned}
& \int_0^t c(s) x'(s - r(s))\, e^{-\int_s^t a(u)du} ds = \\
& = \frac{c(t)}{1 - r'(t)} x(t - r(t)) - \frac{c(0)}{1 - r'(0)} \psi(-r(0))\, e^{-\int_0^t a(u)du} - \\
& - \int_0^t \mu(s) x(s - r(s))\, e^{-\int_s^t a(u)du} ds,
\end{aligned} \tag{2.4}
$$

where $\mu(s)$ is given by (2.2). Finally, substituting (2.4) in (2.3) ends the proof. $\square$

T.A. Burton studied the theorem of Krasnoselskii and observed (see [3–5] and [9]) that Krasnoselskii's result can be more interesting in applications with certain changes and formulated the Theorem 2.3 below (see [3] for its proof).

Let (M, d) be a metric space and $F : M \to M$. F is said to be a large contraction if $\varphi, \psi \in M$, with $\varphi \neq \psi$ then $d(F\varphi, F\psi) < d(\varphi, \psi)$ and if for each $\varepsilon > 0$ there exists $\eta < 1$ such that

$$
[\varphi, \psi \in M,\ \ d(\varphi, \psi) \geq \varepsilon] \Rightarrow d(F\varphi, F\psi) \leq \eta d(\varphi, \psi).
$$

Theorem 2.2 (Burton)**.** *Let (M, d) be a complete metric space and F be a large contraction. Suppose there is an $x \in M$ and an $\rho > 0$ such that $d(x, F^n x) \leq \rho$ for all $n \geq 1$. Then F has a unique fixed point in M.*

Below we state Krasnoselskii-Burton's hybrid fixed point theorem which will enable us to establish a stability result of the trivial solution of (1.1). For more details on Krasnoselskii's captivating theorem we refer to smart [22] or [2].

Theorem 2.3 (Krasnoselskii-Burton). *Let M be a bounded, closed and convex nonempty subset of a Banach space $(S, \|\cdot\|)$. Suppose that A, B map M into M and that*

(i) *$Ax + By \in M$ for all $x, y \in M$,*
(ii) *A is continuous and AM is contained in a compact subset of M,*
(iii) *B is a large contraction.*

Then there is a $z \in M$ with $z = Az + Bz$.

It is worth mentioning that the third author with H. Deham (see [15, 16]) has proved exitence results of periodic solutions, by means of Theorem 2.3, of equations very close to (1.1) and (1.3).

Here we manipulate function spaces defined on infinite t-intervals. So, for compactness we need an extension of the Arzelà-Ascoli theorem. This extension is taken from [2, Theorem 1.2.2 p. 20] and is as follows.

Theorem 2.4. *Let $q : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous function such that $q(t) \to 0$ as $t \to \infty$. If $\{\varphi_n(t)\}$ is an equicontinuous sequence of $\mathbb{R}^m$-valued functions on $\mathbb{R}_+$ with $|\varphi_n(t)| \le q(t)$ for $t \in \mathbb{R}_+$, then there is a subsequence that converges uniformly on $\mathbb{R}_+$ to a continuous function $\varphi(t)$ with $|\varphi(t)| \le q(t)$ for $t \in \mathbb{R}_+$, where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^m$.*

3. STABILITY BY THE KRASNOSELSKII-BURTON'S THEOREM

From the existence theory, which can be found in [18] or [2], we conclude that for each continuous initial function $\psi : [m_0, 0] \to \mathbb{R}$, there exists a continuous solution $x(t, 0, \psi)$ which satisfies (1.1) on an interval $[0, \sigma)$ for some $\sigma > 0$ and $x(t, 0, \psi) = \psi(t)$, $t \in [m_0, 0]$. We refer to [2] for the stability definitions.

Let S be the Banach space of bounded continuous functions $\varphi : [m_0, \infty) \to \mathbb{R}$ with the supremum norm $\|\cdot\|$.

To apply Theorem 2.3, we have to construct two mappings, a large contraction and a compact operator. So, let S be the Banach space of continuous bounded functions with the supremum norm $\|\cdot\|$. Let $L = \frac{1}{\sqrt{3}}$ and define the set

$$\begin{aligned} S_\psi := \{\, &\varphi \in S \,|\, \varphi \text{ is Lipschitzian, } |\varphi(t)| \le L,\ t \in [m_0, \infty)\,, \\ &\varphi(t) = \psi(t) \text{ if } t \in [m_0, 0] \text{ and } \varphi(t) \to 0 \text{ as } t \to \infty \,\}\,. \end{aligned}$$

Clearly, if $\{\varphi_n\}$ is a sequence of k-Lipschitzian functions converging to some function φ, then

$$\begin{aligned} |\varphi(u) - \varphi(v)| &\le |\varphi(u) - \varphi_n(u)| + |\varphi_n(u) - \varphi_n(v)| + |\varphi_n(v) - \varphi(v)| \le \\ &\le \|\varphi - \varphi_n\| + k\,|u - v| + \|\varphi - \varphi_n\|\,. \end{aligned}$$

Consequently, as $n \to \infty$, we see that φ is k-Lipschitzian. It is clear that S_ψ is convex, bounded and complete endowed with $\|\cdot\|$.

For $\varphi \in S_\psi$ and $t \geq 0$, we define the maps A, B and H on S_ψ as follows:

$$\begin{aligned} A\varphi(t) := {} & \frac{c(t)}{1-r'(t)}\varphi(t-r(t)) + \int_0^t b(s)\varphi^3(s-r(s))\, e^{-\int_s^t a(u)du} ds + \\ & + \int_0^t \mu(s)\varphi(s-r(s))\, e^{-\int_s^t a(u)du} ds, \end{aligned} \tag{3.1}$$

$$\begin{aligned} B\varphi(t) := {} & \left[\psi(0) - \frac{c(0)}{1-r'(0)}\psi(-r(0))\right] e^{-\int_0^t a(u)du} + \\ & + \int_0^t a(s)\left[\varphi(s) - \varphi^3(s)\right] e^{-\int_s^t a(u)du} ds, \end{aligned} \tag{3.2}$$

and

$$H\varphi(t) := A\varphi(t) + B\varphi(t). \tag{3.3}$$

If we are able to prove that H possesses a fixed point φ on the set S_ψ, then $x(t,0,\psi) = \varphi(t)$ for $t \geq 0$, $x(t,0,\psi) = \psi(t)$ on $[m_0, 0]$, $x(t,0,\psi)$ satisfies (1.1) when its derivative exists and $x(t,0,\psi) \to 0$ as $t \to \infty$.

Let $\alpha(t) = \frac{c(t)}{1-r'(t)}$ and assume that there are constants $k_1, k_2, k_3 > 0$ such that for $0 \leq t_1 < t_2$

$$\left|\int_{t_1}^{t_2} a(u)du\right| \leq k_1 |t_2 - t_1|, \tag{3.4}$$

$$|r(t_2) - r(t_1)| \leq k_2 |t_2 - t_1|, \tag{3.5}$$

and

$$|\alpha(t_2) - \alpha(t_1)| \leq k_3 |t_2 - t_1|. \tag{3.6}$$

Suppose that for $t \geq 0$,

$$|\mu(t)| \leq \delta a(t), \tag{3.7}$$

$$|b(t)|\, L^2 \leq \beta a(t), \tag{3.8}$$

$$\sup_{t\geq 0} |\alpha(t)| = \alpha, \tag{3.9}$$

and that

$$J(\alpha + \beta + \delta) < 1, \tag{3.10}$$

where α, β, δ and J are constants with $J > 3$.

Choose $\gamma > 0$ small enough such that

$$\left(1 + \left|\frac{c(0)}{1 - r'(0)}\right|\right) \gamma e^{-\int_0^t a(u)du} + \frac{L}{J} + \frac{2L}{3} \leq L. \tag{3.11}$$

The chosen γ in the relation (3.11) will be used below in Lemma 3.3 to show that if $\varepsilon = \sqrt{3}/3$ and if $\|\psi\| < \gamma$, then the solutions satisfies $|x(t, 0, \psi)| < \varepsilon$.

Assume further that

$$t - r(t) \to \infty \quad \text{as} \quad t \to \infty \quad \text{and} \quad \int_0^t a(u)du \to \infty \quad \text{as} \quad t \to \infty, \tag{3.12}$$

$$\alpha(t) \to 0 \quad \text{as} \quad t \to \infty, \tag{3.13}$$

$$\frac{\mu(t)}{a(t)} \to 0 \quad \text{as} \quad t \to \infty, \tag{3.14}$$

and

$$\frac{b(t)}{a(t)} \to 0 \quad \text{as} \quad t \to \infty. \tag{3.15}$$

First we begin with the following example (see [2, 6]).

Example 3.1. Let $\|\cdot\|$ be the supremum norm,

$$M := \left\{\varphi : \mathbb{R} \to \mathbb{R} \,|\, \varphi \text{ is continuous, } \|\varphi\| \leq \sqrt{3}/3\right\},$$

and define $(F\varphi)(t) := \varphi(t) - \varphi^3(t)$. Then F is a large contraction of the set M.

Indeed, for each $t \in \mathbb{R}$, we have for φ, ψ real functions

$$\begin{aligned}|(F\varphi)(t) - (F\psi)(t)| &= \left|\varphi(t) - \varphi^3(t) - \psi(t) + \psi^3(t)\right| = \\ &= |\varphi(t) - \psi(t)| \left|1 - \left(\varphi^2(t) + \varphi(t)\psi(t) + \psi^2(t)\right)\right|.\end{aligned}$$

Then for

$$|\varphi(t) - \psi(t)|^2 = \varphi^2(t) - 2\varphi(t)\psi(t) + \psi^2(t) \leq 2(\varphi^2(t) + \psi^2(t)),$$

and for $\varphi^2(t) + \psi^2(t) < 1$, we have

$$\begin{aligned}|(F\varphi)(t) - (F\psi)(t)| &= |\varphi(t) - \psi(t)| \left[1 - \left(\varphi^2(t) + \psi^2(t)\right) + |\varphi(t)\psi(t)|\right] \leq \\ &\leq |\varphi(t) - \psi(t)| \left[1 - \left(\varphi^2(t) + \psi^2(t)\right) + \frac{\varphi^2(t) + \psi^2(t)}{2}\right] \leq \\ &\leq |\varphi(t) - \psi(t)| \left[1 - \frac{\varphi^2(t) + \psi^2(t)}{2}\right].\end{aligned}$$

Thus, F is pointwise a large contraction. But application F is still a large contraction for the supremum norm. Let $\varepsilon \in (0, 1)$ be given and let $\varphi, \psi \in M$ with $\|\varphi - \psi\| \geq \varepsilon$.

a) Suppose that for some t we have

$$\varepsilon/2 \leq |\varphi(t) - \psi(t)|.$$

Then

$$(\varepsilon/2)^2 \leq |\varphi(t) - \psi(t)|^2 \leq 2(\varphi^2(t) + \psi^2(t)),$$

that is

$$\varphi^2(t) + \psi^2(t) \geq \varepsilon^2/8.$$

For all such t we have

$$|(F\varphi)(t) - (F\psi)(t)| \leq |\varphi(t) - \psi(t)| \left[1 - \frac{\varepsilon^2}{16}\right] \leq$$
$$\leq \|\varphi - \psi\| \left[1 - \frac{\varepsilon^2}{16}\right].$$

b) Suppose that for some t we have

$$|\varphi(t) - \psi(t)| \leq \varepsilon/2.$$

Then

$$|(F\varphi)(t) - (F\psi)(t)| \leq |\varphi(t) - \psi(t)| \leq (1/2)\|\varphi - \psi\|.$$

Consequently, we obtain

$$\|F\varphi - F\psi\| \leq \min\left\{\frac{1}{2}, 1 - \frac{\varepsilon^2}{16}\right\} \|\varphi - \psi\|.$$

By a series of steps we will prove the fulfillment of (i), (ii) and (iii) in Theorem 2.3.

Lemma 3.2. *Suppose that* (3.7)–(3.10) *and* (3.12) *are true. For A defined in* (3.1)*, if $\varphi \in S_\psi$, then $|A\varphi(t)| \leq L/J < L$. Moreover, $A\varphi(t) \to 0$ as $t \to \infty$.*

Proof. Using the conditions (3.7)–(3.10) and the expression (3.1) of the map A, we get

$$|A\varphi(t)| \leq \left|\frac{c(t)}{1 - r'(t)}\varphi(t - r(t))\right| + \int_0^t \left|b(s)\varphi^3(s - r(s))\right| e^{-\int_s^t a(u)du} ds +$$
$$+ \int_0^t |\mu(s)\varphi(s - r(s))| e^{-\int_s^t a(u)du} ds \leq$$
$$\leq \alpha L + L \int_0^t L^2 |b(s)| e^{-\int_s^t a(u)du} ds + L \int_0^t |\mu(s)| e^{-\int_s^t a(u)du} ds \leq$$
$$\leq L\left\{\alpha + \int_0^t \beta a(s) e^{-\int_s^t a(u)du} ds + \int_0^t \delta a(s) e^{-\int_s^t a(u)du} ds\right\} \leq$$
$$\leq L(\alpha + \beta + \delta) \leq L/J < L.$$

So, AS_ψ is bounded by L as required.

Let $\varphi \in S_\psi$ be fixed. We will prove that $A\varphi(t) \to 0$ as $t \to \infty$. It is obvious, due to the conditions $t - r(t) \to \infty$ as $t \to \infty$ in (3.12) and (3.9), that the first term in the right-hand side of A tends to 0 as $t \to \infty$, that is,

$$\left|\frac{c(t)}{1-r'(t)}\varphi(t-r(t))\right| \le \alpha\left|\varphi(t-r(t))\right| \to 0 \quad \text{as} \quad t \to \infty.$$

It is left to show that the two remaining integral terms of A go to zero as $t \to \infty$.

Let $\varepsilon > 0$ be given. Find T such that $|\varphi(t-r(t))| < \varepsilon$, for $t \ge T$. Then we have

$$\begin{aligned}
&\left|\int_0^t \mu(s)\varphi(s-r(s))\,e^{-\int_s^t a(u)du}ds\right| \le \\
&\le \int_0^T \left|\mu(s)\varphi(s-r(s))\right| e^{-\int_s^t a(u)du}ds + \int_T^t \left|\mu(s)\varphi(s-r(s))\right| e^{-\int_s^t a(u)du}ds \le \\
&\le Le^{-\int_T^t a(u)du}\int_0^T |\mu(s)|\,e^{-\int_s^T a(u)du}ds + \varepsilon\int_T^t |\mu(s)|\,e^{-\int_s^t a(u)du}ds \le \\
&\le L\delta e^{-\int_T^t a(u)du} + \varepsilon\delta.
\end{aligned}$$

The term $L\delta e^{-\int_T^t a(u)du}$ is, as $t \to \infty$, arbitrarily small because of (3.12). The remaining integral term in A goes to zero by a similar argument. This ends the proof. $\square$

Lemma 3.3. *Let* (3.7)–(3.10) *and* (3.12) *hold. For A, B defined in* (3.1) *and* (3.2), *if $\phi, \varphi \in S_\psi$ are arbitrary, then*

$$|B\varphi + A\phi| \le L.$$

Moreover, B is a large contraction on S_ψ with a unique fixed point in S_ψ and $B\varphi(t) \to 0$ as $t \to \infty$.

Proof. Note first that the condition $|z| \le \frac{\sqrt{3}}{3}$ implies $\left|z - z^3\right| \le \frac{2\sqrt{3}}{9} = \frac{2}{3}L$. Now, using the definitions (3.1), (3.2) of A and B and applying (3.7)–(3.10), we obtain

$$\begin{aligned}
|B\varphi(t) + A\phi(t)| &\le \left(1 + \left|\frac{c(0)}{1-r'(0)}\right|\right)\|\psi\|\,e^{-\int_0^t a(u)du} + \alpha L + L\int_0^t |\mu(s)|\,e^{-\int_s^t a(u)du}ds + \\
&\quad + \frac{2L}{3}\int_0^t a(s)e^{-\int_s^t a(u)du}ds + \int_0^t |b(s)|\,L^3 e^{-\int_s^t a(u)du}ds \le \\
&\le \left(1 + \left|\frac{c(0)}{1-r'(0)}\right|\right)\|\psi\|\,e^{-\int_0^t a(u)du} + (\alpha+\beta+\delta)\,L + \frac{2L}{3} \le \\
&\le \left(1 + \left|\frac{c(0)}{1-r'(0)}\right|\right)\|\psi\|\,e^{-\int_0^t a(u)du} + \frac{L}{J} + \frac{2L}{3}.
\end{aligned}$$

So, by choosing the initial function ψ having small norm, say $\|\psi\| < \gamma$, then, from the above inequality and referring to (3.11), we obtain

$$|B\varphi(t) + A\phi(t)| \leq \left(1 + \left|\frac{c\,(0)}{1 - r'\,(0)}\right|\right) \gamma e^{-\int_0^t a(u)du} + \frac{L}{J} + \frac{2L}{3} \leq L.$$

Since $0 \in S_\psi$, we have also proved that $|B\varphi\,(t)| \leq L$. The proof that $B\varphi$ is Lipschitzian is similar to that of the map $A\varphi$ below. To see that B is a large contraction on S_ψ with a unique fixed point, we know from Example 3.1 that $F\varphi = \varphi - \varphi^3$ is a large contraction within the integrand. Thus, for the ε of the proof of that example, we have found η such that

$$\begin{aligned}|B\varphi(t) - B\phi(t)| &\leq \int_0^t a(s)\,|F\varphi(s) - F\phi(s)|\, e^{-\int_s^t a(u)du} ds \leq \\ &\leq \eta \int_0^t a(s)\,\|\varphi - \phi\|\, e^{-\int_s^t a(u)du} ds \leq \eta\,\|\varphi - \phi\|\,.\end{aligned}$$

To prove that $B\varphi(t) \to 0$ as $t \to \infty$ we use (3.12) for the first term and for the second term we argue as above for the map A. $\square$

Lemma 3.4. *Suppose* (3.7)–(3.10) *hold. Then the mapping A is continuous on S_ψ.*

Proof. Let $\varphi, \phi \in S_\psi$. Then

$$\begin{aligned}|A\varphi(t) - A\phi(t)| \leq \Bigg\{ &\alpha|\varphi(t - r(t)) - \phi(t - r(t))| + \\ &+ \left|\int_0^t b(s)\left[\varphi^3\,(s - r(s)) - \phi^3\,(s - r(s))\right] e^{-\int_s^t a(u)du} ds\right| + \\ &+ \left|\int_0^t \mu(s)\left[\varphi\,(s - r(s)) - \phi\,(s - r(s))\right] e^{-\int_s^t a(u)du} ds\right|\Bigg\} \leq \\ \leq\ & \alpha\,\|\varphi - \phi\| + 3\int_0^t L^2\,|b(s)|\,|\varphi\,(s - r(s)) - \phi\,(s - r(s))|\, e^{-\int_s^t a(u)du} ds + \\ &+ \|\varphi - \phi\| \int_0^t |\mu(s)|\, e^{-\int_s^t a(u)du} ds \leq \\ \leq\ & (\alpha + 3\beta + \delta)\,\|\varphi - \phi\| \int_0^t a(s) e^{-\int_s^t a(u)du} ds \leq \\ \leq\ & (\alpha + 3\beta + \delta)\,\|\varphi - \phi\| \leq (3/J)\,\|\varphi - \phi\|\,.\end{aligned}$$

Let $\varepsilon > 0$ be arbitrary. Define $\eta = \dfrac{\varepsilon J}{3}$. Then for $\|\varphi - \phi\| \leq \eta$, we obtain

$$\|A\varphi - A\phi\| \leq \frac{3}{J}\|\varphi - \phi\| \leq \varepsilon.$$

Therefore A is continuous. $\square$

Lemma 3.5. *Let* (3.4)–(3.9) *and* (3.13)–(3.15) *hold. The function $A\varphi$ is Lipschitzian and the operator A maps S_ψ into a compact subset of S_ψ.*

Proof. Let $\varphi \in S_\psi$ and let $0 \leq t_1 < t_2$. Then

$$\begin{aligned}
&|A\varphi(t_2) - A\varphi(t_1)| \leq \\
&\leq \left| \frac{c(t_2)}{1 - r'(t_2)} \varphi(t_2 - r(t_2)) - \frac{c(t_1)}{1 - r'(t_1)} \varphi(t_1 - r(t_1)) \right| + \\
&\quad + \left| \int_0^{t_2} \mu(s) \varphi(s - r(s)) e^{-\int_s^{t_2} a(u)du} ds - \right. \\
&\quad \left. - \int_0^{t_1} \mu(s) \varphi(s - r(s)) e^{-\int_s^{t_1} a(u)du} ds \right| + \\
&\quad + \left| \int_0^{t_2} b(s) \varphi^3(s - r(s)) e^{-\int_s^{t_2} a(u)du} ds - \right. \\
&\quad \left. - \int_0^{t_1} b(s) \varphi^3(s - r(s)) e^{-\int_s^{t_1} a(u)du} ds \right|.
\end{aligned} \tag{3.16}$$

By hypotheses (3.5)–(3.6), we have

$$\begin{aligned}
&|\alpha(t_2)\varphi(t_2 - r(t_2)) - \alpha(t_1)\varphi(t_1 - r(t_1))| \leq \\
&\leq |\alpha(t_2)| \, |\varphi(t_2 - r(t_2)) - \varphi(t_1 - r(t_1))| + \\
&\quad + |\varphi(t_1 - r(t_1))| \, |\alpha(t_2) - \alpha(t_1)| \leq \\
&\leq \alpha k \, |(t_2 - t_1) - (r(t_2) - r(t_1))| + Lk_3 \, |t_2 - t_1| \leq \\
&\leq (\alpha k + \alpha k k_2 + Lk_3) \, |t_2 - t_1|,
\end{aligned} \tag{3.17}$$

where k is the Lipschitz constant of φ. By hypotheses (3.4) and (3.7), we have

$$
\begin{aligned}
&\left| \int_0^{t_2} \mu(s)\varphi\left(s-r(s)\right) e^{-\int_s^{t_2} a(u)du} ds - \int_0^{t_1} \mu(s)\varphi\left(s-r(s)\right) e^{-\int_s^{t_1} a(u)du} ds \right| = \\
&= \left| \int_0^{t_1} \mu(s)\varphi\left(s-r\left(s\right)\right) e^{-\int_s^{t_1} a(u)du} \left(e^{-\int_{t_1}^{t_2} a(u)du} - 1\right) ds + \right. \\
&\quad \left. + \int_{t_1}^{t_2} \mu(s)\varphi\left(s-r(s)\right) e^{-\int_s^{t_2} a(u)du} ds \right| \leq \\
&\leq L \left| e^{-\int_{t_1}^{t_2} a(u)du} - 1 \right| \int_0^{t_1} \delta a(s) e^{-\int_s^{t_1} a(u)du} + L \int_{t_1}^{t_2} \left|\mu(s)\right| e^{-\int_s^{t_2} a(u)du} ds \leq \\
&\leq L\delta \int_{t_1}^{t_2} a(s)ds + L \int_{t_1}^{t_2} e^{-\int_s^{t_2} a(u)du} d\left(\int_{t_1}^{s} \left|\mu(v)\right| dv\right) \leq \\
&\leq L\delta \int_{t_1}^{t_2} a(s)ds + L \left\{ \left[e^{-\int_s^{t_2} a(u)du} \int_{t_1}^{s} \left|\mu(v)\right| dv \right]_{t_1}^{t_2} + \right. \\
&\quad \left. + \int_{t_1}^{t_2} a(s) e^{-\int_s^{t_2} a(u)du} \int_{t_1}^{s} \left|\mu(v)\right| dv ds \right\} \leq \\
&\leq L\delta \int_{t_1}^{t_2} a(s)ds + L \int_{t_1}^{t_2} \left|\mu(s)\right| ds \left(1 + \int_{t_1}^{t_2} a(s) e^{-\int_s^{t_2} a(u)du} ds\right) \leq \\
&\leq L\delta \int_{t_1}^{t_2} a(s)ds + 2L \int_{t_1}^{t_2} \left|\mu(s)\right| ds \leq \\
&\leq L\delta \int_{t_1}^{t_2} a(s)ds + 2L\delta \int_{t_1}^{t_2} a\left(s\right) ds \leq 3L\delta k_1 |t_2 - t_1|.
\end{aligned}
\tag{3.18}
$$

Similarly, by (3.4) and (3.8), we deduce

$$
\begin{aligned}
&\left| \int_0^{t_2} b(s)\varphi^3\left(s-r(s)\right)e^{-\int_s^{t_2} a(u)du} ds - \int_0^{t_1} b(s)\varphi^3\left(s-r(s)\right)e^{-\int_s^{t_1} a(u)du} ds \right| = \\
&= \left| \int_0^{t_1} b(s)\varphi^3\left(s-r\left(s\right)\right)e^{-\int_s^{t_1} a(u)du}\left(e^{-\int_{t_1}^{t_2} a(u)du}-1\right) ds + \right. \\
&\quad \left. + \int_{t_1}^{t_2} b(s)\varphi^3\left(s-r(s)\right)e^{-\int_s^{t_2} a(u)du} ds \right| \le \\
&\le L\left|e^{-\int_{t_1}^{t_2} a(u)du}-1\right| \int_0^{t_1} \beta a(s)e^{-\int_s^{t_1} a(u)du} + L^3 \int_{t_1}^{t_2} \left|b(s)\right| e^{-\int_s^{t_2} a(u)du} ds \le \\
&\le L\beta \int_{t_1}^{t_2} a(u)du + L^3 \int_{t_1}^{t_2} e^{-\int_s^{t_2} a(u)du} d\left(\int_{t_1}^{s} \left|b(v)\right| dv\right) \le \\
&\le L\beta \int_{t_1}^{t_2} a(u)du + L^3 \left\{ \left[e^{-\int_s^{t_2} a(u)du} \int_{t_1}^{s} \left|b(v)\right| dv \right]_{t_1}^{t_2} + \right. \\
&\quad \left. + \int_{t_1}^{t_2} a(s)e^{-\int_s^{t_2} a(u)du} \int_{t_1}^{s} \left|b(v)\right| dv ds \right\} \le \\
&\le L\beta \int_{t_1}^{t_2} a(u)du + L^3 \int_{t_1}^{t_2} \left|b(s)\right| ds \left(1 + \int_{t_1}^{t_2} a(s)e^{-\int_s^{t_2} a(u)du} ds\right) \le \\
&\le L\beta \int_{t_1}^{t_2} a(u)du + 2L^3 \int_{t_1}^{t_2} \left|b(s)\right| ds \le \\
&\le L\beta \int_{t_1}^{t_2} a(u)du + 2L\beta \int_{t_1}^{t_2} a\left(s\right) ds \le 3L\beta k_1 |t_2-t_1|.
\end{aligned}
\tag{3.19}
$$

Thus, by substituting (3.17)–(3.19) in (3.16), we obtain

$$
\begin{aligned}
&|A\varphi(t_2) - A\varphi(t_1)| \le \\
&\le \left(\alpha k + \alpha k k_2 + L k_3\right)|t_2 - t_1| + 3L\delta k_1 |t_2 - t_1| + 3L\beta k_1 |t_2 - t_1| \le \\
&\le K|t_2 - t_1|,
\end{aligned}
\tag{3.20}
$$

for some constant $K > 0$. This shows that $A\varphi$ is Lipschitzian if φ is and that AS_ψ is equicontinuous. Next, we notice that for arbitrary $\varphi \in S_\psi$ we have

$$
\begin{aligned}
|A\varphi(t)| \leq & \left|\frac{c(t)}{1-r'(t)}\varphi(t-r(t))\right| + \int_0^t \left|b(s)\varphi^3(s-r(s))\right| e^{-\int_s^t a(u)du}ds+ \\
& + \int_0^t |\mu(s)\varphi(s-r(s))| e^{-\int_s^t a(u)du}ds \leq \\
& \leq L\alpha(t) + L^3 \int_0^t a(s)\left[|b(s)|/a(s)\right] e^{-\int_s^t a(u)du}ds+ \\
& + L\int_0^t a(s)\left[|\mu(s)|/a(s)\right] e^{-\int_s^t a(u)du}ds := q(t),
\end{aligned}
$$

because of (3.13)–(3.15) and using a method like the one used for the map A we see that $q(t) \to 0$ as $t \to \infty$. By Theorem 2.4, we conclude that the set AS_ψ resides in a compact set. □

Theorem 3.6. *Let $L = \frac{\sqrt{3}}{3}$. Suppose that the conditions* (1.2) *and* (3.4)–(3.15) *hold. If ψ is a given initial function which is sufficiently small, then there is a solution $x(t,0,\psi)$ of* (1.1) *with $|x(t,0,\psi)| \leq L$ and $x(t,0,\psi) \to 0$ as $t \to \infty$.*

Proof. From Lemmas 3.2 and 3.5 we deduce that A is bounded by L, Lipschitzian and $A\phi(t) \to 0$ as $t \to \infty$. So, A maps S_ψ into S_ψ. By Lemmas 3.3 and 3.5 we have, for arbitrary $\phi, \varphi \in S_\psi$, $B\varphi + A\phi \in S_\psi$, since both $A\phi$ and $B\varphi$ are Lipschitzian, bounded by L and $B\varphi(t) \to 0$ as $t \to \infty$. From Lemmas 3.4 and 3.5, we proved that A is continuous and AS_ψ resides in a compact set. Thus, all the conditions of Theorem 2.3 are satisfied. Therefore, there exists a solution of (1.1) with $|x(t,0,\psi)| \leq L$ and $x(t,0,\psi) \to 0$ as $t \to \infty$. □

4. STABILITY AND COMPACTNESS

Referring to Burton [2], except for the fixed point method, we know of no other way proving that solutions of (1.1) (and particularly (1.3)) converge to zero. Nevertheless, if all we need is stability and not asymptotic stability, then we can avoid conditions (3.13)–(3.15) and still use Krasnoselskii-Burton's theorem on a Banach space endowed with a weighted norm.

Let $h : [m_0, \infty) \to [1, \infty)$ be any strictly increasing and continuous function with $h(m_0) = 1$, $h(s) \to \infty$ as $s \to \infty$. Let $(S, |\cdot|_h)$ be the Banach space of continuous $\varphi : [m_0, \infty) \to \mathbb{R}$ for which

$$|\varphi|_h := \sup_{t \geq m_0} |\varphi(t)/h(t)| < \infty,$$

exists. We continue to use $\|\cdot\|$ as the supremum norm of any $\varphi \in S$, provided φ is bounded. Also, we use $\|\psi\|$ as the bound of the initial function. Further, we can, modulo a slight modification, prove that the function $F\varphi = \varphi - \varphi^3$ is still a large contraction with the norm $|\cdot|_h$.

Theorem 4.1. *If the conditions of Theorem 3.6 hold, except for* (3.13)–(3.15)*, then the zero solution of* (1.1) *is stable.*

Proof. We prove the stability starting at $t_0 = 0$. Let $\varepsilon > 0$ be given such that $0 < \varepsilon < \sqrt{3}/3$, then for $|x| \le \varepsilon$, we find a γ^* with $\left|x - x^3\right| \le \gamma^*$, and choose a number γ such that

$$\gamma + \gamma^* + \left(\varepsilon^3/J\right) \le \varepsilon. \tag{4.1}$$

In fact, since $x - x^3$ is increasing on $\left(-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$, we may take $\gamma^* = \varepsilon - \varepsilon^3$. Thus, inequality (4.1) allows $\gamma > 0$. Now, remove the condition $\varphi(t) \to 0$ as $t \to 0$ from S_ψ defined previously and consider the set

$$\begin{gathered} M_\psi := \{\varphi \in S \,|\, \varphi \text{ is Lipschitzian, } |\varphi(t)| \le \varepsilon,\ t \in [m_0, \infty) \\ \text{and } \varphi(t) = \psi(t) \text{ if } t \in [m_0, 0]\}. \end{gathered}$$

Define A, B on M_ψ, as before by (3.1), (3.2). We easily check that, if $\varphi \in M_\psi$, then $|A\varphi(t)| < \varepsilon$, and B is a large contraction on M_ψ. Also, by choosing $\|\psi\| < \gamma$ and referring to (4.1), we verify that for $\varphi, \phi \in M_\psi$, $|B\varphi(t) + A\phi(t)| \le \varepsilon$ and $|B\varphi(t)| < \varepsilon$. AM_ψ is equicontinuous set. According to Theorem 4.0.1 in [2], in the space $(S, |\cdot|_h)$ the set AM_ψ resides in a compact subset of M_ψ. Moreover, the operator $A : M_\psi \to M_\psi$ is continuous. Indeed, for $\varphi, \phi \in S_\psi$, then

$$\begin{aligned} &|A\varphi(t) - A\phi(t)|/h(t) \le \\ &\le (1/h(t)) \Bigg\{ \alpha \left|\varphi(t - r(t)) - \phi(t - r(t))\right| + \\ &\quad + \left| \int_0^t b(s) \left[\varphi^3(s - r(s)) - \phi^3(s - r(s))\right] e^{-\int_s^t a(u)du} ds \right| + \\ &\quad + \left| \int_0^t \mu(s) \left[\varphi(s - r(s)) - \phi(s - r(s))\right] e^{-\int_s^t a(u)du} ds \right| \Bigg\} \le \\ &\le \alpha |\varphi - \phi|_h + 3 \int_0^t L^2 |b(s)| \left|\varphi(s - r(s)) - \phi(s - r(s))\right| / h(t)\, e^{-\int_s^t a(u)du} ds + \\ &\quad + \int_0^t |\mu(s)| \left|\varphi(s - r(s)) - \phi(s - r(s))\right| / h(t) e^{-\int_s^t a(u)du} ds \le \end{aligned}$$

$$\leq \alpha \left|\varphi-\phi\right|_h + 3\beta \left|\varphi-\phi\right|_h \int_0^t a(s)\left[h\left(s-r(s)\right)/h(t)\right] e^{-\int_s^t a(u)du} ds +$$
$$+ \delta \left|\varphi-\phi\right|_h \int_0^t a\left(s\right)\left[h\left(s-r(s)\right)/h(t)\right] e^{-\int_s^t a(u)du} ds \leq (3/J) \left|\varphi-\phi\right|_h .$$

The conditions of Theorem 2.3 are satisfied on M_ψ and so there exists a fixed point which solve (1.1) and lying in M_ψ. □

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