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PRE-ASSEMBLY OF FEM N-TH ORDER TRIANGULAR ELEMENTS

The paper presents a method of preparing data, which can be used to simplify the procedure of formulating local stiffness matrices in the finite element method (the part of preparing this data is hence referred to as the pre-assembly). For the presentation of the methodology non-curvilinear triangular elements have been chosen, where it has been taken into account that any order can be selected for the elements. The paper presents a mathematical basis of how the auxiliary data is prepared. The appropriate relations to the final FEM equations are also given.

KEYWORDS: assembly, finite element method, higher order elements

1. INTRODUCTION

When solving an electromagnetic field problem with the use of the finite element method, certain steps need to be performed. These can be generally divided into (in order of their execution):

- definition of the problem geometry and environment parameters,
- meshing,
- assembly,
- analysis (i.e. computations of the field),
- post-processing.

The study that is presented concerns the assembly part, where local stiffness matrices of elements are obtained, then the global stiffness matrix is built according to them and a defined local to global enumeration is designated. The local stiffness matrices are built with a dependence on:

- the element type,
- the element order,
- the base element shape,
- the element placement in the global coordinate system.

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The matter discussed in this paper is a possibility of preparing a database that would store the dependency of applied types of elements only on the coordinates of the element's essential vertices (i.e. the nodes that are responsible for the geometrical approximation). This could reduce the computational weight of the finite element method in the assembly part. This pre-assembly will also allow to skip most of the assembly part by applying the derived dependencies along with prepared coefficient values.

The presented study is planned to be presented in a series of articles, where the current one only concerns triangular non-curvilinear elements of an arbitrary order (an analysis of this type for curvilinear elements will be discussed in a future paper).

2. TRANSIENT MAGNETIC FIELD EQUATIONS

The current study concerns only 2D problems. In this case the transient magnetic field is described by the following differential equation for the magnetic vector potential component (assuming a linear medium):

$$\frac{1}{\mu} \bar{\nabla}^2 A - \gamma \frac{\partial A}{\partial t} = -J_{\text{ext}}. \quad (1)$$

From the above the following weak formulation can be derived [1, 2]:

$$\oint_{\partial\Omega} \frac{1}{\mu} \frac{\partial A}{\partial n} w \, d\Gamma - \int_{\Omega} \frac{1}{\mu} \left(\frac{\partial A}{\partial x} \frac{\partial w}{\partial x} + \frac{\partial A}{\partial y} \frac{\partial w}{\partial y} \right) dx dy - \int_{\Omega} \gamma \frac{\partial A}{\partial t} w \, dx dy + \int_{\Omega} J_{\text{ext}} w \, dx dy = 0, \quad (2)$$

where w is the test function, Ω is the considered domain, J_{ext} is an externally enforced current density. $\frac{\partial A}{\partial n}$ is the derivative across the direction normal to the boundary $\partial\Omega$ that covers the region Ω .

When applying the finite element formulation, in order to obtain the equations and their dependencies on the appropriate degrees of freedom then the following three basic components must be designated for each element of the equation (2):

- concerning the second component:

$$\Lambda_{a,b} = - \int_{\Omega'} \left(\frac{\partial N_a}{\partial x} \frac{\partial N_b}{\partial x} + \frac{\partial N_a}{\partial y} \frac{\partial N_b}{\partial y} \right) dx dy, \quad (3)$$

where N denotes shape functions of the element and Ω' is the region of the element,

- concerning the third and fourth component:

$$K_{a,b} = \int_{\Omega'} N_a N_b \, dx dy, \quad (4)$$

- additionally if the element is in contact with the boundary then the following component is also important:

$$d_a = \oint_{\partial\Omega'} N_a|_{\eta=0} \frac{1}{\mu} \frac{\partial A}{\partial n} d\xi, \quad (5)$$

where in the local coordinate system (ξ, η) it is assumed that $\eta = 0$ is the element face that represents the boundary fragment. If the normal derivative $\frac{\partial A}{\partial n}$ is nonhomogeneous on the boundary then the following equation can be introduced:

$$d_a = \oint_{\partial\Omega'} N_a|_{\eta=0} \frac{1}{\mu} \frac{\partial A}{\partial n} d\xi = \sum_{c=1}^m d_{a,c} \frac{1}{\mu} \frac{\partial A_c}{\partial n}. \quad (6)$$

Hence it can be derived that:

$$d_{a,c} = \int_{\xi=0}^1 \left(\prod_{\substack{k=1 \\ k \neq a}}^n \frac{\xi - \xi_k}{\xi_a - \xi_k} \right) \left(\prod_{\substack{k=1 \\ k \neq c}}^n \frac{\xi - \xi_k}{\xi_c - \xi_k} \right) d\xi, \quad (7)$$

where $m = p - 1$ with p being the order of the element.

In total the triangular element contains m' degrees of freedom ($m' = \frac{m^2 + m}{2}$) hence, in conclusion, when applying the finite element method one obtains m' local equations for each element (where each equation has a unique index a) of the following form subject to (3), (4) and (7):

$$\delta_a \Gamma \sum_{c=1}^m d_{a,c} \frac{1}{\mu} \frac{\partial A_c}{\partial n} + \frac{1}{\mu} \sum_{b=1}^{m'} \Lambda_{a,b} A_b + \sum_{b=1}^{m'} K_{a,b} \left(-\gamma_b \frac{\partial A_b}{\partial t} + J_{\text{ext}b} \right) = 0, \quad (8)$$

where δ_a is equal to 1 only if the degree of freedom with the index a is placed on the problem boundary (otherwise it equals 0). Γ is the length of the boundary fragment that is covered by the considered element.

3. FORMULATION OF AUXILIARY EQUATIONS

The section provides information on the formulae to obtain the terms $\Lambda_{a,b}$ and $K_{a,b}$. Subject to the relation between the local and global coordinates in a 2D space [1, 2] the following equation can be obtained:

$$\Lambda_{a,b} = - \int_{\Omega'} \left(\left(\frac{\partial y}{\partial \eta} \frac{\partial N_a}{\partial \xi} - \frac{\partial y}{\partial \xi} \frac{\partial N_a}{\partial \eta} \right) \left(\frac{\partial y}{\partial \eta} \frac{\partial N_b}{\partial \xi} - \frac{\partial y}{\partial \xi} \frac{\partial N_b}{\partial \eta} \right) + \left(-\frac{\partial x}{\partial \eta} \frac{\partial N_a}{\partial \xi} + \frac{\partial x}{\partial \xi} \frac{\partial N_a}{\partial \eta} \right) \left(-\frac{\partial x}{\partial \eta} \frac{\partial N_b}{\partial \xi} + \frac{\partial x}{\partial \xi} \frac{\partial N_b}{\partial \eta} \right) \right) \frac{1}{\det J} d\xi d\eta, \quad (9)$$

where J is the Jacobian matrix of transformation between the local and global coordinates:

$$J = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}. \quad (10)$$

Since the presented study deals so far with non-curvilinear triangular elements then the global coordinates x and y can be respectively expressed by:

$$x = x_1(1 - \xi - \eta) + x_2\eta + x_3\xi, \quad (11)$$

and:

$$y = y_1(1 - \xi - \eta) + y_2\eta + y_3\xi. \quad (12)$$

The coordinates (x_1, y_1) , (x_2, y_2) and (x_3, y_3) represent the element's essential vertices (triangle corners). The Jacobian matrix determinant is hence:

$$\det J = \frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta} \frac{\partial y}{\partial \xi} = (x_3 - x_1)(y_2 - y_1) - (x_2 - x_1)(y_3 - y_1). \quad (13)$$

Due to $\det J$ being a constant value – its reciprocal can be taken into account as a separate multiplier independent of the integration. When introducing the notation for the geometric basis functions:

$$\begin{cases} G_1 = (1 - \xi - \eta), \\ G_2 = \eta, \\ G_3 = \xi, \end{cases} \quad (14)$$

equation (9) takes the form:

$$A_{a,b} = -\frac{1}{\det J} \int_{\Omega'} \left(\begin{array}{l} \left(\left(\sum_{l=1}^3 y_l \frac{\partial G_l}{\partial \eta} \right) \frac{\partial N_a}{\partial \xi} - \left(\sum_{l=1}^3 y_l \frac{\partial G_l}{\partial \eta} \right) \frac{\partial N_b}{\partial \xi} \right) + \\ \left(- \left(\sum_{l=1}^3 y_l \frac{\partial G_l}{\partial \xi} \right) \frac{\partial N_a}{\partial \eta} - \left(\sum_{l=1}^3 y_l \frac{\partial G_l}{\partial \xi} \right) \frac{\partial N_b}{\partial \eta} \right) \\ + \\ \left(- \left(\sum_{l=1}^3 x_l \frac{\partial G_l}{\partial \eta} \right) \frac{\partial N_a}{\partial \xi} + \left(- \left(\sum_{l=1}^3 x_l \frac{\partial G_l}{\partial \eta} \right) \frac{\partial N_b}{\partial \xi} + \right. \right. \\ \left. \left. + \left(\sum_{l=1}^3 x_l \frac{\partial G_l}{\partial \xi} \right) \frac{\partial N_a}{\partial \eta} \right) \left(+ \left(\sum_{l=1}^3 x_l \frac{\partial G_l}{\partial \xi} \right) \frac{\partial N_b}{\partial \eta} \right) \right) \end{array} \right) d\xi d\eta. \quad (15)$$

Next, an auxiliary notation is introduced:

$$Z_{a,l} = \left(\frac{\partial G_l}{\partial \eta} \frac{\partial N_a}{\partial \xi} - \frac{\partial G_l}{\partial \xi} \frac{\partial N_a}{\partial \eta} \right), \quad (16)$$

which allows to write:

$$\Lambda_{a,b} = -\frac{1}{\det J} \int_{\Omega'} \left(\sum_{l=1}^3 y_l Z_{a,l} \sum_{l=1}^3 y_l Z_{b,l} + \sum_{l=1}^3 x_l Z_{a,l} \sum_{l=1}^3 x_l Z_{b,l} \right) d\xi d\eta. \quad (17)$$

The above equation in turn can be presented in the form:

$$\Lambda_{a,b} = -\frac{1}{\det J} \int_{\Omega'} \left(\sum_{l=1}^3 \sum_{k=1}^3 y_k Z_{b,k} y_l Z_{a,l} + \sum_{l=1}^3 \sum_{k=1}^3 x_k Z_{b,k} x_l Z_{a,l} \right) d\xi d\eta. \quad (18)$$

Another auxiliary notation is introduced for:

$$\lambda_{a,b,l,k} = \begin{cases} \int_{\Omega'} -Z_{a,l} Z_{b,l} & l = k, \\ \int_{\Omega'} -Z_{a,l} Z_{b,k} - Z_{a,k} Z_{b,l} & l \neq k. \end{cases} \quad (19)$$

which allows to write the final form:

$$\Lambda_{a,b} = \sum_{l=1}^3 \sum_{k=1}^l (x_l x_k + y_l y_k) \cdot \lambda_{a,b,l,k}. \quad (20)$$

Subject to the previously derived dependencies it can be observed that:

$$\lambda_{a,b,l,k} = \lambda_{b,a,l,k}, \quad (21)$$

hence it is only necessary to obtain the terms $\lambda_{a,b,l,k}$ for $b \leq a$ and $k \leq l$ so that the final form of a 4D matrix λ can be presented as a triangularly filled structure as depicted in Fig. 1.

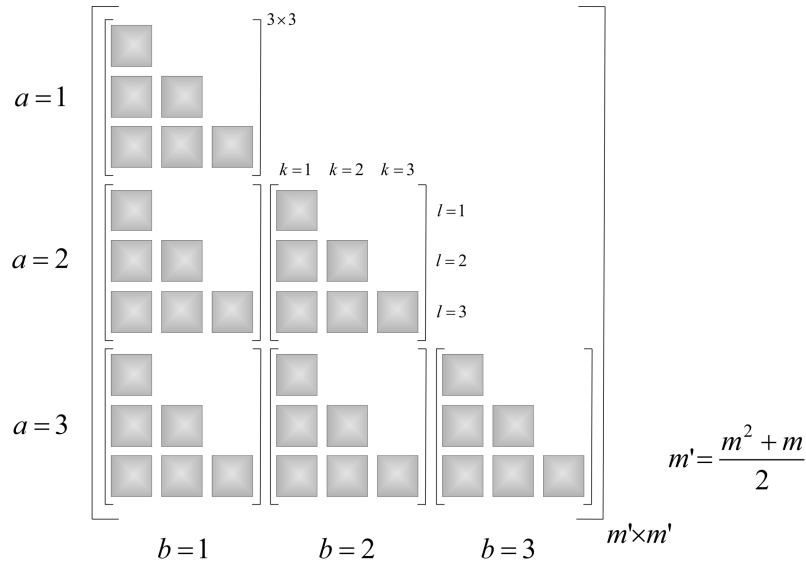


Fig. 1. The $\lambda_{a,b,l,k}$ terms depicted in the form of a 4D structure

In order to obtain the terms $K_{a,b}$ much simpler formulae need to be used in comparison to the ones for $\Lambda_{a,b}$. Since $\det J$ is a constant value then it can be again placed outside of the integral. The auxiliary term is introduced:

$$\kappa_{a,b} = \int_{\Omega'} (N_a N_b) d\xi d\eta, \quad (22)$$

so that one can use the formula:

$$K_{a,b} = \det J \kappa_{a,b}. \quad (23)$$

A special feature of the values expressed by $d_{a,c}$, $\kappa_{a,b}$ and $\lambda_{a,b,l,k}$ is that they are independent of the triangle's essential vertices (i.e. (x_i, y_i) , $i = 1, 2, 3$) so that they can be prepared in a pre-assembly and placed in auxiliary data sheets.

4. CONCLUSIONS

The effect of the work so far is that if one wants to obtain a local stiffness matrix of a non-curvilinear triangular element then only the triangle corner coordinates need to be known while the remaining data is given in auxiliary data sheets obtained through the presented method. The final values of the stiffness matrix entries can be obtained through the formula (8) while initially the equations (6), (20) and (23) are used, which apply the values of $d_{a,c}$, $\lambda_{a,b,l,k}$ and $\kappa_{a,b}$ respectively. Exemplary data sheets for $m = 2$ and $m = 3$ (which stand for elements of order 1 and 2) are given in Figure 2 and 3.

$m = 2$										
$d = \begin{bmatrix} 1/3 & -1/6 \\ -1/6 & 1/3 \end{bmatrix}$										
$\kappa = \begin{bmatrix} 5/60 \\ 1/24 & 5/60 \\ 1/24 & 1/24 & 5/60 \end{bmatrix}$	$\lambda =$	0	0	0	-0.5	-0.5	0	0	-0.5	0
		0	-0.5	0	0	-0.5	0	0	-0.5	0
		0.5	0	-0.5	-0.5	0.5	1	0	-0.5	-0.5
		-0.5	-0.5	0.5	0.5	-0.5	0	0	-0.5	0
		0.5	-0.5	0	-0.5	0.5	0	0	-0.5	0
		-0.5	0.5	-0.5	0	-0.5	0	1	-0.5	-0.5
		0.5	-0.5	0	-0.5	0.5	0	0	0	0

Fig. 2. Auxiliary data sheet for the computation of local stiffness matrices that result from applying non-curvilinear triangular elements of the first order

