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CONSTANT-SIGN SOLUTIONS FOR A NONLINEAR NEUMANN PROBLEM INVOLVING THE DISCRETE p-LAPLACIAN

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Abstract. In this paper, we investigate the existence of constant-sign solutions for a nonlinear Neumann boundary value problem involving the discrete p-Laplacian. Our approach is based on an abstract local minimum theorem and truncation techniques.

Keywords: constant-sign solution, difference equations, Neumann problem.

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1. INTRODUCTION

Nonlinear discrete problems are important mathematical models in various research fields such as computer science, mechanical engineering, astrophysics, control systems, artificial or biological neural networks, economics, fluid mechanics, image processing and many others. During the last few decades, many authors have intensively investigated various kinds of nonlinear discrete problems by using different tools, as for instance, fixed point theorems and sub-super solutions methods. Of these topics see [4–6, 22] and the reference therein. For general references on difference equations and their applications we also cite [1] and [21]. In particular, by using variational methods, the existence and multiplicity of solutions for nonlinear difference equations have been studied in many papers, usually, under a suitable $(p-1)-$sublinear or $(p-1)-$superlinear growth condition at infinity on the nonlinearities, see [2, 3, 6–10, 12–15, 17, 18, 20, 23, 25]. For a complete overview on variational methods on finite Banach spaces and discrete problems, see [11].

In this paper, we investigate the existence of constant-sign solutions for the following nonlinear discrete Neumann boundary value problem

$$\begin{cases} -\Delta(\phi_p(\Delta u(k-1))) + q(k)\phi_p(u(k)) = \lambda f_k(u(k)), & k \in [1, N], \\ \Delta u(0) = \Delta u(N) = 0, \end{cases} \tag{$N_{\lambda,\underline{f}}$}$$

where λ is a positive parameter, N is a fixed positive integer, $[1, N]$ is the discrete interval $\{1, \dots, N\}$, $\phi_p(s) := |s|^{p-2}s$, $1 < p < +\infty$ and for all $k \in [1, N]$, $q(k) > 0$, $\Delta u(k) := u(k+1) - u(k)$ denotes the forward difference operator and $f_k : \mathbb{R} \to \mathbb{R}$ is a continuous function for all $k \in [1, N]$.

More precisely, we obtain a suitable interval of parameters for which problem $(N_{\lambda,\underline{f}})$ admits constant-sign solutions which are local minimizers of the corresponding Euler-Lagrange functional. First, we consider the case in which problem $(N_{\lambda,\underline{f}})$ does not have the trivial solution and the existence of a nonzero solution is ensured whenever the parameter λ belongs to a well determined interval (Theorem 3.1). Such solutions, roughly speaking, are positive when $f_k(0) \geq 0$ for every $k \in [1, N]$ (Theorem 3.3, Corollary 3.4). We emphasize that to achieve our goal, we do not assume any growth condition at infinity on the nonlinearities.

Next, if problem $(N_{\lambda,\underline{f}})$ admits the trivial solution, then the existence of at least one positive solution is established under the more restrictive condition that the nonlinearities f_k are superlinear at zero (Theorem 3.7).

The existence of a negative solution is also ensured by similar arguments to those described before (Theorem 3.5, Corollary 3.6). Combining the previous two situations, the existence of at least two constant-sign solutions, one positive and one negative, is also shown (Theorem 3.9).

For completeness, we observe that the results given here are new also for a nonlinear discrete problem with Dirichlet boundary conditions involving the p-Laplacian for $p \neq 2$. While, for $p = 2$ and $q(k) = 0$ for every $k \in [1, N]$, similar results have been already given in [11] and for nonlinear algebraic systems in [12]. The existence of two constant-sign solutions for a Dirichlet problem is treated in [15], for $p = 2$ and provided that the functions f_k are superlinear at infinity.

Finally, we point out that multiple solutions for nonlinear discrete depending-parameter problems are investigated in [8, 13, 14, 16, 19].

2. MATHEMATICAL BACKGROUND

In the N-dimensional Banach space

$$X = \{u : [0, N+1] \to \mathbb{R} : \Delta u(0) = \Delta u(N) = 0\},$$

we consider the norm

$$\|u\| := \left(\sum_{k=1}^{N+1} |\Delta u(k-1)|^p + \sum_{k=1}^{N} q(k)|u(k)|^p \right)^{1/p} \quad \text{for all} \quad u \in X.$$

Moreover, we will use also the equivalent norm

$$\|u\|_\infty := \max_{k \in [0, N+1]} |u(k)| \quad \text{for all} \quad u \in X.$$

For our purpose, the following inequality will be useful

$$\|u\|_\infty \leq \|u\| q^{-1/p} \quad \text{for all} \quad u \in X, \quad \text{where} \quad q := \min_{k \in [1, N]} q_k. \tag{2.1}$$

To describe the variational framework of problem $(N_{\lambda,\underline{f}})$, we introduce the following two functions

$$\Phi(u) := \frac{\|u\|^p}{p} \quad \text{and} \quad \Psi(u) := \sum_{k=1}^{N} F_k(u(k)) \quad \text{for all} \quad u \in X, \tag{2.2}$$

where $F_k(t) := \int_0^t f_k(\xi)d\xi$ for every $(k,t) \in [1,N] \times \mathbb{R}$. A direct computation shows that Φ and Ψ are two C^1-functions on X and taking into account that

$$-\sum_{k=1}^{N} \Delta(\phi_p(\Delta u(k-1)))v(k) = \sum_{k=1}^{N+1} \phi_p(\Delta u(k-1))\Delta v(k-1) \quad \text{for all} \quad u, v \in X,$$

it is easy to verify, see also [25], that the following result holds.

Lemma 2.1. *A vector $u \in X$ is a solution of problem $(N_{\lambda,\underline{f}})$ if and only if u is a critical point of the function $I_\lambda = \Phi - \lambda\Psi$.*

For the reader convenience, we recall here the main tool used to achieve our goal, an abstract local minimum theorem given in [11, Theorem 3.3], which is a new version of [10, Theorem 1.5].

Theorem 2.2. *Let $(X, \|\cdot\|)$ be a finite dimensional Banach space and let $I_\lambda : X \to \mathbb{R}$ be a function satisfying the following structure hypothesis:*

(H) *$I_\lambda(u) := \Phi(u) - \lambda\Psi(u)$ for all $u \in X$, where $\Phi, \Psi : X \to \mathbb{R}$ are two functions of class C^1 on X with Φ coercive, i.e. $\lim_{\|u\|\to\infty} \Phi(u) = +\infty$, such that*

$$\inf_X \Phi = \Phi(0) = \Psi(0) = 0,$$

and λ is a real positive parameter.

Then, let $r > 0$, for each $\lambda \in \Lambda := \left(0, \frac{r}{\sup_{\Phi^{-1}([0,r])} \Psi}\right)$, the function $I_\lambda = \Phi - \lambda\Psi$ admits at least a local minimum $\overline{u} \in X$ such that $\Phi(\overline{u}) < r$, $I_\lambda(\overline{u}) \leq I_\lambda(u)$ for all $u \in \Phi^{-1}([0,r])$ and $I'_\lambda(\overline{u}) = 0$.

3. MAIN RESULTS

Now, we give the main results.

Theorem 3.1. *Let c be a positive constant. Assume that $f_k(0) \neq 0$ for some $k \in [1,N]$. Then, for every*

$$\lambda \in \Lambda_c := \left(0, \frac{q}{p} \frac{c^p}{\sum_{k=1}^{N} \max_{s\in[-c,c]} F_k(s)}\right),$$

problem $(N_{\lambda,\underline{f}})$ admits at least one nontrivial solution u such that $\|u\|_\infty < c$.

Proof. Fix λ as in Λ_c. Our aim is to apply Theorem 2.2, by putting Φ and Ψ as in (2.2) on the space X. An easy computation ensures the Φ and Ψ satisfy condition (H). Now, we put

$$r = \frac{q}{p} c^p.$$

Taking into account (2.1), for all $u \in X$ such that $\Phi(u) \le r$, one has

$$\|u\|_\infty \le c. \tag{3.1}$$

Therefore, we have that

$$\frac{\sup_{\Phi(u)\le r} \Psi(u)}{r} \le \frac{p}{q} \frac{\sum_{k=1}^N \max_{s\in[-c,c]} F_k(s)}{c^p}.$$

Hence, owing to Theorem 2.2, for each

$$\lambda < \frac{q}{p} \frac{c^p}{\sum_{k=1}^N \max_{s\in[0,c]} F_k(s)} \le \frac{r}{\sup_{\Phi(u)\le r} \Psi(u)},$$

the functional I_λ admits a non-zero critical point $u \in X$ such that $\Phi(u) < r$. By (3.1) and Lemma 2.1, we have that u is a solution of $(N_{\lambda,\underline{f}})$ such that $\|u\|_\infty < c$. □

Remark 3.2. If we are interested in obtaining the biggest interval of parameters, it is a simple matter to see that Theorem 3.1 ensures the existence of a nontrivial solution if we replace the interval Λ_c with the following

$$\Lambda := \left(0, \frac{q}{p} \sup_{c>0} \frac{c^p}{\sum_{k=1}^N \max_{s\in[-c,c]} F_k(s)}\right).$$

Of course, in this case we lose the estimate on the maximum of the solution.

The next result establishes the existence of a positive solution.

Theorem 3.3. *Let c be a positive constant. Assume that $f_k(0) \ge 0$ for every $k \in [1,N]$ and $f_k(0) \ne 0$ for some $k \in [1,N]$. Then, for every*

$$\lambda \in \Lambda_c^+ := \left(0, \frac{q}{p} \frac{c^p}{\sum_{k=1}^N \max_{s\in[0,c]} F_k(s)}\right),$$

problem $(N_{\lambda,\underline{f}})$, admits at least one positive solution u such that $\|u\|_\infty < c$.

Proof. Since we are interested in obtaining a positive solution for problem $(N_{\lambda,\underline{f}})$, we adopt the following truncation on the functions f_k,

$$f_k^+(s) = \begin{cases} f_k(s), & \text{if } s \ge 0, \\ f_k(0), & \text{if } s < 0. \end{cases}$$

Fixed $\lambda \in \Lambda_c^+$. Working with the truncations f_k^+, since we have that $f_k(0) \ne 0$ for some $k \in [1,N]$, Theorem 3.1 ensures a nontrivial solution for problem $(N_{\lambda,\underline{f}^+})$ such that $\|u\|_\infty < c$. Now, to prove the u is nonnegative, we exploit the fact that u is a critical point of the energy functional $I_\lambda = \Phi - \lambda\Psi$ associated to problem $(N_{\lambda,\underline{f}^+})$. In other words, we have that $u \in X$ satisfies the following condition:

$$\begin{aligned} &\sum_{k=1}^{N+1} \phi_p(\Delta u(k-1))\Delta v(k-1) + \sum_{k=1}^{N} q(k)\phi_p(u(k))v(k) \\ &= \sum_{k=1}^{N} f_k^+(u(k))v(k) \quad \text{for all} \quad u, v \in X. \end{aligned} \tag{3.2}$$

From this, taking as test function $v = -u^-$, it is a simple computation to prove that $\|u^-\| = 0$, that is u is nonnegative. Moreover, arguing by contradiction, we show that u is also a positive solution of problem $(N_{\lambda,\underline{f}})$. Suppose that $u(k) = 0$ for some $k \in [1, N]$. Being u a solution of problem $(N_{\lambda,\underline{f}})$ we have

$$\phi_p(\Delta u(k-1)) - \phi_p(\Delta u(k)) = f_k(0) \geq 0,$$

which implies that

$$0 \geq -|u(k-1)|^{p-2}u(k-1) - |u(k+1)|^{p-2}u(k+1) \geq 0.$$

So, we have that $u(k-1) = u(k+1) = 0$. Hence, iterating this process, we get that $u(k) = 0$ for every $k \in [1, N]$, which contradicts that u is nontrivial and this completes the proof. □

Corollary 3.4. *Let c be a positive constant. Assume that $f_k(0) > 0$ for every $k \in [1, N]$. Then, for every*

$$\lambda \in \widetilde{\Lambda}_c^+ := \left(0, \frac{q}{p} \frac{c^p}{\sum_{k=1}^N F_k(c)}\right),$$

problem $(N_{\lambda,\underline{f}})$, admits at least one positive solution u such that $\|u\|_\infty < c$.

Clearly, with analogous arguments, we can prove the following results on the existence of negative solutions.

Theorem 3.5. *Let c be a positive constant. Assume that $f_k(0) \leq 0$ for every $k \in [1, N]$ and $f_k(0) \neq 0$ for some $k \in [1, N]$. Then, for every*

$$\lambda \in \Lambda_c^- := \left(0, \frac{q}{p} \frac{c^p}{\sum_{k=1}^N \max_{s \in [-c,0]} F_k(s)}\right),$$

problem $(N_{\lambda,\underline{f}})$, admits at least one negative solution u such that $\|u\|_\infty < c$.

Corollary 3.6. *Let c be a positive constant. Assume that $f_k(0) < 0$ for every $k \in [1, N]$. Then, for every*

$$\lambda \in \widetilde{\Lambda}_c^- := \left(0, \frac{q}{p} \frac{c^p}{\sum_{k=1}^N F_k(-c)}\right),$$

problem $(N_{\lambda,\underline{f}})$, admits at least one negative solution u such that $\|u\|_\infty < c$.

Requiring a more restrictive growth condition at zero on the nonlinearities f_k, we can obtain a positive solution also whenever problem $(N_{\lambda,\underline{f}})$ admits the trivial one. For simplicity, we work with nonnegative nonlinearities.

Theorem 3.7. *Assume that $f_k(t) \geq 0$ for all $t \geq 0$ and for all $k \in [1, N]$, and*

(i) $\limsup_{s\to 0^+} \frac{F_k(s)}{s^p} = +\infty$ *for some* $k \in [1, N]$.

Then, for each $\lambda \in \left(0, \frac{q}{p} \sup_{c>0} \frac{c^p}{\sum_{k=1}^N F_k(c)}\right)$, *problem* $(N_{\lambda,\underline{f}})$ *admits at least one positive solution.*

Proof. Fix λ as in the conclusion and $\bar{c} > 0$ such that $\lambda < \frac{q}{p} \frac{\bar{c}^2}{\sum_{k=1}^N F_k(\bar{c})}$. Taking into account the proof of Theorem 3.1, problem $(N_{\lambda,\underline{f}})$ admits a solution u which is a global minimum for the restriction of the function I_λ to the set $\Phi(u) < \frac{q}{p}\bar{c}^p$. On the other hand, from (i) there is $d < (q/\sum_{k=1}^N q_k)\bar{c}$ such that

$$\frac{\sum_{k=1}^N q_k}{p\lambda} < \frac{F_k(d)}{d^p} \leq \frac{\sum_{k=1}^N F_k(d)}{d^p}.$$

Hence, an easy computation gives that $\Phi(w) < \frac{q}{p}c^p$ and $I_\lambda(w) < 0$, being $w \in X$ defined by putting $w(k) = d$ for every $k \in [1, N]$. Therefore, we have $I_\lambda(u) \leq I_\lambda(w) < 0$, which implies that $u \neq 0$. To show that u is positive, we argue as in the proof of Theorem 3.1. □

Remark 3.8. It is interesting to point out that Theorem 3.7 guarantees a positive solution also when $f_k(0) = 0$ for every $k \in [1, N]$. Moreover, we highlight that Theorem 3.7 continues to hold under a slightly less general condition

(i') $\lim_{s\to 0^+} \frac{f_k(s)}{s^{p-1}} = +\infty$ for some $k \in [1, N]$.

Finally, by now it is clear how to prove the following result.

Theorem 3.9. *Assume that $sf_k(s) \geq 0$ for every $s \in \mathbb{R}$, $k \in [1, N]$, and*

(ii) $\limsup_{s\to 0} \frac{F_k(s)}{|s|^p} = +\infty$ *for some* $k \in [1, N]$.

Then, for each $\lambda \in \left(0, \frac{q}{p} \sup_{c>0} \frac{c^p}{\sum_{k=1}^N \max\{F_k(-c), F_k(c)\}}\right)$, *problem* $(N_{\lambda,\underline{f}})$ *admits at least two constant-sign solutions (one positive and one negative).*

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