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Walking the Line. Traveling Forces vs Moving Masses

1 Introduction

In Vehicle Dynamics, Machine Building, Robotics and Mechatronics, we frequently have to deal with moving sub-systems, which interact in one or several points of contact [1,2,3,7,8,9,10]. Often one of the systems is treated, in a very abstract way, as a concentrated force. More recently, the notion of moving masses along a track was proposed. Well known cases are models of train-track interaction, pantograph-traction contact, or crane crab systems [4,5,6]. Most approaches concentrate on linear models, which allow to apply classical methods like Fourier Transform or Floquet Theory, e.g. [6]. However, some singularities occur, which are not well understood. The aim of this presentation is to fill this gap.

The paper shows the limits of the standard approach to moving force problems in the case of flexible structures, see Fig. 1. By introducing a different parametrization of the middle line of a rope, essential nonlinear effects are captured. The obtained solutions are related to the results of the classical approach.

Fig. 1. Human-rope interaction
Rys. 1. Oddziaływanie człowieka z liną

2 Modeling

The most frequently used models in the 1D theory of moving forces and/or masses are in the flexible case the classical string equation

$$\boldsymbol{\rho u_{tt} = P u_{xx} + f}, \tag{1}$$

and in the bending-stiff case the Bernoulli-Euler beam equation

$$\boldsymbol{\rho u_{tt} = -\,(S u_{xx} - P u)_{xx} + f} \tag{2}$$

with bending stiffness $\boldsymbol{S}$ and longitudinal force $\boldsymbol{P}$.

Other terms may be added, e.g. describing an elastic bedding or viscoelastic forces. Essential is the fact that the unknown $u = u(x,t)$ is a scalar quantity, and the moving force

$$\boldsymbol{f(x,t) = -f_1(t)\delta\big(x - x_1(t)\big)} \tag{3}$$

is described as a scalar as well – just one component acting in the negative z-direction at the length coordinate x_1, which in general is time dependent.

The functions f (lateral force density), ρ (mass density) and u (lateral displacement) are defined for nonnegative time t and positions $\in [0,X]$, $X > 0$. Notice that all densities together with the Dirac delta are with respect to the length measure along the string.

In equations (1) and (2) ge-ometrically nonlinear effects are excluded. The angle of inclination of each line or beam element $\alpha = \tan^{-1}(u_x) \approx u_x$ is considered small. This assumption may be violated, see e.g. Figs. 2 and 3. This may be also the case for larger forces and lighter ropes, and when the position of the force is close to an end point, where the rope is suspended, see Figs. 3 and 4. In that case, it is desirable to consider the middle line as a parametric curve in the form

$$\boldsymbol{v = v(\xi, t)}, \tag{4}$$

$$\boldsymbol{w = w(\xi, t)}, \tag{5}$$

where v is the displacement along the x-axis and w that along the z-axis, while ξ parametrizes the rope.

If the assumptions of the linear theory are valid, one should expect $u \approx w$ and $x \approx \xi$.

The unknown functions v and w are defined on $\xi \in [0,L]$ and $t > 0$, they are determined by the following relations:

- initial conditions $v_0(\xi) = v(\xi,0)$, $w_0(\xi) = w(\xi,0)$,
- boundary conditions $v(0,t)$, $v(L,t)$, $w(0,t)$, $w(L,t)$,
- geometric equation defining the strain ε,
- material law defining the internal force σ,
- balance of momentum with external forces $e(\xi,t)$, $f(\xi,t)$.

Fig. 2. Finite angles
Rys. 2. Duże kąty ugięcia

For this paper, we assume $\varepsilon = \sqrt{v_\xi{}^2 + w_\xi{}^2} - 1$ and $\sigma = k\varepsilon$ as measures of the elongation of line elements and internal force in the rope. The material constant k defines the stiffness of the material. Nonlinear material laws may be introduced – the main point in this study, however, is the geometrical nonlinearity.

The counterpart of equation (1) in the vector case is a system of equations, one for each of the components of the momentum ρv_t and ρw_t, defining its time evolution in terms of the acting forces

$$\boldsymbol{\rho v_{tt}} = \left(\boldsymbol{\sigma v_\xi}\right)_\xi + \boldsymbol{e}, \tag{6}$$

$$\boldsymbol{\rho w_{tt}} = \left(\boldsymbol{\sigma w_\xi}\right)_\xi + \boldsymbol{f}. \tag{7}$$

What is new is the nonlinear coupling between the equations by the line force $\sigma(\xi, t)$.

In this paper, we lay out the approach in two dimensions. The extension to three components is straightforward.

3 Solution Techniques

The problem (6), (7) can be treated as a variational problem. In the quasistatic case, assuming slow changes of the external forces, the solution can be obtained by minimizing the potential energy $U(v, w)$ step by step over the time interval of interest.

The total potential energy U has two contributions: the elastic energy stored in the rope due to its stretch and the work of the external forces.

$$\boldsymbol{U_{ela}} = \frac{1}{2}\boldsymbol{k\sigma^2}, \tag{8}$$

$$\boldsymbol{U_{ext}} = \boldsymbol{ev} + \boldsymbol{fw}. \tag{9}$$

So, finally, the expression

$$\boldsymbol{U} = \boldsymbol{U_{ela}} + \boldsymbol{U_{ext}}, \tag{10}$$

is to be minimized with respect to the unknown displacements v and w, given the boundary conditions and the momentary external forces.

Practically, the displacement functions v and w are substituted by vectors of nodal values, which comprise the spatial positions of the rope suspended in the field of external forces at a chosen number $n+1$ of material points ξ_j, $j=0,1,2,\ldots n$, on the interval $[0, L]$.

In the balance equations and in the geometric equation, differentiation is replaced by differences. The obtained discrete model can be interpreted as that of a chain of n elastically extensible, but bending-stiff, links. In the dynamic case, inertia forces could be included as additional external forces, this does, however, not lead to efficient numerical solutions.

In the classical approaches to problems (1) or (2) together with (3), the solution is typically searched for in a given analytical form, composed of trigonometric and/or exponential terms with certain unknown parameters.

Equations for parameters like frequencies and amplitudes are derived, e.g. dispersion relations, by substituting setups like

$$\boldsymbol{u}(\boldsymbol{x},\boldsymbol{t}) = \boldsymbol{a}\cos(\boldsymbol{\omega t} - \boldsymbol{\kappa x}), \tag{11}$$

into the equation of motion and simplification. At the point of contact $x_1(t)$, often $x_1(t) = Vt$ with a constant running speed V, compatibility conditions have to be considered.

In the present formulation, the equations of motion have to be integrated directly, without specified analytical form of the solution.

For practical implementation, in the vector case the previously described space discretization is applied, so that a second order system for $2(n-1)$ coordinates of the free nodal points is obtained. By the standard technique, this system is transformed into a first order system in the $4(n-1)$-dimensional phase space of positions and momentums, see e.g. [7,8].

We write this system in the form:

$$\dot{\boldsymbol{Y}}(\boldsymbol{t}) = \frac{d}{dt}\boldsymbol{Y}(\boldsymbol{t}) = \boldsymbol{F}\big(\boldsymbol{t}, \boldsymbol{Y}(\boldsymbol{t})\big). \tag{12}$$

The Jacobian $J = \nabla_Y F(t, Y)$ describes the linearization of the system around the present point of its trajectory. Its momentary spectrum $spec(J)$ is responsible for the stiffness, respectively non-stiffness, of the problem, and hence for the choice of a suitable method of integration.

In the given case, an implicit solver is recommended, since the ratio of the extreme eigenvalues of J is rather large in the considered examples.

Numerical Results

We start with the quasi-static case of a rope with negligible mass, infinite strength and vanishing bending stiffness.

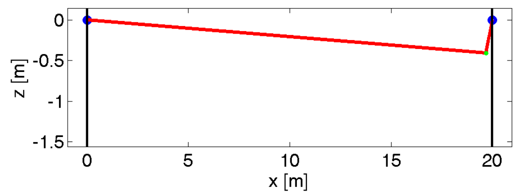

Fig. 3. Weightless unstretchable rope
Rys. 3. Nieważka, nierozciągliwa lina

Fig. 3 shows that a force near the interval end leads to a very steep slope between the force position and the point of support. Hence, the parameter curve approach is essential. Moreover, moving the point of force application further to the right may lead to vertical line segments and even to such running backwards, i.e., there may be instants t such that $\exists\xi \in [0, L]\ \ v(\xi, t) > X$.

It should be mentioned that, unlike in models (1) and (2), a pure vertical external force, i.e. $\equiv 0$, is not compatible in the case of (6), (7). A horizontal force has to be introduced in order to compensate the different angles of the line segments at the particle ξ_1, where the force is applied. This may be an external force, a friction force between the moving object and the rope, or it may be an inertia force – provided that there is a mass assigned to that object. In that case, a variable forward speed should be regarded.

In the case of an extensible line, the solution looks similar as in the inextensible case, but there are some important differences.

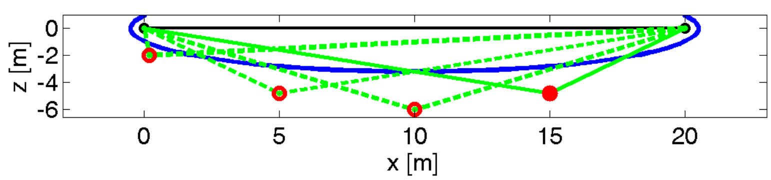

Fig. 4. Elastic rope
Rys. 4. Lina sprężysta

While in the previous case, the point $\big(v(\xi_1(t), t), w(\xi_1(t), t)\big)$ followed an ellipse, now the sum of the lengths of the two segments in Fig. 5 is no longer constant, it exceeds the original length L of the unstrained rope.

Now, if the mass density ϱ is big enough and constant, the previously straight segments become arcs of catenoids, or piecewise linear approximations thereof in the discrete case. It should be stressed that this concerns the idealized case of slow motion – all oscillations are damped, as if there was no inertia, just weight.

Such a simplification may be justified e.g. for a rope being totally or partially immersed in water.

A solution is obtained by solving the force balance at the force position, which is described by a nonlinear algebraic system. Alternatively, a direct solution of the minimum energy formulation may be attempted. Actually, the presented result in Fig. 4 was obtained by a shooting method.

Finally, if inertia effects cannot be neglected, a dynamical approach is required. In this case, results depend on the imposed speed of moving, i.e. on

$$\boldsymbol{V} = (\boldsymbol{1} + \boldsymbol{\epsilon})\frac{\boldsymbol{d\xi}}{\boldsymbol{dt}}. \tag{13}$$

A semi-analytical solution by a modal approach, for the case of 49 and of 149 modes and a speed close to the velocity of elastic waves $V = 0.9999c$, is shown in Fig. 5.

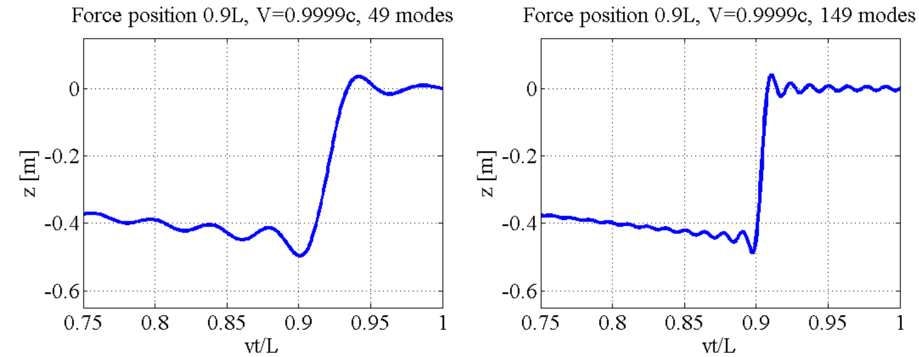

Fig.5. String displacement in case of the force approaching a support
Rys. 5. Przemieszczenia struny w przypadku siły zbliżającej się do podpory

Alternatively, we apply the discrete approach outlined before, using the same number of degrees of freedom, and start from the equilibrium state with zero velocities and positions according to the minimum potential energy of the gravity forces alone.

Here, we illustrate in Fig. 5 the difference between perfectly damped and dynamic motion by an example with a moderate speed of the force.

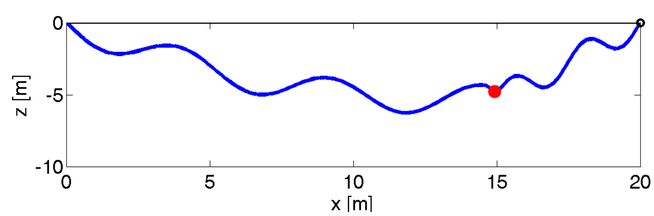

Fig. 6. Inertial rope in the case of speed $V = 10$ m/s
Rys. 6. Lina z bezwładnością w przypadku prędkości $V = 10\ m/s$

Figure 6 captures the moment $t = 1.5\ s$, when the moving force has traveled three quarters of the length of rope of lenght $L = 20\ m$ at a speed $V = 10\ m/s$.

Remark: *In the nonlinear case, definitions of speed may vary. We choose the speed along the line in spatial configuration.*

As an example of a rope (string) subjected to moving loads we can give a catenary system visible on Fig. 7. Technically, in order to overcome problems resulting from wave reflection from supports, the system is composed of two strings – the contact wire and the support wire. As we can see, the wire in contact with the pantograph is not connected directly to the posts. Instead, it is suspended on droppers attaching it to the upper support wire, see [1,5].

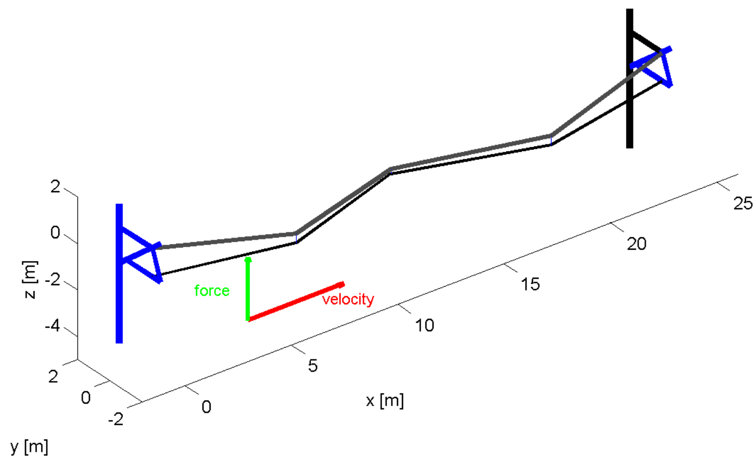

Fig. 7. Catenary system subjected to a moving force exerted by a pantograph

Rys. 7. Sieć zasilająca poddana działaniu ruchomej siły wywieranej przez odbierak prądu

4 Conclusions

In the case of moving contact between a flexible 1D structure and an external massless object or a concentrated mass, results depend on model assumptions. The assumption of small angles may become inadequate near points with a prescribed position. Friction forces may be essential, the longitudinal forces before and behind the point of contact may differ in a significant degree. Analogous results may be expected in the case of very thin membranes, tissues or meshes. The simulated effects diminish with increasing pre-stress, weight and bending stiffness of the considered structure. Soft supports are less prone to singular behavior than rigid ones.

Even in the simple case of a person walking step by step along a line, it is not reasonable to describe the moving object as a lumped mass. The interaction with the slack line can be well modeled as a moving force (or a pair of such forces) with variable value and direction – which depend on the behavior of the person walking the line. In turn, the person on the line will adopt their movements to that of the line, so that there is a coupling both ways.

The moving sub-system, be it a rolling wheel, a pantograph or a pedestrian, has to be modeled as a dynamical system, in general with many more degrees of freedom than suffice in the case of a point mass or a rigid body. If a complete simulation of the coupled system is to complex, also a co-simulation of both systems, modeled apart from each other, should be considered as one of the best options.

Considerable simplifications are possible if the feedback between the subsystems can be neglected. This concerns, to some extent, analytical models of rail-wheel motion. In each application, it has to be considered, whether the substitution of a vehicle by the force it exerts on the track is acceptable. On the other hand, in the context of very flexible manifolds, interacting with a moving rigid lump of mass, models assuming a functional dependence of the vertical displacement on the horizontal coordinate are rarely ever adequate.

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Summary

In railway mechanics, and more general, in civil engineering and vehicle dynamics, the problem of simultaneous modeling of a track and a vehicle has been the subject of many papers. Often one of the coupled subsystems is highly simplified in order to be able to obtain results on the other. For example, when the propagation of waves in the track is the main concern, vehicles such as a train or a taxiing airplane, are treated as an external force, travelling at a certain speed along a given path. In that case, the force is assumed as independent of the motion in the track, which results from the load. On the other hand, dynamical simulations of vehicles typically run on a defined ground, which is given and invariant, whatever the motion of the vehicle.

In order to make a track model more realistic, the vehicle model may be improved, without going as far as to couple a full-fledged vehicle model with a realistic model of a track consisting of rails on sleepers, supported via some subgrade on the ground. A first simple step is to attach an additional mass in the contact point, i.e. the position, where the external force is applied.

Keywords: chord, waves, moving forces, geometrical nonlinearity

Wędrowanie po linie. Ruchome siły i ruchome masy

Streszczenie

W inżynierii kolejowej lub ogólniej - w budownictwie i dynamice pojazdów zagadnienie równoczesnego modelowania toru i pojazdu było przedmiotem wielu badań. Często jeden ze sprzężonych układów był nadmiernie upraszczany, aby uzyskać rozwiązanie problemu. Zagadnienie struny najczęściej rozpatruje się w zakresie małych przesunięć i kątów, poszukiwana jest wówczas funkcja skalarna jednej zmiennej. Gdy konfiguracja aktualna odbiega znacznie od konfiguracji materialnej, wyniki uzyskane mogą być fizycznie nieakceptowalne. W pracy do rozwiązywania tego typu zadań zaleca się podejście parametryczne. Zalety takiego podejścia demonstrowane są na podanych przykładach z zakresu modelowania współpracy pojazdu z trakcją oraz popularnych sportów rekreacyjnych.

Słowa kluczowe: struna, fale, siły wędrujące, nieliniowość geometryczna