Analytical determination of the order of stress field singularity in some configurations of multiwedge systems for the case of antiplane deformation

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Received July 12.2017: accepted September 21.2017

Abstract. In this paper, the possibility of constructing the analytical expressions to determine the order of the stress singularities in multi-wedge composites of the most prevalent geometric configurations for the case of antiplane deformation is considered. Particularly, the analytical solutions of the corresponding characteristic equations are constructed for three-wedge systems whose components have such geometric characteristics: $\alpha_1 = \pi/2$, $\alpha_2 = \pi$, $\alpha_3 = \pi/2$ is a half-plane and attached to it wedges with the such apical angles: $\pi/2$ (in the presence and absence of a slit); $\alpha_1 = \pi/4$, $\alpha_2 = \pi$, $\alpha_3 = 3\pi/4$ is a half-plane and attached to it wedges with such apical angles: $\alpha_1 = \pi/4$, $\alpha_2 = 3\pi/4$ (in the presence and absence of the slit with outlet angle $\alpha = \pi/4$ to the linear materials interface); $\alpha_1 = \pi/4$, $\alpha_2 = \pi$, $\alpha_3 = \pi/4$ is a half-plane and attached to it wedges with such apical angles: $\alpha_1 = \alpha_2 = \pi/4$. The analytical solutions of characteristic equations for composite wedges composed of $n = 3, 4$ elements with identical apical angles are constructed as well. Additional studies, the results of which have not been included in the materials of the article due to their inconvenience, indicate to that there are analytical solutions of the characteristic equation for a composite of this type with more elements. The obtained results make it possible to study the stress-strain state in multi-wedge systems of the considered configurations not restricting ourselves only to the vicinity of the wedges convergence point. In addition, the use of analytical solutions of characteristic equations in systems with a large number of wedges having the same apical angles gives the additional possibilities for analysis the angularly functionally graded materials.

Key words: multi-wedge system, antiplane deformation, order of the stress singularities, analytical solutions, composite wedge.

INTRODUCTION

When designing the units of new models of machines and building structures of different designated purposes (including agricultural), the question of predicting their strength and reliability under operating loads arises. Because the components of such objects usually consist of separate elements interconnected in different ways, they have the so-called stress concentrators such as the breakpoint of the materials contact surfaces, tips of the slit (cracks) and pointed inclusions, wedge-shaped notches and points of contact of several materials, etc (the singular point) [1-20]. In the vicinity of points of this type, the stress fields are singular and clarification of the order of their singularity (the growth rate of stresses) is necessary to predict the reliability of the projected object [1, 4, 5, 9-14].

ANALYSIS OF RECENT RESEARCHES AND PUBLICATIONS

The issue of determination the singularity of stresses in the vicinity of special points has attracted the attention of researchers long. One of the first suggestions concerning the methods of determination the stress singularity is given in the work of K. Wieghardt [18] (1907), and the subsequent development of these themes is in a classical work of M. L. Williams [19]. A detailed overview of the works related to this problem is contained in the publications [1, 9, 10, 12]. It should be noted that the vicinity of singular points is usually modeled by the authors with the help of multi-wedge systems whose elements have a common tip. In their works, the researchers considered the analytical and numerical methods for clarification the stress-strain state in multi-wedge systems and realized the research for their specific configurations [2, 8, 15]. However, in analyzing the behavior of the stress-field they were mainly restricted to the systems, the number of elements of which did not exceed four. This, in the first place, is due to the fact that using the analytical approaches (complex potentials or Mellin's transformation) [17, 15, 20] requires the solution of the $2n$-order system of linear algebraic equations in the case of the antiplane problem and the $4n$-order system for the plane problem (n is the number of elements of the multi-wedge system), and numerical methods [6, 8] require further improvement, especially in the study of systems containing elements with small apical angles [6]. It is important to note that all these methods result in the necessity of using numerical procedure of solution the transcendental equations. Therefore, when determining the order of the stress singularity, we restrict ourselves only to those solutions of the characteristic equation that result in the maximum singularity when approaching a singular point. So, for an arbitrary geometric configuration of a composite wedge, it is almost impossible to determine the entire range of solutions of the characteristic equation and to write down the corresponding expressions for the stress or displacement fields in the whole wedge composite. That is why the analysis of the stress-strain state is limited only by the vicinity of the stress concentrator.

OBJECTIVE

The objective of this study is to identify such geometric configurations of multi wedge systems for which their characteristic equations admit an analytical solution. For this purpose, we use the characteristic equations for a multi-wedge system under the condition of antiplane deformation, constructed in the works [9, 10]. This made it possible to find a series of multi-wedge systems such that the characteristic equations constructed for them can be solved analytically. Then, according to the a known procedure, it is easy to construct the corresponding expressions for the stresses or displacements field in the whole wedge composite.

STATEMENT OF THE PROBLEM AND INITIAL EXPRESSIONS

Consider a composite composed of an arbitrary number n of ideally coupled heterogeneous elastic, isotropic wedges $S_i \left(i=1,2,...,n\right)$ with apical angles $\alpha_i \left(\sum_i^n \alpha_i \le 2\pi\right)$ and shear modulus μ_i, respectively, and a wedge-shaped notch S_{n+1} with apical angles $\alpha_{n+1} = 2\pi - \sum_{i=1}^n \alpha_i \le 2\pi$ (Fig. 1).

We consider that the composite: is under the condition of antiplane deformation $u=0$, $v=0$, $w=w(r,\varphi)$; is referred to the polar coordinate system r, φ centered at the point of the wedges joint O (wedge S_i occupies the area $\varphi_{i-1} < \varphi < \varphi_i$, $0 < r < \infty$, and to connecting lines of the wedges S_i and S_{i+1} the coordinates $\varphi_i = \sum_{i=1}^i \alpha_i \le 2\pi$ correspond). On the edges of the wedge-shaped notch S_{n+1} ($\varphi_0 = 0$ and $\varphi_n = \sum_{k=1}^n \alpha_k$) the conditions of the first, second or mixed problems of the elasticity theory are specified.

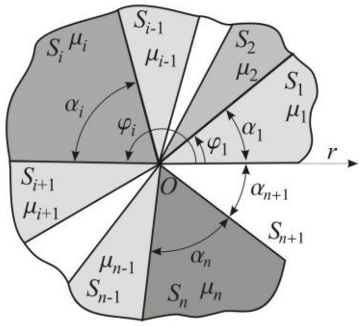


Fig. 1. The scheme of multi-wedge system

In such a system the stress field is described by relations [9, 15]:

$$\begin{aligned}
&\sigma_{\varphi z}\left(r,\varphi\right) = \sum_{i=0}^{\infty} \operatorname{Res}\left(\frac{\partial \tilde{w}\left(p,\varphi\right)}{\partial \varphi}, p_i\right) r^{-\left(p_i+1\right)}, \\
&\sigma_{rz}\left(r,\varphi\right) = -\sum_{i=0}^{\infty} \operatorname{Res}\left(p\tilde{w}\left(p,\varphi\right), p_i\right) r^{-\left(p_i+1\right)},
\end{aligned} \tag{1}$$

where: $\operatorname{Res}\left(\frac{\partial \tilde{w}\left(p,\varphi\right)}{\partial \varphi}, p_i\right)$ and $\operatorname{Res}\left(p\tilde{w}\left(p,\varphi\right), p_i\right)$ are the residues of the corresponding functions; p_i are the poles of the function $\tilde{w}\left(p,\varphi\right)$ ($\operatorname{Re} p_i > -1$) and $\tilde{w}\left(p,\varphi\right)$ is the Mellin transform of the displacement which in this wedge system is defined as follows [9, 10]:

$$\begin{aligned}
\tilde{w}\left(p,\varphi\right) = A_1\left(p\right)&\left[\cos p\varphi - \sum_{i=1}^{n-1} \frac{\mu_{i+1}-\mu_i}{p\mu_{i+1}} L_1^i \times\right. \\
&\left.\times \sin\left[p\left(\varphi-\varphi_i\right)\right] S_+\left(\varphi-\varphi_i\right)\right] + \\
+ B_1\left(p\right)&\left[\sin p\varphi - \sum_{i=1}^{n-1} \frac{\mu_{i+1}-\mu_i}{p\mu_{i+1}} L_2^i \times\right. \\
&\left.\times \sin\left[p\left(\varphi-\varphi_i\right)\right] S_+\left(\varphi-\varphi_i\right)\right],
\end{aligned} \tag{2}$$

$$\begin{aligned}
&L_1^i = -p\sin p\varphi_i - \sum_{k=1}^{i-1} \frac{\mu_{k+1}-\mu_k}{\mu_{k+1}} L_1^k \cos\left[p\left(\varphi_i-\varphi_k\right)\right], \\
&L_2^i = p\cos p\varphi_i - \sum_{k=1}^{i-1} \frac{\mu_{k+1}-\mu_k}{\mu_{k+1}} L_2^k \cos\left[p\left(\varphi_i-\varphi_k\right)\right].
\end{aligned} \tag{3}$$

The functions $A_1\left(p\right), B_1\left(p\right)$ depending on the boundary conditions are defined by such expressions [9]:

1) for the first boundary-value problem of the elasticity theory –

$$\begin{aligned}
&A_1\left(p\right) = \frac{\tilde{\tau}_{n+1}\left(p+1\right)}{\mu_n \Delta_1\left(p\right)} - \frac{\tilde{\tau}_0\left(p+1\right)}{p\mu_1 \Delta_1\left(p\right)} \times \\
&\quad \times \left(p\cos p\varphi_n - \sum_{i=1}^{n-1} \frac{\mu_{i+1}-\mu_i}{\mu_{i+1}} L_2^i \cos\left[p\left(\varphi_n-\varphi_i\right)\right]\right), \\
&B_1\left(p\right) = \frac{\tilde{\tau}_0\left(p+1\right)}{p\mu_1}, \\
&\Delta_1\left(p\right) = -p\sin p\varphi_n - \sum_{i=1}^{n-1} \frac{\mu_{i+1}-\mu_i}{\mu_{i+1}} L_1^i \cos\left[p\left(\varphi_n-\varphi_i\right)\right];
\end{aligned} \tag{4}$$

2) for the second boundary-value problem of the elasticity theory–

$$B_1(p)=\frac{\tilde{w}_{n+1}(p)}{\Delta_2(p)}-\frac{\tilde{w}_0(p)}{\Delta_2(p)}\times$$
$$\times\left(\cos p\varphi_n-\sum_{i=1}^{n-1}\frac{\mu_{i+1}-\mu_i}{p\mu_{i+1}}L_1^i\sin\left[p\left(\varphi_n-\varphi_i\right)\right]\right), \tag{5}$$
$$A_1(p)=\tilde{w}_0(p),$$
$$\Delta_2(p)=\sin p\varphi_n-\sum_{i=1}^{n-1}\frac{\mu_{i+1}-\mu_i}{p\mu_{i+1}}L_2^i\sin\left[p\left(\varphi_n-\varphi_i\right)\right]; \tag{6}$$

3) for the mixed boundary-value problem of the elasticity theory (depending on its conditions) –

$$B_1(p)=\frac{\tilde{\tau}_{n+1}(p+1)}{\mu_n\Delta_3(p)}+\frac{\tilde{w}_0(p)}{\Delta_3(p)}\times$$
$$\times\left(p\sin p\varphi_n+\sum_{i=1}^{n-1}\frac{\mu_{i-1}-\mu_i}{\mu_{i+1}}L_1^i\cos\left[p\left(\varphi_n-\varphi_i\right)\right]\right), \tag{7}$$
$$A_1(p)=\tilde{w}_0(p),$$
$$\Delta_3(p)=p\cos p\varphi_n-\sum_{i=1}^{n-1}\frac{\mu_{i-1}-\mu_i}{\mu_{i+1}}L_2^i\cos\left[p\left(\varphi_n-\varphi_i\right)\right],$$

or

$$A_1(p)=\frac{w_{n+1}}{\Delta_4(p)}-\frac{\tilde{\tau}_0(p+1)}{p\mu_1\Delta_4(p)}\times$$
$$\times\left(\sin p\varphi_n-\sum_{i=1}^{n-1}\frac{\mu_{i+1}-\mu_i}{p\mu_{i+1}}L_2^i\sin\left[p\left(\varphi_n-\varphi_i\right)\right]\right), \tag{8}$$
$$B_1(p)=\frac{\tilde{\tau}_0(p+1)}{p\mu_1},$$
$$\Delta_4(p)=\cos p\varphi_n-\sum_{i=1}^{n-1}\frac{\mu_{i+1}-\mu_i}{p\mu_{i+1}}L_1^i\sin\left[p\left(\varphi_n-\varphi_i\right)\right].$$

Here $\tilde{\tau}_0(p+1)$, $\tilde{\tau}_{n+1}(p+1)$, $\tilde{w}_0(p)$, $\tilde{w}_{n+1}(p)$ are the Mellin transforms of functions describing the loading and displacements, given on the corresponding edges of the wedge-shaped notch.

Thus, for antiplane deformation, the characteristic equation for determination the poles of Mellin transform of the displacement (2), according to (4) - (8), has the following form:

1) under the conditions of the first boundary-value problem of the elasticity theory

$$p\sin p\varphi_n+\sum_{i=1}^{n-1}\frac{\mu_{i+1}-\mu_i}{\mu_{i+1}}L_1^i\cos\left[p(\varphi_n-\varphi_i)\right]=0; \tag{9}$$

2) under the conditions of the second boundary-value problem –

$$\sin p\varphi_n-\sum_{i=1}^{n-1}\frac{\mu_{i+1}-\mu_i}{p\mu_{i+1}}L_2^i\sin\left[p(\varphi_n-\varphi_i)\right]=0; \tag{10}$$

3) under conditions of mixed boundary-value problem –

$$p\cos p\varphi_n-\sum_{i=1}^{n-1}\frac{\mu_{i+1}-\mu_i}{\mu_{i+1}}L_2^i\cos\left[p(\varphi_n-\varphi_i)\right]=0, \tag{11}$$

if the stresses are given at $\varphi=0$, and the displacements at $\varphi=\varphi_n$ (case a) and

$$\cos p\varphi_n-\sum_{i=1}^{n-1}\frac{\left(\mu_{i+1}-\mu_i\right)}{p\mu_{i+1}}L_1^i\sin\left[p\left(\varphi_n-\varphi_i\right)\right]=0, \tag{12}$$

if the forces are given at $\varphi=\varphi_n$, and the displacements at $\varphi=0$ (case b).

If the system is a continuous piecewise-homogeneous body, composed of n wedges, we believe that the notch S_{n+1} turn into a crack ($\alpha_{n+1}=0$), and on its edges the conditions of an ideal mechanical contact $\sigma_{\varphi z}\big|_{\varphi=0}-\sigma_{\varphi z}\big|_{\varphi=2\pi}=0$, $w\big|_{\varphi=0}-w\big|_{\varphi=2\pi}=0$ are specified. In this case, the characteristic equation has the form:

$$\left(\cos 2\pi p-\sum_{i=1}^{n-1}\frac{\left(\mu_{i+1}-\mu_i\right)}{p\mu_{i+1}}L_1^i\sin\left[p\left(2\pi-\varphi_i\right)\right]-1\right)\times$$
$$\times\left[\mu_n\left(p\cos 2\pi p-\sum_{i=1}^{n-1}\frac{\mu_{i+1}-\mu_i}{\mu_{i+1}}L_2^i\cos\left[p(2\pi-\varphi_i)\right]\right)-\right.$$
$$\left.-\mu_1 p\right]-\left(\sin 2\pi p-\sum_{i=1}^{n-1}\frac{\left(\mu_{i+1}-\mu_i\right)}{p\mu_{i+1}}L_2^i\sin\left[p\left(2\pi-\varphi_i\right)\right]\right)\times \tag{13}$$
$$\times\left(-p\sin 2\pi p-\sum_{i=1}^{n-1}\frac{\mu_{i+1}-\mu_i}{\mu_{i+1}}L_1^i\cos\left[p(2\pi-\varphi_i)\right]\right)\mu_n=0.$$

Expressions (9) – (13) give us the ability to determine the order of stress-singularity in a multi-wedge system composed of arbitrary number of elements with arbitrary geometric and mechanical characteristics under the conditions of longitudinal shear.

Note that according to the presentation (1) the order of the stress fields singularity in the vicinity of the vertex of the wedge-shaped notch contained in this system is determined by the relation $\lambda_i=1+\mathrm{Re}\left(p_i\right)$, where p_i are the roots of the characteristic equation $\Delta_j\left(p\right)=0$ $\left(j=\overline{1,4}\right)$, real part of which belongs to the interval $\mathrm{Re}\left(p_i\right)\in\left(-1;0\right)$ [4, 9].

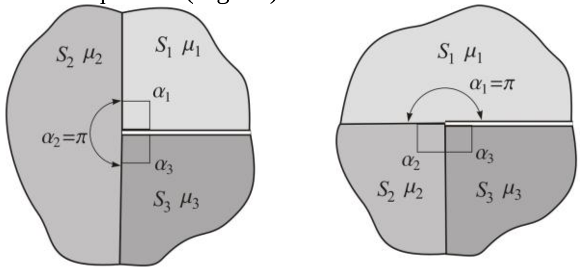
MAIN RESULTS OF THE RESEARCH

The characteristic equations for wedge systems are transcendental and to find their solution the numerical methods are to be used. However, for some geometric configurations of wedge systems, an analytical solution of the corresponding characteristic equation is possible.

1. HALF-PLANE WITH WEDGES ATTACHED TO IT

In engineering, nodal joints in the form of a half-plane and wedges soldered to it occur often. Below we discuss some of the typical geometric configurations of such joint for which the analytic solutions of their characteristic equations are written.

A widespread case of such a system is a **composite composed of a half-plane and quarters attached to it** (wedges with an apical angle $\pi/2$). In such a system, the slit (cracks) can reach the materials interface at right angles (Fig. 2a), or be located between the half-plane and one of the quarters (Fig. 2b).



a b

Fig. 2. The half-plane and quarters attached to it

To construct the characteristic equations for such systems we assume in (9) – (13) $\varphi_1 = \pi/2$, $\varphi_2 = 3\pi/2$, $\varphi_3 = 2\pi$ for a slit that reach the half-plane at right angle (Fig. 2 a) and $\varphi_1 = \pi$, $\varphi_2 = 3\pi/2$, $\varphi_3 = 2\pi$ for a slit that is located between the half-plane and one of the quarters (Fig. 2 b). After a series of algebraic transformations, we obtain the following characteristic equations and their analytical solutions:

for the first boundary-value problem –

$$p\left(b\cos p\pi - a\right)\sin p\pi = 0, \tag{14}$$

$$p = k,\ \ p = \pm\pi^{-1}\arccos\frac{a}{b} + 2k,\ \ k \in Z;$$

for the second boundary-value problem –

$$\left(b\cos p\pi + a\right)\sin p\pi = 0, \tag{15}$$

$$p = k,\ \ p = \pm\pi^{-1}\arccos\left(-\frac{a}{b}\right) + 2k,\ \ k \in Z;$$

for a mixed boundary-value problem (case a) –

$$p(A\cos^2 p\pi + B\cos p\pi + C) = 0, \tag{16}$$

$$p = 0,\ \ p = \pm\pi^{-1}\arccos\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} + 2k,\ \ k \in Z.$$

Here $a = \mu_1\mu_3 - \mu_2^2$, $b = (\mu_1 + \mu_2)(\mu_2 + \mu_3)$, $A = (\mu_1 + \mu_2)(\mu_2 + \mu_3)$, $B = \mu_2(\mu_1 - \mu_3)$, $C = -\mu_2^2 - \mu_1\mu_3$ in the case of a slit reaching the half-plane at right angle (Fig. 2 a) and $a = \mu_1(\mu_3 - \mu_2)$, $b = (\mu_1 + \mu_2)(\mu_2 + \mu_3)$, $A = (\mu_1 + \mu_2)(\mu_2 + \mu_3)$, $B = \mu_1(\mu_2 - \mu_3)$, $C = -\mu_2(\mu_2 + \mu_3)$ in the case of a slit located between the half-plane and the quarter (Fig. 2 b).

For a continuous composite composed of a half-plane and quarters soldered to it, the characteristic equation and its solutions are of the form:

$$p\sin^2\frac{p\pi}{2}\left(b\cos p\pi - a\right) = 0, \tag{17}$$

$$p = 2k,\ \ p = \pm\frac{1}{\pi}\arccos\frac{a}{b} + 2k,\ \ k \in Z,$$

where: $a = -\mu_2(\mu_1 + \mu_3)^2 - \mu_3(\mu_1 + \mu_2)^2$,

$b = (\mu_1 + \mu_2)(\mu_1 + \mu_3)(\mu_2 + \mu_3)$.

We note that the solution of the characteristic equation for a mixed boundary-value problem (16) admits the presence of complex roots.

Interesting from an engineering point of view is **a case when the slit (interfacial crack) reaches a half-plane at an angle** $\pi/4$ (Fig. 3). In this case, you should take in (9) – (13) $\varphi_1 = \pi/4$, $\varphi_2 = 5\pi/4$, $\varphi_3 = 2\pi$.

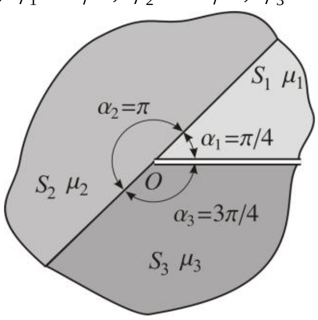


Fig. 3. The slit reaches a half-plane at an angle $\pi/4$

Thus, the characteristic equations for determining the order of the stresses singularity, depending on the boundary conditions, have the following form:

for the first boundary-value problem –

$$p\sin\frac{p\pi}{2}\left(a\cos^3\frac{p\pi}{2} + b\cos^2\frac{p\pi}{2} + c\cos\frac{p\pi}{2} + d\right) = 0, \tag{18}$$

where: $a = 4(\mu_1 + \mu_2)(\mu_2 + \mu_3)$,

$b = -2(\mu_1 - \mu_2)(\mu_2 + \mu_3)$,

$c = -\mu_2(3\mu_2 + \mu_3) - \mu_1(\mu_2 + 3\mu_3)$, $d = \mu_1\mu_3 - \mu_2^2$;

for the second boundary-value problem –

$$\sin\frac{p\pi}{2}\left(a\cos^3\frac{p\pi}{2} + b\cos^2\frac{p\pi}{2} + c\cos\frac{p\pi}{2} + d\right) = 0, \tag{19}$$

where: $a = 4(\mu_1 + \mu_2)(\mu_2 + \mu_3)$,

$b = 2(\mu_1 - \mu_2)(\mu_2 + \mu_3)$,

$c = -\mu_2(3\mu_2 + \mu_3) - \mu_1(\mu_2 + 3\mu_3)$, $d = -(\mu_1\mu_3 - \mu_2^2)$;

for a mixed boundary-value problem (case a) –

$$\begin{gathered} p\left(A\cos^4\frac{p\pi}{2} + B\cos^3\frac{p\pi}{2} + \right. \\ \left. + C\cos^2\frac{p\pi}{2} + D\cos\frac{p\pi}{2} + E\right) = 0, \end{gathered} \tag{20}$$

where: $A = 8(\mu_1 + \mu_2)(\mu_2 + \mu_3)$,

$B = 4(\mu_1 - \mu_2)(\mu_2 + \mu_3)$,

$C = -2\left[\mu_2(5\mu_2 + 3\mu_3) + \mu_1(3\mu_2 + 5\mu_3)\right]$,

$D = 2\mu_2(2\mu_2 + \mu_3) - 2\mu_1(\mu_2 + 2\mu_3)$, $E = \mu_2^2 - \mu_1\mu_3$;

for a continuous composite body –

$$\begin{gathered} p\left(A\cos^4\frac{p\pi}{2} + B\cos^3\frac{p\pi}{2} + \right. \\ \left. + C\cos^2\frac{p\pi}{2} + D\cos\frac{p\pi}{2} + E\right) = 0, \end{gathered} \tag{21}$$

where: $A = -8(\mu_1 + \mu_2)(\mu_1 + \mu_3)(\mu_2 + \mu_3)$,

$B = 4(\mu_1 - \mu_2)(\mu_1 - \mu_3)(\mu_2 + \mu_3)$,

$C = 2(\mu_1 + \mu_3)\left[\mu_2(5\mu_2 + 3\mu_3) + \mu_1(3\mu_2 + 5\mu_3)\right]$,

$D = 2(\mu_3 - \mu_1)\left[\mu_2(2\mu_2 + \mu_3) - \mu_1(\mu_2 + 2\mu_3)\right]$,

$E = -2\left[\mu_3(\mu_1 - \mu_2)^2 + \mu_1(\mu_2 - \mu_3)^2\right]$.

Writing an analytic solution of characteristic equations (18) – (21) requires the solution of algebraic equations of the third and fourth degree. Therefore, we do not record the general form of the analytical solution, but just note the possibility of its construction.

A **multi wedge system, composed of a half-plane, to which two wedges with apical angles** $\pi/4$ **are connected** (Fig. 4), is considered. Having taken in (9) - (12) $\alpha_1 = \pi/4$, $\alpha_2 = \pi$, $\alpha_3 = \pi/4$ we obtain the corresponding characteristic equations for each case of boundary conditions.

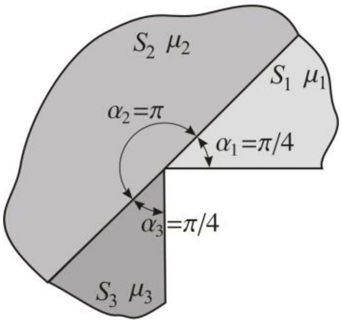


Fig. 4. The half-plane, to which two wedges with apical angles $\pi/4$ are connected

In the case of the first and second boundary-value problems, the characteristic equations and their solutions, respectively, are of such form:

$$p\sin\frac{p\pi}{2}\left(a\cos^2\frac{p\pi}{2}+b\cos\frac{p\pi}{2}+c\right)=0, \qquad (22)$$

$$p=2k,\ \ p=\pm\frac{2}{\pi}\arccos\frac{-b\mp\sqrt{b^2-4ac}}{2a}+4k,\ \ k\in Z;$$

$$p\sin\frac{p\pi}{2}\left(a\cos^2\frac{p\pi}{2}-b\cos\frac{p\pi}{2}+c\right)=0, \qquad (23)$$

$$p=\pm\frac{2}{\pi}\arccos\frac{b\mp\sqrt{b^2-4ac}}{2a}+4k,\ \ k\in Z,$$

where: $a=2(\mu_1+\mu_2)(\mu_2+\mu_3)$, $b=\mu_2^2-\mu_1\mu_3$, $c=-\mu_2(\mu_1+\mu_3)$.

In the case of a mixed problem (case a), the characteristic equation (11) will read:

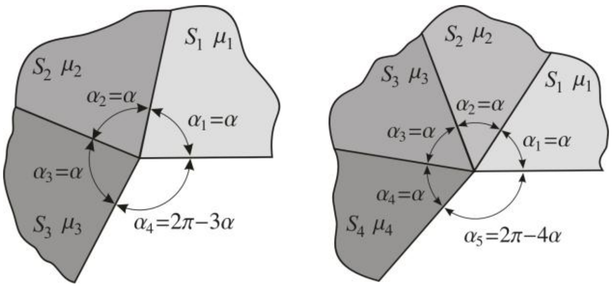
$$p\left(A\cos^3\frac{p\pi}{2}+B\cos^2\frac{p\pi}{2}+C\cos\frac{p\pi}{2}+D\right)=0, \qquad (24)$$

where: $A=2(\mu_1+\mu_2)(\mu_2+\mu_3)$, $B=2\mu_2(\mu_1-\mu_3)$, $C=-\left[\mu_2(2\mu_2+\mu_3)+\mu_1(\mu_2+2\mu_3)\right]$, $D=-\mu_2(\mu_1-\mu_3)$.

In the case of a mixed problem finding the solutions of the characteristic equation (24) requires solution of the third degree equation. Therefore, we will omit in this case the recording of the general form of solution, noting only that it exists and can be found analytically.

2. MULTI-WEDGE COMPOSITES COMPOSED OF WEDGES WITH IDENTICAL APICAL ANGLES

When modeling the multi-wedge composites, composed of wedges with identical apical angles ($\alpha_i=\alpha$, $\varphi_i=i\alpha$, $i=\overline{1,n}$), are often used. Such systems are used in particular in modeling the inserts of functionally graded materials, whose elastic properties change in a transverse direction [7, 8, 16]. In this paper we consider the systems with a small number of wedges ($n\le 4$), which often occur in elements of various types of machines and structures (Fig. 5).



a b

Fig. 5. The multi-wedge composites composed of wedges with identical apical angles

For three-wedge system, the substitution in (9) - (13) the values of apical angles and the coordinates of connection lines (Fig. 5 a) will yield the following characteristic equations and their solutions:

for the first boundary-value problem of elasticity theory –

$$p\left(b\cos 2p\alpha-a\right)\sin p\alpha=0, \qquad (25)$$

$$p=\frac{1}{2\alpha_1}\left(\arccos\left[\frac{a_1}{b}\right]+2\pi k\right),\ \ p=\frac{\pi k}{\alpha},\ \ k\in Z;$$

for the second boundary-value problem –

$$\left(a_2-b\cos 2p\alpha\right)\sin p\alpha=0, \qquad (26)$$

$$p=\frac{1}{2\alpha}\left(\arccos\frac{c}{b}+2\pi k\right),\ \ p=\frac{\pi k}{\alpha},\ \ k\in Z;$$

for a mixed boundary-value problem (case a) –

$$p\left(b\cos 2p-a_3\right)\cos p\alpha=0, \qquad (27)$$

$$p=\frac{1}{2\alpha_1}\left(\arccos\frac{a_3}{b}+2\pi k\right),\ p=0,\ p=\frac{\pi}{2\alpha}+\frac{\pi k}{\alpha},\ k\in Z.$$

Here $a_1=\mu_1(\mu_3-\mu_2)-\mu_2(\mu_2+\mu_3)$, $b=(\mu_1+\mu_2)(\mu_2+\mu_3)$, $a_2=\mu_2(\mu_3-\mu_2)-\mu_1(\mu_2+\mu_3)$, $a_3=\mu_1(\mu_2-\mu_3)+\mu_1(\mu_2+\mu_3)$.

In the case of a continuous body composed of three heterogeneous wedges with apical angles $\alpha_i=2\pi/3$ the characteristic equation will have the form:

$$p\sin^2\frac{p\pi}{3}\left(\cos^2\frac{2p\pi}{3}+\cos\frac{2p\pi}{3}+c\right)=0, \qquad (28)$$

$$p=\pm\frac{3}{2\pi}\arccos\left[0{,}5\left(-1\pm\sqrt{1-4c}\right)\right]+3k,\ \ p=3k,\ \ k\in Z,$$

where: $c=2\mu_2\mu_1\mu_3\left[(\mu_1+\mu_2)(\mu_1+\mu_3)(\mu_2+\mu_3)\right]^{-1}$.

We note that in this case the complex roots of the characteristic equation are also possible.

For a system composed of four heterogeneous wedges (Fig. 5 b), based on representations (9) - (13), the following equations are obtained:

for the first boundary-value problem of elasticity theory –

$$p\sin 2p\alpha\left(a_1-b\cos 2p\alpha\right)=0, \qquad (29)$$

$$p=\pm\frac{1}{2\alpha_1}\arccos\frac{a_1}{b}+\frac{\pi k}{\alpha_1},\ \ p=\frac{\pi k}{2\alpha},\ \ k\in Z;$$

for the second boundary-value problem –

$$\sin\left[2p\alpha\right]\left(a_2-b\cos 2p\alpha\right)=0, \qquad (30)$$

$$p=\frac{\pi k}{2\alpha},\ \ p=\pm\frac{1}{2\alpha}\arccos\frac{a}{b}+\frac{\pi k}{\alpha},\ \ k\in Z.$$

Here $b=(\mu_1+\mu_2)(\mu_2+\mu_3)(\mu_3+\mu_4)$,

$$a_1 = \mu_2\left[\mu_2\left(\mu_4-\mu_3\right)-\mu_3\left(\mu_4+\mu_3\right)\right]+$$
$$+\mu_1\left[\mu_2\left(\mu_4-\mu_3\right)+\mu_3\left(\mu_4+\mu_3\right)\right],$$
$$a_2 = \mu_2\left[\mu_3\left(\mu_3-\mu_4\right)+\mu_2\left(\mu_3+\mu_4\right)\right]-$$
$$-\mu_1\left(\mu_3\left(\mu_4-\mu_3\right)+\mu_2\left(\mu_3+\mu_4\right)\right).$$

For a mixed boundary-value problem (case a), the characteristic equation (11) will read:

$$p\left(a\cos^2 2p\alpha + b\cos 2p\alpha + c\right)=0 \tag{31}$$

$$p=\frac{1}{2\alpha}\left(\pm\arccos\frac{-b\mp\sqrt{b^2-4ac}}{2a}+2\pi k\right),\ p=0,\ k\in Z.$$

Here $a=\left(\mu_1+\mu_2\right)\left(\mu_2+\mu_3\right)\left(\mu_3+\mu_4\right)$,

$$b=2\mu_2\left(\mu_1\mu_3-\mu_2\mu_4\right),$$
$$c=\mu_1\left[\mu_2\left(\mu_3-\mu_4\right)-\mu_3\left(\mu_3+\mu_4\right)\right]+$$
$$+\mu_2\left[\mu_2\left(\mu_4-\mu_3\right)-\mu_3\left(\mu_3+\mu_4\right)\right].$$

For a continuous composite body ($\alpha=\pi/2$) the characteristic equation (11) will read –

$$p\sin^2\frac{p\pi}{2}\left(b-a\cos p\pi\right)=0, \tag{32}$$

$$p=2k,\ \frac{1}{\pi}\arccos\frac{b}{a},\ k\in Z.$$

Here $a=-\left(\mu_1+\mu_2\right)\left(\mu_2+\mu_3\right)\left(\mu_1+\mu_4\right)\left(\mu_3+\mu_4\right)$,

$$b=\mu_1^2\left[\mu_3\left(\mu_4-\mu_3\right)+\mu_2\left(\mu_3+\mu_4\right)\right]+$$
$$+\mu_2\mu_4\left[\mu_2\left(\mu_3-\mu_4\right)+\mu_3\left(\mu_3+\mu_4\right)\right]+$$
$$+\mu_1\mu_2^2\left(\mu_3+\mu_4\right)+\mu_1\mu_3\mu_4\left(\mu_3+\mu_4\right)+$$
$$+\mu_2\mu_1\left(\mu_3^2+6\mu_3\mu_4+\mu_4^2\right).$$

We note that the systems considered are symmetric in their geometric parameters. Thus, the characteristic equation for the case of mixed boundary conditions (case b) is obtained formally by replacing the displacement modulus in the characteristic equation for case a μ_i by μ_i', where $\mu_i'=\mu_{n+1-i}$ $i=\overline{1,n}$.

CONCLUSIONS

The results above were obtained on the basis of expressions (4) - (8), the legitimacy of their using to determine the order of stresses singularity is substantiated in the works [9, 10]. Thus, a series of multi-wedge systems, for which it is possible to construct analytical solutions of the corresponding characteristic equations, is determined.

The paper presents the expressions of analytical solutions of characteristic equations only for some systems in which the number of elements does not exceed $n=4$, as those that are most commonly found in engineering. In particular, the analytical expressions that allow to determine the order of the stresses singularity in the vicinity of the tip of a slit (crack), reaching at right angles and at an angle $\pi/4$ the linear materials interface are given. Additional studies, the results of which are not presented here, suggest that the analytical solutions exist for other systems, in particular for systems composed of wedges with identical apical angles with more elements.

The analytical determination of the complete set of the roots of the characteristic equation leads to construction on the basis of representations (1) - (3) the expressions for stress and displacement fields in a multi-system system not limiting only by the vicinity of the wedges convergence point. In addition, the use of analytical solutions of characteristic equations in the systems with a large number of wedges, having the identical apical angles, opens wide possibilities for researching the materials with angular gradientness [7, 8].

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