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ON SOME CHARACTERIZATION OF AN INVERSE PROPORTIONALITY TYPE FUNCTION

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Abstract

We deal with a functional equation of the form

$$f(x+y) = F(f(x), f(y))$$

(so called addition formula) assuming that the given binary operation F is associative but its domain is not connected. The aim of the present paper is to discuss solutions of the equation

$$f(x+y) = \frac{f(x)f(y)}{f(x)+f(y)}.$$

It turns out that this functional equation characterized an inverse proportionality type function, but if the domain of the unknown function has no neutral element. In this paper we admit fairly general structure in the domain of the unknown function.

1. Introduction

The functional equation

$$f(x+y) = \frac{f(x)f(y)}{f(x)+f(y)}$$

is the particular case (with $\alpha = 0$) of the equation, which was considered by the author in [3]. In this work for this case we obtain only trivial solutions.

A map $F : \{(x,y) \in \mathbb{R} : \quad x \neq -y\} \longrightarrow \mathbb{R}$ of the form

$$F(u,v) = \frac{uv}{u+v}$$

is a rational two-place real-valued function defined on a disconnected subset of the real plane $\mathbb{R}^2$, that satisfies the equation

$$F(F(x,y),z) = F(x, F(y,z))$$

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for all $(x, y, z) \in \mathbb{R}^3$ such that sums $x + y, y + z, F(x, y) + z, x + F(y, z)$ are not equal to 0. Rational functions with such or similar properties are termed associative operations. The class of the associative operations was described by A. Chéritat [1] and his work was followed by paper [2] of the author.

For every constant $c \in \mathbb{R} \setminus \{0\}$ a homographic function $\varphi : \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}$ given by the formula

$$\varphi(x) = \frac{c}{x}, \quad x \neq 0$$

satisfies the functional equation

$$f(x + y) = \frac{f(x)f(y)}{f(x) + f(y)}$$

for every pair $(x, y) \in \mathbb{R}^2 \setminus D$, where

$$D = \{(x, -x) : \quad x \in \mathbb{R}\} \cup \{(x, 0) : \quad x \in \mathbb{R}\} \cup \{(0, x) : \quad x \in \mathbb{R}\}.$$

We shall determine all functions $f : G \longrightarrow \mathbb{R}$, where $(G, +)$ is a group, that satisfy the functional equation

$$f(x + y) = \frac{f(x)f(y)}{f(x) + f(y)}. \tag{1}$$

A neutral element of the group $(G, +)$ will be written as 0.

By a solution of the functional equation (1) we understand here any function $f : G \longrightarrow \mathbb{R}$ that satisfies equality (1) for every pair $(x, y) \in G^2$ such that $f(x) + f(y) \neq 0$. Thus we deal with the following conditional functional equation:

$$f(x) + f(y) \neq 0 \quad \text{implies} \quad f(x + y) = \frac{f(x)f(y)}{f(x) + f(y)} \tag{E1}$$

for all $x, y \in G$. In [3] we can find the following:

Theorem 1. *Let $(G, +)$ be a group. The only solution $f : G \longrightarrow \mathbb{R}$ of the equation* (E1) *is $f = 0$.*

We want to obtain a non-trivial solution of this equation.

2. Main result

We obtain non constant solutions of the equation (E1) if we consider the unknown function not on the whole group, but on complement of the neutral element. Thus we deal with the following conditional functional equation with a restricted domain:

$$f(x)+f(y)\neq 0 \quad \text{implies} \quad f(x+y)=\frac{f(x)f(y)}{f(x)+f(y)}, \quad x,y,x+y\in G\setminus\{0\}, \tag{E}$$

for functions $f : G\setminus\{0\}\longrightarrow \mathbb{R}$, where $(G,+)$ is a group with the neutral element 0. First we observe that solutions of (E) vanishing at the non-zero element a are trivial.

Lemma. *Let* $(G,+)$ *be a group and let* $f : G\setminus\{0\}\longrightarrow\mathbb{R}$ *be a solution of the equation* (E) *such that* $f(a)=0$ *for some* $a\neq 0$*. Then*

$$G\setminus\{0,-a\}\subset S\cup(S-a),$$

where

$$S := \{x\in G\setminus\{0\} : \quad f(x)=0\}.$$

Proof. Let $f : G\setminus\{0\}\longrightarrow\mathbb{R}$ be a solution of the equation (E). On setting $y=a$ in (E) we have

$$f(x)=0 \quad \text{or} \quad f(x+a)=0, \quad x\neq -a, x\neq 0$$

which was to be shown. □

In the light of Lemma, we consider such solutions of (E) which are defined on $G\setminus\{0\}$ and have 0 off their ranges.

Theorem 2. *Let* $(G,+)$ *be a group. A function* $f : G\setminus\{0\}\longrightarrow\mathbb{R}\setminus\{0\}$ *yields a non-constant solution to the equation* (E) *if and only if there exist a homomorphism* $A : G\longrightarrow\mathbb{R}$ *such that* $0\notin A(G\setminus\{0\})$ *and*

$$f(x)=\frac{1}{A(x)}, \quad x\in G\setminus\{0\}.$$

Proof. Let $f : G\setminus\{0\}\longrightarrow\mathbb{R}\setminus\{0\}$ fulfils the equation (E). Then

$$f(x)+f(y)\neq 0 \quad \text{implies} \quad \frac{1}{f(x+y)}=\frac{f(x)+f(y)}{f(x)f(y)}=\frac{1}{f(x)}+\frac{1}{f(y)}$$

for $x,y,x+y\neq 0$. This states that a function $g : G\longrightarrow\mathbb{R}$ of the form

$$g(x) := \begin{cases} \frac{1}{f(x)} & \text{for } x\in G\setminus\{0\} \\ \textit{arbitrary} & \text{for } x=0. \end{cases}$$

yields a solution of the equation

$$g(x)+g(y)\neq 0 \quad \text{implies} \quad g(x+y)=g(x)+g(y) \quad x,y,x+y\neq 0.$$

We omit here a simple calculation showing that $f(x)+f(y)=0$ if and only if $g(x)+g(y)=0$, for all $x,y\in G\setminus\{0\}$.

From the theorems proved by R. Ger [5], J. G. Dhombres, R. Ger [4] we conclude that there exist a homomorphism $A : G \longrightarrow \mathbb{R}$ groups $(G, +)$ and $(\mathbb{R}, +)$ such that

$$g(x) = A(x), \quad x \in G \setminus \{0\}.$$

Clearly, $0 \notin A(G \setminus \{0\})$ and

$$\frac{1}{f(x)} = A(x) \quad \text{for} \quad x \neq 0,$$

whence

$$f(x) = \frac{1}{A(x)}, \quad x \neq 0.$$

It is easy to check that the above formula establishes a solution of the equation (E). Thus the proof has been completed. □

Remark. *Continuous solutions of the equation* (E) *are inverse proportions.*

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