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EXISTENCE OF POSITIVE CONTINUOUS WEAK SOLUTIONS FOR SOME SEMILINEAR ELLIPTIC EIGENVALUE PROBLEMS

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Abstract. Let D be a bounded $C^{1,1}$-domain in $\mathbb{R}^d$, $d \geq 2$. The aim of this article is twofold. The first goal is to give a new characterization of the Kato class of functions $K(D)$ that was defined by N. Zeddini for $d = 2$ and by H. Mâagli and M. Zribi for $d \geq 3$ and adapted to study some nonlinear elliptic problems in D. The second goal is to prove the existence of positive continuous weak solutions, having the global behavior of the associated homogeneous problem, for sufficiently small values of the nonnegative constants λ and μ to the following system $\Delta u = \lambda f(x, u, v)$, $\Delta v = \mu g(x, u, v)$ in D, $u = \phi_1$ and $v = \phi_2$ on ∂D, where ϕ_1 and ϕ_2 are nontrivial nonnegative continuous functions on ∂D. The functions f and g are nonnegative and belong to a class of functions containing in particular all functions of the type $f(x, u, v) = p(x)u^\alpha h_1(v)$ and $g(x, u, v) = q(x)h_2(u)v^\beta$ with $\alpha \geq 1$, $\beta \geq 1$, h_1, h_2 are continuous on $[0, \infty)$ and p, q are nonnegative functions in $K(D)$.

Keywords: Green function, Kato class, nonlinear elliptic systems, positive solution, maximum principle, Schauder fixed point theorem.

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1. INTRODUCTION

Let D be a bounded $C^{1,1}$-domain of $\mathbb{R}^d$ ($d \geq 2$). In this paper, we study the existence of positive continuous solutions of the following semilinear elliptic system

$$\begin{cases} \Delta u = \lambda\, f(\cdot, u, v) & \text{in } D \quad \text{(in the sense of distributions)}, \\ \Delta v = \mu\, g(\cdot, u, v) & \text{in } D \quad \text{(in the sense of distributions)}, \\ u = \phi_1 \text{ and } v = \phi_2 & \text{on } \partial D, \end{cases} \tag{1.1}$$

where ϕ_1, ϕ_2 are two nontrivial nonnegative continuous functions on the boundary ∂D, $\lambda \geq 0$, $\mu \geq 0$ and f, g are two nontrivial nonnegative functions defined on $D \times [0,\infty) \times [0,\infty)$. This problem was investigated, recently, in particular cases of nonlinearities f, g by many authors (see for example [2,6,11,19] and the references therein). In [11], the authors considered the particular case where $f(x,u,v) = p(x)g_1(v)$ and $g(x,u,v) = q(x)f_2(u)$, where f_2, g_1 are nonnegative continuous functions that are both nondecreasing or both non-increasing and p, q are nonnegative measurable functions belonging to the Kato class $K(D)$ introduced and studied in [20] for $d = 2$ and in [15] for $d \geq 3$. Under some conditions on ϕ_1 and ϕ_2, the existence of positive continuous solutions having the global behavior of the associated homogeneous system is established. System (1.1) has been also studied in [2] for the particular cases $f(x,u,v) = p(x)u^\alpha v^r$ and $g(x,u,v) = q(x)u^s v^\beta$, where $\alpha \geq 1$, $\beta \geq 1$, $r \geq 0$, $s \geq 0$ and p, q are two nonnegative measurable functions that belong to the class $K(D)$. In [19], the author considered the case where the nonnegative nonlinearities f and g are both nondecreasing with respect to the second and the third variables or both non increasing with respect to the second and the third variables and such that for each c_1, $c_2 > 0$ the functions $f(\cdot, c_1, c_2)$ and $g(\cdot, c_1, c_2)$ are in the class $K(D)$. Under a condition of positivity of two constants defined by means of f, g, ϕ_1, ϕ_2 and exploiting the monotonicity assumption of f and g, the author extends the results of [11] by proving the existence of positive continuous solutions for (1.1). This also was done by investigating the properties of the Kato class $K(D)$.

Our aim in this paper is twofold. The first goal is to give a new characterization of the Kato class $K(D)$ as it will be stated in Theorem 2.2 in the sequel. This explains in a certain manner the optimality of the 3G-inequality (2.4), satisfied by the Green function and established in [13] and [17]. The second goal is to extend the results of [2,11,19] to a class of nonlinearities f and g including in particular those where f is nondecreasing with respect to u but not necessarily monotone with respect to v and g is nondecreasing with respect to v but not necessarily monotone with respect to u. The proof will differs from those in [11] and [19]. Namely, we will establish and exploit an existence result of a positive continuous solution for the problem

$$\begin{cases} \Delta u = \lambda f(x,u) & \text{in } D \quad \text{(in the sense of distributions)}, \\ u = \phi & \text{on } \partial D, \end{cases} \tag{1.2}$$

where $\lambda \geq 0$, ϕ is a nontrivial nonnegative continuous function on ∂D and the function f belongs to a class of functions containing in particular those of the form $p(x)u^\alpha$ with $\alpha \geq 1$ and this will be an extension of the results of [18] established, for $d \geq 3$, in the case where the variables are separated and $f(x,u) = p(x)h(u)$.

We note that different types of weak solutions for problems that are more general than (1.2) can be defined and existence results for these solutions can established by different methods such as variational methods or topological methods (see [12,16]), or via global invertibility (see [4]). These problems are also treated in the case of the lack of compactness (see [10]). Here we restrict ourselves to the case of continuous weak solutions combining topological and approximation methods and potential theory tools.

Our paper is organized as follows. Section 2 is devoted to give a new characterization of the Kato class $K(D)$ and to recall some properties of this class that will be used in the study of (1.2) and (1.1). In Section 3, we prove the existence of a positive continuous solution for (1.2). The last section is devoted to the study of the existence of a positive continuous solutions for the system (1.1).

Next, we give some notations that will be used in the sequel. We denote by $B(D)$ the set of all Borel measurable functions in D, by $B^+(D)$ the set of nonnegative ones and by $B_b(D)$ the set of bounded ones. We denote also by $C_0(D)$ the set of continuous functions in D having limit zero at the boundary ∂D and by $C(\overline{D})$ be the set of all functions in $B(D)$ that are continuous in $\overline{D}$. Let G^D be the Green function of the Laplace operator in D with Dirichlet boundary conditions. For any $p \in B^+(D)$, we denote by ${}^D Gp$ the Green potential of p defined on D by

$$ {}^D Gp(x) = \int_D G^D(x,y)p(y)\,dy $$

and we recall that if $p \in L^1_{loc}(D)$ and ${}^D Gp \in L^1_{loc}(D)$, then we have in the sense of distributions (see [8, p. 52])

$$ \Delta({}^D Gp) = -p \quad \text{in } D. \tag{1.3} $$

For any nonnegative continuous function ϕ on ∂D, we denote by $H_D\phi$ the unique solution $u \in C^2(D) \cap C(\overline{D})$ of the Dirichlet problem

$$ \begin{cases} \Delta u = 0 & \text{in } D, \\ u = \phi & \text{on } \partial D. \end{cases} $$

Let $(X_t)_{t\geq 0}$ be the canonical Brownian motion defined on $C\left([0,\infty);\mathbb{R}^d\right)$, P^x be the probability measure on the Brownian continuous paths starting at x and τ_D be the exist time of D. For any $q \in B^+(D)$, we define (see [7] or [8, p. 84]), the subordinate Green kernel ${}^D G_q$ by

$$ {}^D G_q(p)(x) = \frac{1}{2}\, E^x \left(\int_0^{\tau_D} e^{-\frac{1}{2}\int_0^t q(X_s)ds} p(X_t)\,dt \right) \quad \text{for } p \in B(D), \tag{1.4} $$

where E^x is the expectation on P^x. Moreover, for $q \in B^+(D)$ such that ${}^D Gq < \infty$ we have (see [5, 8, 14]) the resolvent equation

$$ {}^D G = {}^D G_q + {}^D G_q \left(q\, {}^D G \right). \tag{1.5} $$

So for each $u \in B(D)$ such that ${}^D G(q|u|) < \infty$, we have

$$ \left[I + {}^D G(q\,\cdot)\right]\left[I - {}^D G_q(q\,\cdot)\right] u = \left[I - {}^D G_q(q\,\cdot)\right][I + {}^D G(q\,\cdot)]u = u \tag{1.6} $$

and for every $u \in B^+(D)$ we have

$$ 0 \leq {}^D G_q(u) \leq {}^D G(u). \tag{1.7} $$

We close this section by adopting the following notation. If S is a nonempty set and f, g are two nonnegative functions defined on S, we write $f \sim g$ if there exist a positive constant C such that $\frac{1}{C} f(x) \leq g(x) \leq C f(x)$ for every $x \in S$. We note also that as long of this paper the positive constant C may vary from line to line.

2. THE KATO CLASS OF FUNCTIONS

In [21], Zhao have established the following important estimates and inequalities for the Green function G^D of a $C^{1,1}$-bounded domain D. Let $\rho_0(x) = dis(x, \partial D)$ be the Euclidean distance from x to ∂D. Then Zhao proved, for $d \geq 3$, that there exists a positive constant C such that for each $x, y, z \in D$.

$$\frac{\rho_0(y)}{\rho_0(x)} G^D(x,y) \leq \frac{C}{\|x-y\|^{d-2}}, \tag{2.1}$$

$$G^D(x,y) \sim \frac{1}{\|x-y\|^{d-2}} \min\left(1, \frac{\rho_0(x)\rho_0(y)}{\|x-y\|^2}\right) \tag{2.2}$$

and

$$\frac{G^D(x,z)G^D(z,y)}{G^D(x,y)} \leq C\left[\frac{1}{\|x-z\|^{d-2}} + \frac{1}{\|y-z\|^{d-2}}\right]. \tag{2.3}$$

Inequality (2.3) has been improved by Kalton and Verbitsky in [13] for $d \geq 3$ and by Selmi in [17] for $d = 2$. More precisely, they proved that there exists a positive constant C_0 such that

$$\frac{G^D(x,z)G^D(z,y)}{G^D(x,y)} \leq C_0\left[\frac{\rho_0(z)}{\rho_0(x)} G^D(x,z) + \frac{\rho_0(z)}{\rho_0(y)} G^D(y,z)\right]. \tag{2.4}$$

This was exploited by Zeddini in [20] for $d = 2$ and by Mâagli and Zribi in [15] for $d \geq 3$ to define a new Kato class on the bounded domain D which has been adapted to study some semilinear elliptic boundary value problems using some potential theory tools. More precisely, this class was defined as follows.

Definition 2.1 ([15,20])**.** A measurable function q belongs to the Kato class $K(D)$ if q satisfies the following condition

$$\lim_{r\to 0}\left(\sup_{x\in D} \int_{D\cap B(x,r)} \frac{\rho_0(z)}{\rho_0(x)} G^D(x,z)|q(z)|dz\right) = 0. \tag{2.5}$$

Our main goal in this section is to give a new characterization of this class of functions by means of the left hand side term of inequality (2.4). This was motivated by giving an answer to the question of comparing the Kato class defined by means of (2.5) with an eventual new class $K_0(D)$ that may be defined by (2.6) below, after noticing that the properties satisfied by functions in $K(D)$ are also satisfied by those in $K_0(D)$. Thus we prove here, in a nontrivial manner, the equality between these two classes.

Theorem 2.2. *Let q be a Borel measurable function in D. Then $q \in K(D)$ if and only if*

$$\lim_{r\to 0}\left(\sup_{(x,y)\in D\times D}\int_{D\cap(B(x,r)\cup B(y,r))}\frac{G^D(x,z)G^D(z,y)}{G^D(x,y)}|q(z)|dz\right)=0. \tag{2.6}$$

To prove this theorem we need to recall and establish some preliminary results.

Proposition 2.3 ([15,17])**.** *For $x, y \in D$ we have*

$$G^D(x,y)\sim\begin{cases} Log\left(1+\frac{\rho_0(x)\rho_0(y)}{\|x-y\|^2}\right) & if\, d=2,\\ \frac{\rho_0(x)\rho_0(y)}{\|x-y\|^{d-2}(\|x-y\|^2+\rho_0(x)\rho_0(y))} & if\, d\geq 3,\end{cases} \tag{2.7}$$

and

$$\rho_0(x)\rho_0(y)\leq C\,G^D(x,y). \tag{2.8}$$

The following lemma will be also used.

Lemma 2.4. *Let $x, y \in D$. Then we have the following properties:*

(1) *If $\rho_0(x)\rho_0(y) \leq \|x-y\|^2$, then*

$$\max(\rho_0(x),\rho_0(y))\leq\frac{1+\sqrt{5}}{2}\|x-y\|.$$

(2) *If $\|x-y\|^2 \leq \rho_0(x)\rho_0(y)$, then for every $z \in D^c$ we have*

$$\frac{3-\sqrt{5}}{2}\|y-z\|\leq\|x-z\|\leq\frac{3+\sqrt{5}}{2}\|y-z\|.$$

In particular, we have

$$\frac{3-\sqrt{5}}{2}\,\rho_0(y)\leq\rho_0(x)\leq\frac{3+\sqrt{5}}{2}\,\rho_0(y).$$

(3) $\frac{1}{2}\left(\|x-y\|^2+\rho_0^2(x)+\rho_0^2(y)\right)\leq\|x-y\|^2+\rho_0(x)\rho_0(y)\leq\|x-y\|^2+\rho_0^2(x)+\rho_0^2(y).$

Proof. (1) and (2) were proved in [3].

(3) Squaring the well known inequality $|\rho_0(x)-\rho_0(y)|\leq\|x-y\|$ we obtain

$$\rho_0^2(x)+\rho_0^2(y)\leq\|x-y\|^2+2\rho_0(x)\rho_0(y).$$

This together with the fact that $ab \leq a^2 + b^2$ gives

$$\begin{aligned}|x-y\|^2+\rho_0^2(x)+\rho_0^2(y)&\leq 2\left[\|x-y\|^2+\rho_0(x)\rho_0(y)\right]\\&\leq 2\left[\|x-y\|^2+\rho_0^2(x)+\rho_0^2(y)\right].\end{aligned}$$

This completes the proof. □

The following result is the key of the proof of the new characterization of the class $K(D)$.

Proposition 2.5. *There exists a constant $C > 0$ such that for all $r > 0$ and all $x, y \in D$ we have*

$$\int\limits_{D\cap(B(x,r)\cup B(y,r))} \frac{G^D(x,z)G^D(z,y)}{G^D(x,y)}|q(z)|dz \leq C \int\limits_{D\cap B(x,3r)} \frac{\rho_0(z)}{\rho_0(x)} G^D(x,z)|q(z)|\,dz + C \int\limits_{D\cap B(y,3r)} \frac{\rho_0(z)}{\rho_0(y)} G^D(y,z)|q(z)|\,dz.$$

Proof. Let $r > 0$ and $x, y \in D$. Then we have

$$\begin{aligned}
&\int\limits_{D\cap(B(x,r)\cup B(y,r))} \frac{G^D(x,z)G^D(z,y)}{G^D(x,y)}|q(z)|dz \\
&= \int\limits_{D\cap B(x,r)\cap B(y,r)} \frac{G^D(x,z)G^D(z,y)}{G^D(x,y)}|q(z)|dz \\
&\quad + \int\limits_{D\cap B(x,r)\cap B^c(y,r)} \frac{G^D(x,z)G^D(z,y)}{G^D(x,y)}|q(z)|dz \\
&\quad + \int\limits_{D\cap B(y,r)\cap B^c(x,r)} \frac{G^D(x,z)G^D(z,y)}{G^D(x,y)}|q(z)|dz \\
&= I_1(x,y) + I_2(x,y) + I_3(x,y).
\end{aligned}$$

Using the inequality (2.4), we obtain

$$\begin{aligned}
I_1(x,y) := &\int\limits_{D\cap B(x,r)\cap B(y,r)} \frac{G^D(x,z)G^D(z,y)}{G^D(x,y)}|q(z)|dz \\
\leq C_0 &\int\limits_{D\cap B(x,r)\cap B(y,r)} \left[\frac{\rho_0(z)}{\rho_0(x)} G^D(x,z) + \frac{\rho_0(z)}{\rho_0(y)} G^D(y,z)\right]|q(z)|dz \\
\leq C_0 &\int\limits_{D\cap B(x,r)} \frac{\rho_0(z)}{\rho_0(x)} G^D(x,z)|q(z)|dz \\
&+ C_0 \int\limits_{D\cap B(y,r)} \frac{\rho_0(z)}{\rho_0(y)} G^D(y,z)|q(z)|dz.
\end{aligned}$$

Next, we estimate $I_2(x,y)$ and $I_3(x,y)$. To this aim we will discuss two cases:
Case 1. $B(x,r) \cap B(y,r) \neq \emptyset$.
Choose $z_0 \in B(x,r) \cap B(y,r)$. Then for every $z \in B(x,r) \cap B^c(y,r)$ we have

$$\|z - y\| \leq \|z - x\| + \|x - z_0\| + \|z_0 - y\| \leq 3r.$$

Similarly for every $z \in B(y,r) \cap B^c(x,r)$ we have

$$\|z-x\| \le \|z-y\| + \|y-z_0\| + \|z_0-x\| \le 3r.$$

Hence

$$B(x,r) \cap B^c(y,r) \subset B(x,r) \cap B(y,3r)$$

and

$$B(y,r) \cap B^c(x,r) \subset B(y,r) \cap B(x,3r).$$

So we obtain

$$\begin{aligned}
I_2(x,y) &:= \int\limits_{D\cap B(x,r)\cap B^c(y,r)} \frac{G^D(x,z)G^D(z,y)}{G^D(x,y)}|q(z)|dz \\
&\le \int\limits_{D\cap B(x,r)\cap B(y,3r)} \frac{G^D(x,z)G^D(z,y)}{G^D(x,y)}|q(z)|dz \\
&\le C_0 \int\limits_{D\cap B(x,r)\cap B(y,3r)} \left[\frac{\rho_0(z)}{\rho_0(x)}G^D(x,z) + \frac{\rho_0(z)}{\rho_0(y)}G^D(y,z)\right]|q(z)|dz \\
&\le C_0 \int\limits_{D\cap B(x,r)} \frac{\rho_0(z)}{\rho_0(x)}G^D(x,z)|q(z)|dz \\
&\quad + C_0 \int\limits_{D\cap B(y,3r)} \frac{\rho_0(z)}{\rho_0(y)}G^D(y,z)|q(z)|dz
\end{aligned}$$

and

$$\begin{aligned}
I_3(x,y) &:= \int\limits_{D\cap B(y,r)\cap B^c(x,r)} \frac{G^D(x,z)G^D(z,y)}{G^D(x,y)}|q(z)|dz \\
&\le \int\limits_{D\cap B(y,r)\cap B(x,3r)} \frac{G^D(x,z)G^D(z,y)}{G^D(x,y)}|q(z)|dz \\
&\le C_0 \int\limits_{D\cap B(y,r)\cap B(x,3r)} \left[\frac{\rho_0(z)}{\rho_0(x)}G^D(x,z) + \frac{\rho_0(z)}{\rho_0(y)}G^D(y,z)\right]|q(z)|dz \\
&\le C_0 \int\limits_{D\cap B(x,3r)} \frac{\rho_0(z)}{\rho_0(x)}G^D(x,z)|q(z)|dz \\
&\quad + C_0 \int\limits_{D\cap B(y,r)} \frac{\rho_0(z)}{\rho_0(y)}G^D(y,z)|q(z)|dz.
\end{aligned}$$

Case 2. $B(x,r) \cap B(y,r) = \emptyset$.
In this case $B(x,r) \subset B^c(y,r)$ and $B(y,r) \subset B^c(x,r)$. For every $z \in B(x,r)$ we have

$$\|y-z\| \le \|y-x\| + \|x-z\| \le \|y-x\| + r \le 2\|x-y\|$$

and

$$\|x-y\| \le \|x-z\| + \|y-z\| \le r + \|y-z\| \le 2\|y-z\|.$$

So, in this case

$$\frac{1}{2}\|y-z\| \le \|x-y\| \le 2\|y-z\|. \tag{2.9}$$

Similarly for every $z \in B(y,r)$ we have

$$\|x-z\| \le \|x-y\| + \|y-z\| \le \|x-y\| + r \le 2\|x-y\|$$

and

$$\|x-y\| \le \|x-z\| + \|y-z\| \le \|x-z\| + r \le 2\|x-z\|.$$

Also, in this case

$$\frac{1}{2}\|x-z\| \le \|x-y\| \le 2\|x-z\|. \tag{2.10}$$

Now, using (2.7) we obtain

$$\frac{G^D(x,z)G^D(z,y)}{G^D(x,y)} \sim \begin{cases} \dfrac{Log\left(1+\dfrac{\rho_0(y)\rho_0(z)}{\|z-y\|^2}\right)}{Log\left(1+\frac{\rho_0(x)\rho_0(y)}{\|x-y\|^2}\right)}\, G^D(x,z) & \text{if } d=2, \\ \dfrac{\|x-y\|^{d-2}}{\|z-y\|^{d-2}} \dfrac{(\|x-y\|^2+\rho_0(x)\rho_0(y))}{(\|z-y\|^2+\rho_0(z)\rho_0(y))} \dfrac{\rho_0(z)}{\rho_0(x)}\, G^D(x,z) & \text{if } d\ge 3, \end{cases}$$

and

$$\frac{G^D(x,z)G^D(z,y)}{G^D(x,y)} \sim \begin{cases} \dfrac{Log\left(1+\frac{\rho_0(x)\rho_0(z)}{\|z-x\|^2}\right)}{Log\left(1+\frac{\rho_0(x)\rho_0(y)}{\|x-y\|^2}\right)}\, G^D(y,z) & \text{if } d=2 \\ \dfrac{\|x-y\|^{d-2}}{\|z-x\|^{d-2}} \dfrac{(\|x-y\|^2+\rho_0(x)\rho_0(y))}{(\|z-x\|^2+\rho_0(z)\rho_0(x))} \dfrac{\rho_0(z)}{\rho_0(y)}\, G^D(y,z) & \text{if } d\ge 3. \end{cases}$$

So we will discuss two subcases.

Subcase 1. If $\rho_0(x)\rho_0(y) \le \|x-y\|^2$.

In this case we have $\|x-y\|^2 + \rho_0(x)\rho_0(y) \le 2\|x-y\|^2$. So for $d \ge 3$, we use this fact and (2.9) to obtain

$$\begin{aligned} \frac{\|x-y\|^{d-2}}{\|z-y\|^{d-2}} \frac{(\|x-y\|^2+\rho_0(x)\rho_0(y))}{(\|z-y\|^2+\rho_0(z)\rho_0(y))} &\le \frac{\|x-y\|^{d-2}(\|x-y\|^2+\rho_0(x)\rho_0(y))}{\|z-y\|^d} \\ &\le 2\frac{\|x-y\|^d}{\|z-y\|^d} \le 2^{d+1}. \end{aligned}$$

And for $d=2$, we use (2.9) and the inequalities $\frac{1}{2}\,t \le Log(1+t)$ for $t \in [0,1]$ and $Log(1+t) \le t$ for $t \ge 0$ to obtain

$$\frac{Log\left(1+\frac{\rho_0(y)\rho_0(z)}{\|z-y\|^2}\right)}{Log\left(1+\frac{\rho_0(x)\rho_0(y)}{\|x-y\|^2}\right)} \le 2\,\frac{\|x-y\|^2}{\rho_0(x)\rho_0(y)}\,\frac{\rho_0(y)\rho_0(z)}{\|z-y\|^2} \le 8\,\frac{\rho_0(z)}{\rho_0(x)}.$$

Consequently, for every $z \in B(x,r)$ we have

$$\frac{G^D(x,z)G^D(z,y)}{G^D(x,y)} \le C\,\frac{\rho_0(z)}{\rho_0(x)}G^D(x,z)$$

and

$$I_2(x,y) = \int\limits_{D\cap B(x,r)} \frac{G^D(x,z)G^D(z,y)}{G^D(x,y)}|q(z)|dz \le C \int\limits_{D\cap B(x,r)} \frac{\rho_0(z)}{\rho_0(x)}G^D(x,z)|q(z)|dz.$$

Similarly for every $z \in B(y,r)$ we obtain by using (2.10) that

$$\frac{G^D(x,z)G^D(z,y)}{G^D(x,y)} \le C\,\frac{\rho_0(z)}{\rho_0(y)}G^D(y,z)$$

and

$$I_3(x,y) = \int\limits_{D\cap B(y,r)} \frac{G^D(x,z)G^D(z,y)}{G^D(x,y)}|q(z)|dz \le C \int\limits_{D\cap B(y,r)} \frac{\rho_0(z)}{\rho_0(y)}G^D(y,z)|q(z)|dz.$$

Subcase 2. If $\|x-y\|^2 \le \rho_0(x)\rho_0(y)$.
In this case we obtain from Lemma 2.4 that

$$\frac{3-\sqrt{5}}{2}\rho_0(y) \le \rho_0(x) \le \frac{3+\sqrt{5}}{2}\rho_0(y), \tag{2.11}$$

and we will treat the cases $d \ge 3$ and $d = 2$ separately. If $d \ge 3$, then we deduce from (2.11), (2.9) and property (3) of Lemma 2.4 that for every $z \in B(x,r)$ we have

$$\begin{aligned}
\frac{\|x-y\|^{d-2}}{\|z-y\|^{d-2}}\frac{(\|x-y\|^2+\rho_0(x)\rho_0(y))}{(\|z-y\|^2+\rho_0(z)\rho_0(y))} &\le 2^{d-2}\,\frac{\|x-y\|^2+\rho_0(x)\rho_0(y)}{\|z-y\|^2+\rho_0(z)\rho_0(y)}\\
&\le 2^d\,\frac{\|x-y\|^2+\rho_0^2(x)+\rho_0^2(y)}{\|z-y\|^2+\rho_0^2(z)+\rho_0^2(y)}\\
&\le 2^d\,\frac{\left(1+\left(\frac{3+\sqrt{5}}{2}\right)^2\right)\left(\|x-y\|^2+\rho_0^2(y)\right)}{\|z-y\|^2+\rho_0^2(z)+\rho_0^2(y)}\\
&\le 2^d\left(\frac{9+3\sqrt{5}}{2}\right)\frac{\|x-y\|^2+\rho_0^2(y)}{\|z-y\|^2+\rho_0^2(y)}\\
&\le 2^{d+1}\Big(9+3\sqrt{5}\Big).
\end{aligned}$$

Consequently, for every $z \in B(x,r)$ we have

$$\frac{G^D(x,z)G^D(z,y)}{G^D(x,y)} \le C\,\frac{\rho_0(z)}{\rho_0(x)}G^D(x,z)$$

and

$$I_2(x,y) = \int\limits_{D\cap B(x,r)} \frac{G^D(x,z)G^D(z,y)}{G^D(x,y)}|q(z)|dz$$
$$\leq C \int\limits_{D\cap B(x,r)} \frac{\rho_0(z)}{\rho_0(x)} G^D(x,z)|q(z)|dz.$$

Similarly, for $z \in B(y,r)$ we use (2.10) and similar arguments as above to obtain

$$\frac{G^D(x,z)G^D(z,y)}{G^D(x,y)} \leq C \, \frac{\rho_0(z)}{\rho_0(y)} G^D(y,z)$$

and

$$I_3(x,y) = \int\limits_{D\cap B(y,r)} \frac{G^D(x,z)G^D(z,y)}{G^D(x,y)}|q(z)|dz$$
$$\leq C \int\limits_{D\cap B(y,r)} \frac{\rho_0(z)}{\rho_0(y)} G^D(y,z)|q(z)|dz.$$

Finally, for $d = 2$ we will discuss two subcases.

(i) If $\|x-z\|^2 \leq \rho_0(x)\rho_0(z)$ or $\|y-z\|^2 \leq \rho_0(y)\rho_0(z)$. Then taking into account (2.11) and using Lemma 2.4 we obtain in this case that

$$\frac{3-\sqrt{5}}{2}\,\rho_0(x) \leq \rho_0(z) \leq \frac{3+\sqrt{5}}{2}\,\rho_0(x)$$
$$\text{and} \qquad \Big(\frac{3-\sqrt{5}}{2}\Big)^2 \rho_0(y) \leq \rho_0(z) \leq \Big(\frac{3+\sqrt{5}}{2}\Big)^2 \rho_0(y)\,,$$

or

$$\frac{3-\sqrt{5}}{2}\,\rho_0(y) \leq \rho_0(z) \leq \frac{3+\sqrt{5}}{2}\,\rho_0(y)$$
$$\text{and} \qquad \Big(\frac{3-\sqrt{5}}{2}\Big)^2 \rho_0(x) \leq \rho_0(z) \leq \Big(\frac{3+\sqrt{5}}{2}\Big)^2 \rho_0(x).$$

Using these facts, (2.9) and the fact that for $\alpha > 0$ and $t \geq 0$ we have

$$\min(1,\alpha)\, Log(1+t) \leq Log(1+\alpha t) \leq \max(1,\alpha)\, Log(1+t),$$

we obtain for $z \in B(x,r)$ that

$$\begin{aligned}\frac{Log\left(1+\frac{\rho_0(y)\rho_0(z)}{\|z-y\|^2}\right)}{Log\left(1+\frac{\rho_0(x)\rho_0(y)}{\|x-y\|^2}\right)} &\leq \frac{Log\left(1+\left(\frac{3+\sqrt{5}}{2}\right)\frac{\rho_0(x)\rho_0(z)}{\|z-y\|^2}\right)}{Log\left(1+\left(\frac{3-\sqrt{5}}{2}\right)^2\frac{\rho_0(x)\rho_0(z)}{4\,\|z-y\|^2}\right)} \\ &\leq \left(\frac{3+\sqrt{5}}{2}\right)\frac{16}{\left(3-\sqrt{5}\right)^2}\frac{Log\left(1+\frac{\rho_0(x)\rho_0(z)}{\|z-y\|^2}\right)}{Log\left(1+\frac{\rho_0(x)\rho_0(z)}{\|z-y\|^2}\right)} \\ &\leq \left(3+\sqrt{5}\right)^3 \\ &\leq \left(3+\sqrt{5}\right)^3\left(\frac{3+\sqrt{5}}{2}\right)^2\frac{\rho_0(z)}{\rho_0(x)}.\end{aligned}$$

Hence for every $z \in B(x,r)$ we obtain

$$\frac{G^D(x,z)G^D(z,y)}{G^D(x,y)} \leq C\,\frac{\rho_0(z)}{\rho_0(x)}G^D(x,z)$$

and

$$I_2(x,y) = \int\limits_{D\cap B(x,r)} \frac{G^D(x,z)G^D(z,y)}{G^D(x,y)}|q(z)|dz \leq C \int\limits_{D\cap B(x,r)} \frac{\rho_0(z)}{\rho_0(x)}G^D(x,z)|q(z)|dz.$$

Similarly, for $z \in B(y,r)$ we use (2.10) and similar arguments as above to obtain

$$\frac{G^D(x,z)G^D(z,y)}{G^D(x,y)} \leq C\,\frac{\rho_0(z)}{\rho_0(y)}G^D(y,z)$$

and

$$I_3(x,y) = \int\limits_{D\cap B(y,r)} \frac{G^D(x,z)G^D(z,y)}{G^D(x,y)}|q(z)|dz \leq C \int\limits_{D\cap B(y,r)} \frac{\rho_0(z)}{\rho_0(y)}G^D(y,z)|q(z)|dz.$$

(ii) If $\|x-z\|^2 \geq \rho_0(x)\rho_0(z)$ and $\|y-z\|^2 \geq \rho_0(y)\rho_0(z)$, then in this case we have $\max(\rho_0(x),\rho_0(z)) \leq \|x-z\|$ and $\max(\rho_0(y),\rho_0(z)) \leq \|y-z\|$. Hence it follows from the inequalities $\frac{t}{1+t} \leq Log(1+t) \leq t$ for $t \geq 0$, that

$$\frac{\rho_0(x)\rho_0(y)}{\|x-y\|^2+\rho_0(x)\rho_0(y)} \leq Log\left(1+\frac{\rho_0(x)\rho_0(y)}{\|x-y\|^2}\right).$$

Hence

$$\frac{Log\left(1+\frac{\rho_0(y)\rho_0(z)}{\|z-y\|^2}\right)}{Log\left(1+\frac{\rho_0(x)\rho_0(y)}{\|x-y\|^2}\right)} \le \frac{\|x-y\|^2+\rho_0(x)\rho_0(y)}{\|y-z\|^2}\ \frac{\rho_0(z)}{\rho_0(x)}$$

$$\le \frac{\|x-y\|^2+\left(\frac{3+\sqrt{5}}{2}\right)(\rho_0(y))^2}{\|y-z\|^2}\ \frac{\rho_0(z)}{\rho_0(x)}$$

$$\le \left(\frac{3+\sqrt{5}}{2}\right)\frac{\|x-y\|^2+(\rho_0(y))^2}{\|y-z\|^2}\ \frac{\rho_0(z)}{\rho_0(x)}$$

$$\le \left(\frac{3+\sqrt{5}}{2}\right)\frac{\|x-y\|^2+\|y-z\|^2}{\|y-z\|^2}\ \frac{\rho_0(z)}{\rho_0(x)}$$

and similarly

$$\frac{Log\left(1+\frac{\rho_0(x)\rho_0(z)}{\|x-z\|^2}\right)}{Log\left(1+\frac{\rho_0(x)\rho_0(y)}{\|x-y\|^2}\right)} \le \left(\frac{3+\sqrt{5}}{2}\right)\frac{\|x-y\|^2+\|x-z\|^2}{\|x-z\|^2}\ \frac{\rho_0(z)}{\rho_0(y)}.$$

So, using (2.9) we obtain for $z\in B(x,r)$ we get

$$\frac{G^D(x,z)G^D(z,y)}{G^D(x,y)} \le C\,\frac{\rho_0(z)}{\rho_0(x)}G^D(x,z)$$

and

$$I_2(x,y)=\int_{D\cap B(x,r)}\frac{G^D(x,z)G^D(z,y)}{G^D(x,y)}|q(z)|dz \le C\int_{D\cap B(x,r)}\frac{\rho_0(z)}{\rho_0(x)}G^D(x,z)|q(z)|dz.$$

Similarly, for $z\in B(y,r)$ we use (2.10) and similar arguments as above to obtain

$$\frac{G^D(x,z)G^D(z,y)}{G^D(x,y)} \le C\,\frac{\rho_0(z)}{\rho_0(y)}G^D(y,z)$$

and

$$I_3(x,y)=\int_{D\cap B(y,r)}\frac{G^D(x,z)G^D(z,y)}{G^D(x,y)}|q(z)|dz \le C\int_{D\cap B(y,r)}\frac{\rho_0(z)}{\rho_0(y)}G^D(y,z)|q(z)|dz.$$

This completes the proof of the proposition. □

Proof of Theorem 2.2. Assume that $q\in K(D)$. Then by Proposition 2.5, we deduce that (2.6) is satisfied. To prove the converse, we deduce from (2.2), (2.7) and (2.8) that

$$\frac{G^D(x,z)G^D(z,y)}{G^D(x,y)}|q(z)| = \frac{G^D(z,y)}{G^D(x,y)}\,G^D(x,z)|q(z)|$$

$$\ge C\|x-y\|^d\ \frac{\rho_0(z)}{\rho_0(x)}\,G^D(x,z)|q(z)|.$$

Let $r > 0$ and $x, y \in D$. Then

$$\begin{aligned}
&C\|x-y\|^d \int\limits_{D\cap B(x,r)} \frac{\rho_0(z)}{\rho_0(x)} G^D(x,z)|q(z)|dz \\
&\leq \int\limits_{D\cap(B(x,r)\cup B(y,r))} \frac{G^D(x,z)G^D(z,y)}{G^D(x,y)}|q(z)|dz \\
&\leq \sup_{(\xi,\zeta)\in D\times D} \int\limits_{D\cap(B(\xi,r)\cup B(\zeta,r))} \frac{G^D(\xi,z)G^D(z,\zeta)}{G^D(\xi,\zeta)}|q(z)|dz.
\end{aligned}$$

Let L be the diameter of D. Then for every $x \in D$, there exists $y = y_x \in D$ such that $\|x - y_x\| \geq \frac{L}{4}$. If this is not true, then there exists $x_0 \in D$ such that for every $y \in D$ we have $\|x_0 - y\| \leq \frac{L}{4}$. So we obtain $D \subset B(x_0, \frac{L}{4})$, which gives a contradiction with the definition of the diameter L. Using this fact we deduce that for every $r > 0$ and $x \in D$ we have

$$\begin{aligned}
&C\left(\frac{L}{4}\right)^d \int\limits_{D\cap B(x,r)} \frac{\rho_0(z)}{\rho_0(x)} G^D(x,z)|q(z)|dz \\
&\leq \sup_{(\xi,\zeta)\in D\times D} \int\limits_{D\cap(B(\xi,r)\cup B(\zeta,r))} \frac{G^D(\xi,z)G^D(z,\zeta)}{G^D(\xi,\zeta)}|q(z)|dz.
\end{aligned}$$

This shows that if (2.6) is satisfied then (2.5) is also satisfied. The proof is complete. □

Next, we recall some important properties that will be used in the study of the boundary value problems (1.2) and (1.1). The proofs of these properties can be found in references [15, 20] and [2].

Proposition 2.6. *Let $q \in K(D)$. Then the following assertions hold.*

(1) *Letting r tends to infinity in Proposition 2.5 and using the results established in* [20] *and* [15] *stating that $\sup_{x\in D}\int_D \frac{\rho_0(z)}{\rho_0(x)} G^D(x,z)|q(z)|dz < \infty$, we deduce that*

$$N_D(q) = \sup_{(x,y)\in D\times D} \int\limits_D \frac{G^D(x,z)G^D(z,y)}{G^D(x,y)}|q(z)|dz < \infty. \tag{2.12}$$

(2) *For any nonnegative superharmonic function h and every $x \in D$ we have*

$$\int\limits_D G^D(x,z)h(z)|q(z)|dz \leq N_D(q)h(x). \tag{2.13}$$

(3) *The function $y \longrightarrow \rho_0(y)q(y) \in L^1(D)$. In particular $q \in L^1_{loc}(D)$.*
(4) *The Green potential ${}^DGq$ belongs to $C_0(D)$.*

The following results will also play an important role in the sequel.

Proposition 2.7. *Let h be a nonnegative superharmonic function in D and q be a nonnegative functions in $K(D)$. Then for each $x \in D$ such that $0 < h(x) < \infty$, we have*

$$\exp(-N_D(q))\, h(x) \le h(x) - {}^D G_q(q\,h)(x) \le h(x). \tag{2.14}$$

Proposition 2.8. *Let q be a nonnegative function in $K(D)$. Then the family of functions*

$$\mathcal{F}_q = \{{}^D Gp\,;\, |p| \le q\}$$

is equicontinuous in D and consequently it is relatively compact in $C_0(D)$.

We close this section by giving a fundamental example of functions belonging to $K(D)$ that was given in [20] and [15].

Example 2.9. Let $\beta \in \mathbb{R}$ and define $q(x) = \dfrac{1}{(\rho_0(x))^{\beta}}$ for $x \in D$. Then

$$q \in K(D) \quad \text{if and only if} \quad \beta < 2.$$

3. EXISTENCE OF POSITIVE SOLUTIONS FOR SOME SEMILINEAR ELLIPTIC EQUATIONS

The aim of this section is to study the existence of positive continuous weak solutions of the following semilinear elliptic Dirichlet problem (1.2). First we begin by introducing the notion of continuous weak solutions for this problem.

Denote by $C_c^\infty(D)$ the set of all infinitely differentiable functions in D with compact support in D.

Definition 3.1. A function u is called a continuous weak solution of (1.2) if

(i) $u \in C(\overline{D}, \mathbb{R})$,

(ii) $\displaystyle\int_D u(x)\Delta\varphi(x) - \lambda f(x, u(x))\varphi dx = 0$ for every $\varphi \in C_c^\infty(D)$,

(iii) $\displaystyle\lim_{\substack{x \to \xi \in \partial D \\ x \in D}} u(x) = \phi(\xi)$.

The following result ensure the uniqueness of an eventual continuous weak solution for (1.2) in the case where $f \ge 0$ and nondecreasing and continuous with respect to the second variable.

Proposition 3.2. *Let $f : D \times [0, \infty) \longrightarrow [0, \infty)$ be a Borel measurable function such that $f(\cdot, c) \in L^1_{loc}(D)$ for each $c \ge 0$ and for each $x \in D$ the function $t \to f(x,t)$ is nondecreasing and continuous on $[0, \infty)$. For any nontrivial nonnegative continuous function ϕ on the boundary ∂D and $\lambda \ge 0$, problem (1.2) has at most one nonnegative continuous weak solution.*

Proof. Assume that there exist two nonnegative continuous weak solutions u_1, u_2 of (1.2) with $u_1 \neq u_2$. We suppose that there exists $x_0 \in D$ such that $u_1(x_0) > u_2(x_0)$. Put $w = u_1 - u_2$ and denote by $E = \{x \in D : \; w(x) > 0\}$. Then E is a nonempty open set and from the fact that f is nondecreasing with respect to the second variable, we obtain

$$\begin{cases} \Delta w = \lambda\,[f(x,u_1(x)) - f(x,u_2(x))] \geq 0 & \text{in } E \quad \text{(in the sense of distributions)}, \\ w \leq 0 & \text{on } \partial E. \end{cases}$$

Hence by the weak maximum principle (see [9, p. 333–334]) we get $w \leq 0$ in E. This contradicts the definition of E and achieves the proof. □

In order to state an existence result for (1.2) for λ sufficiently small, we assume that f satisfy the following hypotheses:

(H_1) The function $f(\cdot,0)$ belongs to $K(D)$.
(H_2) $f : D \times [0,\infty) \longrightarrow [0,\infty)$ is Borel measurable such that for each $x \in D$, the map $t \to f(x,t)$ is continuous and satisfying the following condition: For each $M > 0$, there exists a nonnegative function $q_M \in K(D)$ such that for each $x \in D$, the map $t \to q_M(x)\,t - f(x,t)$ is nondecreasing on $[0,M]$.
(H_3) $\sigma_0 := \inf\limits_{x\in D} \left[\dfrac{H_D\phi(x)}{{}^D G f(\cdot,0)(x)}\right] > 0.$

Remarks 3.3.

(1) The conditions (H_1) and (H_2) are satisfied in the particular case $f(x,t) = p(x)\,g(t)$, where $p \in K(D)$ and $g(t) = t^\alpha$, $\alpha \geq 1$ or more generally $g : [0,\infty) \to [0,\infty)$ is continuous and satisfying for each $M > 0$, there exists a constant $b = b(M) \geq 0$ such that $g(t) - g(s) \leq b\,(t-s)$ for $0 \leq s < t \leq M$. Indeed in this case (H_2) is satisfied with $q_M = b(M)\,p$.
(2) Let $p \in K(D)$ and $g(t) = \frac{1}{1+\sqrt{t}}$. Then the function $f(x,t) = p(x)\,g(t)$ satisfy (H_2) with $q_M = 0$ despite the derivative $g'(t) = \frac{-1}{2\sqrt{t}(1+\sqrt{t})^2}$ for $t > 0$ is not bounded near zero.
(3) The hypothesis (H_3) is satisfied in the particular case where $f(\cdot,0) = 0$ with $\sigma_0 = \infty$.

Example 3.4. Let $\alpha \geq 1$, and $\beta,\gamma \in \mathbb{R}$ such that $\beta < 2 + \min(\gamma,0)$. Define

$$f(x,t) = \frac{1}{(\rho_0(x))^\beta}\,(\rho_0(x)+t)^\gamma\; t^\alpha \quad \text{for} \quad (x,t) \in D \times [0,\infty).$$

Then f satisfy hypotheses (H_1)–(H_3). Indeed, since $f(x,0) = 0$ then (H_1) is satisfied and (H_3) is satisfied with $\sigma_0 = \infty$. Next we prove that f satisfy (H_2). To this aim we consider $M > 0$, $0 \leq s \leq t \leq M$ and $\eta \in [s,t]$ such that

$$(\rho_0(x)+t)^\gamma\, t^\alpha - (\rho_0(x)+s)^\gamma\, s^\alpha = (t-s)\left[\gamma(\rho_0(x)+\eta)^{\gamma-1}\,\eta^\alpha + \alpha(\rho_0(x)+\eta)^\gamma\,\eta^{\alpha-1}\right].$$

We will discuss two cases.

Case 1. $\gamma \geq 0$.
In this case since $\gamma + \alpha - 1 \geq 0$, so

$$\begin{aligned}(\rho_0(x) + \eta)^{\gamma-1}\,\eta^{\alpha} &\leq \max\left((L+\eta)^{\gamma-1}, \eta^{\gamma-1}\right)\,\eta^{\alpha}\\ &= \max\left((L+\eta)^{\gamma-1}\eta^{\alpha}\,,\,\eta^{\gamma+\alpha-1}\right)\\ &\leq \max\left((L+\eta)^{\gamma+\alpha-1}\,,\,\eta^{\gamma+\alpha-1}\right) = (L+M)^{\gamma+\alpha-1}\,.\end{aligned}$$

Hence

$$f(x,t) - f(x,s) \leq \frac{(\alpha+\gamma)(L+M)^{\gamma+\alpha-1}}{(\rho_0(x))^{\beta}}(t-s).$$

Consequently, (H_3) is satisfied with

$$q_M(x) = \frac{(\alpha+\gamma)(L+M)^{\gamma+\alpha-1}}{(\rho_0(x))^{\beta}}.$$

Case 2. $\gamma < 0$.
In this case we have

$$\begin{aligned}\gamma(\rho_0(x)+\eta)^{\gamma-1}\,\eta^{\alpha} + \alpha(\rho_0(x)+\eta)^{\gamma}\,\eta^{\alpha-1} &\leq \alpha(\rho_0(x)+\eta)^{\gamma}\,\eta^{\alpha-1}\\ &\leq \alpha M^{\alpha-1}(\rho_0(x))^{\gamma}\end{aligned}$$

Hence

$$f(x,t) - f(x,s) \leq \frac{\alpha M^{\alpha-1}}{(\rho_0(x))^{\beta-\gamma}}(t-s).$$

Since $\beta < 2+\gamma$, then $\frac{1}{(\rho_0(x))^{\beta-\gamma}} \in K(D)$ and consequently (H_3) is satisfied with

$$q_M(x) = \frac{\alpha M^{\alpha-1}}{(\rho_0(x))^{\beta-\gamma}}.$$

Combining Cases 1 and 2 we deduce that (H_2) is satisfied with

$$q_M(x) = \frac{[\alpha + \max(\gamma,0)]\,(L+M)^{\alpha-1+\max(\gamma,0)}}{(\rho_0(x))^{\beta-\min(\gamma,0)}}.$$

Example 3.5. As it assumed as long of this paper D is a bounded $C^{1,1}$-domain and we consider $\beta\,,\,\gamma \in \mathbb{R}$ such that $\beta < \min(1+\gamma, 2)$. Define

$$f(x,t) = \frac{1}{(\rho_0(x))^{\beta}}\,(\rho_0(x)+t)^{\gamma} \quad \text{for} \quad (x,t) \in D \times [0,\infty).$$

Then f satisfy hypotheses (H_1)–(H_3). Indeed, since $\beta - \gamma < 1$ then $f(x,0) = \frac{1}{(\rho_0(x))^{\beta-\gamma}} \in K(D)$ and (H_1) is satisfied. To prove that f verify (H_2), we consider $M > 0$, $0 \leq s \leq t \leq M$ and $\eta \in [s,t]$ such that

$$(\rho_0(x)+t)^{\gamma} - (\rho_0(x)+s)^{\gamma} = (t-s)\left[\gamma(\rho_0(x)+\eta)^{\gamma-1}\right].$$

We will discuss three cases.

Case 1. $\gamma \leq 0$.
In this case we have $f(x,t) - f(x,s) \leq 0$. So we can take $q_M = 0$.

Case 2. $\gamma \geq 1$.
In this case we have

$$f(x,t) - f(x,s) = \frac{\gamma(\eta + \rho_0(x))^{\gamma-1}}{(\rho_0(x))^{\beta}}\,(t-s) \leq \frac{\gamma(L+M)^{\gamma-1}}{(\rho_0(x))^{\beta}}\,(t-s).$$

So we can take

$$q_M = \frac{\gamma(L+M)^{\gamma-1}}{(\rho_0(x))^{\beta}} \in K(D).$$

Case 3. $0 < \gamma < 1$.
In this case

$$f(x,t) - f(x,s) = \frac{\gamma(\eta + \rho_0(x))^{\gamma-1}}{(\rho_0(x))^{\beta}}\,(t-s) \leq \frac{\gamma}{(\rho_0(x))^{\beta-\gamma+1}}\,(t-s).$$

Since $\beta < 1 + \gamma$, then

$$q_M = \frac{\gamma}{(\rho_0(x))^{\beta-\gamma+1}} \in K(D)$$

and this shows that (H_2) is satisfied.

Finally we prove that f satisfy (H_3). Since $f(x,0) = \dfrac{1}{(\rho_0(x))^{\beta-\gamma}}$ and $\beta - \gamma < 1$, then it follows from Proposition 5 in [15] that there exists $C_1 > 0$ such that

$$\frac{1}{C_1}\,\rho_0(x) \leq {}^{D}Gf(\cdot,0)(x) \leq C_1\,\rho_0(x)$$

for every $x \in D$. On the other hand, since D is a bounded $C^{1,1}$-domain and $H_D\phi$ is positive and harmonic in D, then it follows from Corollary 6.2 in [1] that there exists $C_2 > 0$ depending only on ϕ and D such that $C_2\,\rho_0(x) \leq H_D\phi(x)$ for every $x \in D$. Consequently for $x \in D$ we have

$$\frac{H_D\phi(x)}{{}^{D}Gf(\cdot,0)(x)} \geq \frac{C_2\,\rho_0(x)}{C_1\,\rho_0(x)} = \frac{C_2}{C_1} > 0$$

and then $\sigma_0 > 0$.

Remark 3.6. If f satisfies (H_1)–(H_2), then for each $c > 0$ we have $f(\cdot,c) \in K(D)$. Indeed, for $c > 0$ we deduce from (H_2) that there exists $q_c \in K(D)$ such that for each $x \in D$ and $0 \leq t_1 \leq t_2 \leq c$ we have

$$t_1\,q_c(x) - f(x,t_1) \leq t_2\,q_c(x) - f(x,t_2).$$

By taking $t_1 = 0$ and $t_2 = c$ we obtain

$$0 \leq f(x,c) \leq f(x,0) + c\,q_c(x).$$

This together with (H_1) shows that $f(\cdot,c) \in K(D)$.

The second main result of this paper will be an extension of a result established in [18] for the case $f(x,u) = p(x)h(u)$. The proof in [18] will be adapted here and we will give it for the sake of completeness.

Theorem 3.7. *Let ϕ be a nontrivial nonnegative continuous function on ∂D and assume that hypotheses (H_1), (H_2) and (H_3). Then there exists $\lambda_0 > 0$ such that for $\lambda \in [0, \lambda_0)$, problem (1.2) has a positive continuous weak solution u satisfying the following global behavior*

$$c_\lambda \, H_D\phi(x) \le u(x) \le H_D\phi(x) \quad \textit{for each } x \in \overline{D}, \tag{3.1}$$

where $c_\lambda \in [0, 1)$.

Proof. Put

$$M = \|H_D\phi\|_\infty = \sup_{x\in\overline{D}} |H_D\phi(x)|.$$

Since $H_D\phi$ is harmonic in D with boundary value ϕ, then it follows from the maximum principle that

$$M = \|\phi\|_\infty = \sup_{x\in\partial D} |\phi(x)|.$$

Since $\phi \ne 0$, then $M > 0$. From hypothesis (H_2) there exists $q = q_M \in K(D)$ such that for each $x \in D$ and $0 \le s < t \le M$ we have

$$\frac{f(x,t) - f(x,s)}{t - s} \le q_M(x).$$

Consider the function $\theta : \lambda \to \lambda \exp(\lambda N_D(q))$. Then θ is a bijection from $[0,\infty)$ to $[0,\infty)$. Put $\lambda_0 = \theta^{-1}(\sigma_0) > 0$, with the convention that $\lambda_0 = \infty$ if $\sigma_0 = \infty$. For $\lambda \in [0, \lambda_0)$, we define the nonempty closed convex set

$$\Lambda = \left\{ u \in B_b(D) : \left(1 - \frac{\theta(\lambda)}{\sigma_0}\right) \exp(-\lambda N_D(q))\, H_D\phi(x) \le u(x) \le H_D\phi(x) \right\}.$$

Let T be the operator defined on Λ by

$$Tu = H_D\phi - {}^D G_{\lambda q}(\lambda q \, H_D\phi) + {}^D G_{\lambda q}(\lambda q \, u - \lambda f(, u)).$$

We will prove that Λ is invariant under T and T has a fixed point in Λ which is a solution of the integral equation

$$u = H_D\phi - {}^D G(\lambda f(\cdot, u)) \qquad \text{in } D. \tag{3.2}$$

For each $u \in \Lambda$, we have

$$\begin{aligned} Tu &= H_D\phi - \lambda^D G_{\lambda q}(q H_D\phi) + \lambda^D G_{\lambda q}(q\, u - f(\cdot, u)) \\ &\le H_D\phi - \lambda^D G_{\lambda q}(q H_D\phi) + \lambda^D G_{\lambda q}(q\, u) \\ &\le H_D\phi. \end{aligned}$$

Now, since for each $y \in D$ the function $t \longrightarrow q\,t - f(y,t)$ is nondecreasing on $[0, \|\phi\|_\infty]$, then for each $y \in D$ and $t \in [0, \|\phi\|_\infty]$ we have $q\,t + f(y,0) - f(y,t) \geq 0$. So using Proposition 2.7, hypothesis (H_1) and (1.7) we get

$$\begin{aligned}
Tu &= H_D\phi - {}^D G_{\lambda q}(\lambda q H_D \phi) - \lambda {}^D G_{\lambda q}(f(\cdot,0)) + \lambda {}^D G_{\lambda q}(q\,u + f(,0) - f(\cdot,u)) \\
&\geq e^{-N_D(\lambda q)} H_D\phi - \lambda {}^D G_{\lambda q}(f(\cdot,0)) \\
&\geq e^{-\lambda N_D(q)} H_D\phi - \lambda {}^D G(f(\cdot,0)) \\
&\geq e^{-\lambda N_D(q)} H_D\phi - \lambda \frac{{}^D G(f(\cdot,0))}{H_D\phi} H_D\phi \\
&\geq e^{-\lambda N_D(q)} H_D\phi - \lambda \sup_{x\in\overline{D}} \left[\frac{{}^D Gf(\cdot,0)(x)}{H_D\phi(x)}\right] H_D\phi \\
&\geq e^{-\lambda N_D(q)} H_D\phi - \frac{\lambda}{\inf\limits_{x\in\overline{D}} \left[\dfrac{H_D\phi(x)}{{}^D Gf(\cdot,0)(x)}\right]} H_D\phi \\
&\geq \exp(-\lambda N_D(q)) \left[1 - \frac{\theta(\lambda)}{\sigma_0}\right] H_D\phi.
\end{aligned}$$

Consequently $T\Lambda \subset \Lambda$. Next, we prove that T is a nondecreasing operator on Λ. For this aim, we consider $u, v \in \Lambda$ such that $u \leq v$. Then using hypothesis (H_2), we get

$$\begin{aligned}
Tu - Tv &= \lambda {}^D G_{\lambda q}\left(qu - f(\cdot,u) - qv + f(\cdot,v)\right) \\
&= \lambda {}^D G_{\lambda q}\left(f(\cdot,v) - f(\cdot,u) - q(v-u)\right) \leq 0.
\end{aligned}$$

Next, we consider the sequence $(u_n)_{n\geq 0}$ defined by

$$u_0 = H_D\phi - \lambda {}^D G_{\lambda q}(qH_D\phi) - \lambda\, {}^D G_{\lambda q}(f(\cdot,0)) \quad \text{and} \quad u_{n+1} = Tu_n \text{ for } n \geq 0.$$

Using the monotonicity of T, we obtain

$$u_0 \leq u_1 \leq \ldots \leq u_n \leq u_{n+1} \leq H_D\phi.$$

It follows from the dominated convergence theorem and the continuity of f that the sequence $(u_n)_{n\geq 0}$ converges to a function $u \in \Lambda$ satisfying $Tu = u$, or equivalently

$$u = H_D\phi - {}^D G_{\lambda q}(\lambda q H_D\phi) - \lambda {}^D G_{\lambda q}(f(\cdot,0)) + \lambda {}^D G_{\lambda q}(q\,u + f(\cdot,0) - f(\cdot,u)).$$

This implies that

$$(I - {}^D G_{\lambda q}(\lambda q.))u = (I - {}^D G_{\lambda q}(\lambda q.))H_D\phi - {}^D G_{\lambda q}(\lambda f(\cdot,u)).$$

After applying the operator $(I + {}^D G(\lambda q.))$ on the last equation, we deduce by (1.5) and (1.6) that u is a solution of the integral equation (3.2). Now, using hypothesis (H_2) we obtain

$$0 \leq f(y,u(y)) \leq f(y,0) + q\,u \leq f(y,0) + \|\phi\|_\infty q.$$

Since $f(\cdot,0), q \in K(D)$, we obtain $f(\cdot,u) \in K(D)$. So, using property 4 of Proposition 2.6 we obtain ${}^D G(f(\cdot,u)) \in C_0(D)$. This together with the definition of $H_D\phi$ imply that $u \in C(\overline{D})$ and $u = \phi$ on ∂D. Applying (1.3) we deduce that u is a continuous weak solution of (1.2). $\square$

4. EXISTENCE OF POSITIVE SOLUTIONS FOR SOME SEMILINEAR ELLIPTIC SYSTEMS

In this section we deal with the existence of positive continuous weak solutions for the semilinear elliptic system (1.1). We assume that the functions f, g satisfy the following hypotheses

(H_4) $f, g : D \times [0,\infty) \times [0,\infty) \to [0,\infty)$ are Borel measurable functions such that for each $x \in D$ the function $(u,v) \to (f(x,u,v), g(x,u,v))$ is continuous on $[0,\infty) \times [0,\infty)$ and for each $(x,v) \in D \times [0,\infty)$ the function $u \to f(x,u,v)$ is nondecreasing and for each $(x,u) \times D \times [0,\infty)$ the function $v \to g(x,u,v)$ is nondecreasing.

(H_5) The functions $f(\cdot,0,0)$, $g(\cdot,0,0)$ belong to the class $K(D)$.

(H_6) For each $M > 0$, there exist a nonnegative function $q_M \in K(D)$ and two continuous functions $g_1, f_2 : [0,\infty) \to [0,\infty)$ such that for every $0 \le t_1 \le t_2 \le M$, $0 \le s_1 \le s_2 \le M$ and $x \in D$ we have

$$|f(x,t_2,s_2) - f(x,t_1,s_1)| \le q_M(x)\left[(t_2 - t_1) + |g_1(s_2) - g_1(s_1)|\right],$$
$$|g(x,t_2,s_2) - g(x,t_1,s_1)| \le q_M(x)\left[(s_2 - s_1) + |f_2(t_2) - f_2(t_1)|\right].$$

(H_7)

$$\sigma_1 = \inf_{x\in\overline{D}} \frac{H_D\phi_1(x)}{{}^D G(\omega_1)(x)} > 0 \quad \text{and} \quad \sigma_2 = \inf_{x\in\overline{D}} \frac{H_D\phi_2(x)}{{}^D G(\omega_2)(x)} > 0,$$

where

$$\omega_1(x) = f(x,0,0) + q_M(x)\left(\max_{0\le s\le\|\phi_2\|_\infty} g_1(s)\right),$$
$$\omega_2(x) = g(x,0,0) + q_M(x)\left(\max_{0\le t\le\|\phi_1\|_\infty} f_2(t)\right),$$
$$M = \max\left(\|\phi_1\|_\infty, \|\phi_2\|_\infty\right)$$

and the functions q_M, g_1, f_2 are given in (H_6).

Definition 4.1. A pair (u,v) is said to be a positive continuous weak solution for (1.1) if

(i) $(u,v) \in C(\overline{D}) \times C(\overline{D})$ and $u > 0$, $v > 0$ in D,

(ii) for every $\varphi \in C_c^\infty(D)$ we have

$$\int_D u(x)\Delta\varphi(x) - f(x,u(x),v(x))\varphi(x)\,dx = 0$$

and

$$\int_D v(x)\Delta\varphi(x) - g(x,u(x),v(x))\varphi(x)\,dx = 0,$$

(iii) $\lim_{\substack{x\to\xi\in\partial D\\ x\in D}} u(x) = \phi_1(\xi)$ and $\lim_{\substack{x\to\xi\in\partial D\\ x\in D}} v(x) = \phi_2(\xi)$.

Under the precedent hypotheses we prove the following theorem.

Theorem 4.2. *Let ϕ_1, ϕ_2 be two nontrivial nonnegative continuous functions on ∂D and $f, g : D \times [0,\infty) \times [0,\infty) \to [0,\infty)$ be two Borel measurable functions satisfying hypotheses (H_4)–(H_7). Then there exist $\lambda_0 > 0$ and $\mu_0 > 0$ such that for each $\lambda \in [0,\lambda_0)$ and $\mu \in [0,\mu_0)$, system* (1.1) *has a continuous weak solution (u,v) satisfying*

$$\widetilde{c}_{1,\lambda}\, H_D\phi_1 \le u \le H_D\phi_1 \quad \textit{and} \quad \widetilde{c}_{2,\mu}\, H_D\phi_2 \le v \le H_D\phi_2 \qquad \textit{in} D, \tag{4.1}$$

where $\widetilde{c}_{1,\lambda}, \widetilde{c}_{2,\mu} \in [0,1)$.

Remark 4.3. Let $u, v \in C(\overline{D})$ such that $0 \le u \le H_D\phi_1$, $0 \le v \le H_D\phi_2$ and assume that hypotheses (H_4)–(H_7) are satisfied. Then from hypothesis (H_6) we have

$$0 \le f(x,0,v) \le \omega_1(x) \text{ and } 0 \le g(x,u,0) \le \omega_2(x)\,, \text{ for every } x \in D.$$

Hence

$$\sigma_0' = \inf_{x\in\overline{D}} \left[\frac{H_D\phi_1(x)}{{}^D G f(\cdot,0,v)(x)}\right] \ge \inf_{x\in\overline{D}} \left[\frac{H_D\phi_1(x)}{{}^D G(\omega_1)(x)}\right] = \sigma_1 > 0$$

and

$$\sigma_0'' = \inf_{x\in\overline{D}} \left[\frac{H_D\phi_2(x)}{{}^D G g(\cdot,u,0)(x)}\right] \ge \inf_{x\in\overline{D}} \left[\frac{H_D\phi_2(x)}{{}^D G(\omega_2)(x)}\right] = \sigma_2 > 0.$$

Define the positive constants

$$\lambda_1 = \theta^{-1}(\sigma_1) \quad \text{and} \quad \mu_1 = \theta^{-1}(\sigma_2) \quad \text{for } \theta(r) = r\exp\left(r\, N_D(q_M)\right).$$

Then it follows from Theorem 3.7, the fact that f is nondecreasing with respect to the second variable, the fact that g is nondecreasing with respect to the third variable and Proposition 3.2 that for $0 \le \lambda < \lambda_1$ and $0 \le \mu < \mu_1$ the problem

$$\begin{cases} \Delta y = \lambda\, f(x,y,v) & \text{in } D \quad \text{(in the sense of distributions)},\\ \Delta z = \mu\, g(x,u,z) & \text{in } D \quad \text{(in the sense of distributions)},\\ y = \phi_1 \text{ and } z = \phi_2 & \text{on } \partial D \end{cases} \tag{4.2}$$

has a unique pair (y,z) of continuous weak solution satisfying

$$\widetilde{c}_{1,\lambda}\, H_D\phi_1 \le y \le H_D\phi_1 \quad \text{and} \quad \widetilde{c}_{2,\mu}\, H_D\phi_2 \le z \le H_D\phi_2 \quad \text{in } D, \tag{4.3}$$

where

$$\widetilde{c}_{1,\lambda} = \left[1 - \frac{\theta(\lambda)}{\sigma_1}\right]\exp(-\lambda N_D(q_M)) \quad \text{and} \quad \widetilde{c}_{2,\mu} = \left[1 - \frac{\theta(\mu)}{\sigma_2}\right]\exp(-\mu N_D(q_M)).$$

Proof of Theorem 4.2. Let $\lambda_1 = \theta^{-1}(\sigma_1)$, $\mu_1 = \theta^{-1}(\sigma_2)$, $\widetilde{c}_{1,\lambda}$ and $\widetilde{c}_{2,\mu}$ be the constants defined in the Remark 4.3. For any $(\lambda,\mu) \in [0,\lambda_1) \times [0,\mu_1)$, we consider the nonempty closed convex set

$$\Gamma = \left\{(u,v) \in C(\overline{D}) \times C(\overline{D}) : \widetilde{c}_{1,\lambda}\, H_D\phi_1 \le u \le H_D\phi_1;\, \widetilde{c}_{2,\mu}\, H_D\phi_2 \le v \le H_D\phi_2\right\}$$

and define the operator T on Γ by

$$T(u,v) = (T_1(u,v), T_2(u,v)) = (y,z),$$

the unique positive continuous weak solution of the problem (4.2). Then by Theorem 3.7, the solution $(y,z) \in C(\overline{D}) \times C(\overline{D})$, satisfies (4.3) and the integral the equations

$$y = H_D\phi_1 - \lambda^D G\left(f(\cdot, y, v)\right), \quad z = H_D\phi_2 - \mu^D G\left(g(\cdot, u, z)\right).$$

In particular we deduce that Γ is invariant under T. In order to use the Schauder fixed point theorem, we will prove that T is a compact operator on Γ. First, we prove that $T_1(\Gamma)$ and $T_2(\Gamma)$ are equicontinuous on $\overline{D}$. Let $x, x' \in \overline{D}$. Then for any $(u,v) \in \Gamma$ such that $T(u,v) = (T_1(u,v), T_2(u,v)) = (y,z)$ we have

$$|y(x) - y(x')| \leq |H_D\phi_1(x) - H_D\phi_1(x')| + \lambda \left|{}^D G(f(\cdot, y, v))(x) - {}^D G(f(\cdot, y, v))(x')\right|$$

and

$$|z(x) - z(x')| \leq |H_D\phi_2(x) - H_D\phi_2(x')| + \mu \left|{}^D G(g(\cdot, u, z))(x) - {}^D G(g(\cdot, u, z))(x')\right|.$$

Since u, v, y, z have range in $[0, M]$ in $\overline{D}$ and g_1, f_2 are continuous then for each $x \in D$ we have

$$0 \leq f(x, y, v) \leq f(x, 0, 0) + q_M(x)\left(\|\phi_1\|_\infty + \max_{0 \leq s \leq \|\phi_2\|_\infty} g_1(s)\right) := \rho_1(x)$$

and

$$0 \leq g(x, u, z) \leq g(x, 0, 0) + q_M(x)\left(\|\phi_2\|_\infty + \max_{0 \leq t \leq \|\phi_1\|_\infty} f_2(t)\right) := \rho_2(x).$$

Since $f(\cdot, 0, 0)$, $g(\cdot, 0, 0)$ and q_M belong to $K(D)$, then it follows from Proposition 2.8 that the families $\mathcal{F}_{\rho_1}$ and $\mathcal{F}_{\rho_2}$ are equicontinuous in $\overline{D}$. This together with the fact that $H_D\phi_1$ and $H_D\phi_2$ are continuous in $\overline{D}$ imply that for every $\varepsilon > 0$, there exists $\eta > 0$ such that for every $x, x' \in \overline{D}$ with $\|x - x'\| < \eta$ we have

$$\|T_1(u,v)(x) - T_1(u,v)(x')\| < \varepsilon \quad \text{and} \quad \|T_2(u,v)(x) - T_2(u,v)(x')\| < \varepsilon$$

for every $(u,v) \in \Gamma$. Which means that $T_1(\Gamma)$ and $T_2(\Gamma)$ are equicontinuous in $\overline{D}$. Using this fact and the boundedness of $T_1(\Gamma)$ and $T_2(\Gamma)$, we deduce that they are relatively compact in $C(\overline{D}) \times C(\overline{D})$. Next, we prove that T is continuous. To this aim, we consider a sequence $(u_k, v_k)_k$ in Γ that converges to $(u,v) \in \Gamma$ with respect to the norm $\|\cdot\|_\infty + \|\cdot\|_\infty$. Put

$$(y,z) = T(u,v) = (T_1(u,v), T_2(u,v))$$

and

$$(y_k, z_k) = T(u_k, v_k) = (T_1(u_k, v_k), T_2(u_k, v_k)).$$

Then we have

$$\begin{aligned} y_k - y &= \lambda^D G(f(\cdot, y, v)) - \lambda^D G(f(\cdot, y_k, v_k)) \\ &= \lambda^D G\left(f(\cdot, y, v) - f(\cdot, y_k, v)\right) - \lambda^D G\left(f(\cdot, y_k, v_k) - f(\cdot, y_k, v)\right). \end{aligned}$$

So

$$(y_k - y) + \lambda^D G\left(f(\cdot, y_k, v) - f(\cdot, y, v)\right) = \lambda^D G\left(f(\cdot, y_k, v) - f(\cdot, y_k, v_k)\right). \tag{4.4}$$

Put

$$h_k(x) = \begin{cases} \dfrac{f(x, y_k(x), v(x)) - f(x, y(x), v(x))}{y_k(x) - y(x)} & \text{if } y_k(x) \neq y(x), \\ 0 & \text{if } y_k(x) = y(x). \end{cases}$$

Then

$$f(x, y_k(x), v(x)) - f(x, y(x), v(x)) = (y_k(x) - y(x))h_k(x),$$

for every $x \in D$, and from hypotheses (H_4) and (H_6) we obtain $0 \leq h_k(x) \leq q_M(x)$ for each k and each $x \in D$. Since $q_M \in K(D)$, then λh_k. Thus equation (4.4) can be written

$$(y_k - y) + {}^D G\left(\lambda h_k(y_k - y)\right) = \lambda^D G\left(f(\cdot, y_k, v) - f(\cdot, y_k, v_k)\right) \tag{4.5}$$

with $\lambda h_k \in K(D)$. This allows us to apply $\left(I - {}^D G_{\lambda h_k}(\lambda h_k(\,\cdot))\right)$ to equation (4.4) to obtain

$$y_k - y = \lambda^D G_{\lambda h_k}\left(f(\cdot, y_k, v_k) - f(\cdot, y_k, v)\right).$$

Hence it follows from hypothesis (H_6), inequality (1.7) and the fourth property of Proposition 2.6 that there exists $C > 0$ independent of k such that

$$\|y_k - y\|_\infty \leq C\,\lambda \|{}^D G\left(q_M\right)\|_\infty \|g_1(v) - g_1(v_k)\|_\infty.$$

Now, since $v(x), v_k(x) \in [0, \|\phi_2\|_\infty]$ for each $x \in \overline{D}$ and each k, then using the uniform continuity of g_1 on $[0, \|\phi_2\|_\infty]$ and the uniform convergence of $(v_k)_k$ to v in $\overline{D}$, we deduce that $(g_1(v_k))_k$ converges uniformly to $g_1(v)$ in $\overline{D}$. Hence $\lim_{k\to\infty} \|y_k - y\|_\infty = 0$. This proves that T_1 is continuous. In the same manner we prove that $\lim_{k\to\infty} \|z_k - z\|_\infty = 0$, and so T_2 is also continuous. Consequently T is continuous on Γ. From the Schauder fixed point theorem we deduce that there exist $(u, v) \in \Gamma$ such that $T(u, v) = (u, v)$. Equivalently,

$$u = H_D\phi_1 - \lambda\,{}^D G\left(f(\cdot, u, v)\right) \quad \text{and} \quad v = H_D\phi_2 - \mu\,{}^D G\left(g(\cdot, u, v)\right).$$

The pair (u, v) is a positive continuous weak solution of (1.1) satisfying (4.1). □

Example 4.4. Let $f : D \times [0, \infty) \times [0, \infty) \to [0, \infty)$ be defined by $f(x, t, s) = p(x)f_1(t)g_1(s)$ with $p \in K(D)$, $g_1 : [0, \infty) \to [0, \infty)$ is continuous and $f_1 : [0, \infty) \to [0, \infty)$ is continuous and nondecreasing and for each $M > 0$, there exists $b = b(M) > 0$ such that

$$f_1(t_2) - f_1(t_1) \leq b(t_2 - t_1) \;\text{ for } 0 \leq t_1 \leq t_2 \leq M.$$

Then hypotheses (H_4)–(H_6) are satisfied.

Example 4.5. Let α_1, β_1, γ_1, $\nu_1 \in \mathbb{R}$ such that $\alpha_1 \geq 0$, $\alpha_1 + \gamma_1 \geq 0$ and define $f : D \times [0, \infty) \times [0, \infty) \to [0, \infty)$ by

$$f(x, t, s) = \frac{1}{(\rho_0(x))^{\beta_1}} \big(\rho_0(x) + t + s\big)^{\gamma_1} \big(\rho_0(x) + t\big)^{\alpha_1} \big(\rho_0(x) + s\big)^{\nu_1}.$$

The condition $\alpha_1 \geq 0$ and $\alpha_1 + \gamma_1 \geq 0$ is a necessary and sufficient condition in order that f is nondecreasing with respect to the variable t for each $(x, s) \in D \times [0, \infty)$. Next, we assume that this condition is satisfied and we will prove that f satisfy (H_5) and (H_6) if $\beta_1 < 2 + \min(\gamma_1 + \nu_1 - 1, \gamma_1 - 1, \nu_1 - 1, \alpha_1 - 1, 0)$. Under this condition we clearly see that

$$f(x, 0, 0) = \frac{1}{\big(\rho_0(x)\big)^{\beta_1 - \gamma_1 - \alpha_1 - \nu_1}} \in K(D)$$

and (H_5) is satisfied. To verify (H_6), we consider $M > 0$, $0 \leq t_1 \leq t_2 \leq M$ and $0 \leq s_1 \leq s_2 \leq M$. Then, there exist $\eta_1 \in (t_1 + s_1, t_2 + s_2)$, $\eta_2 \in (t_1, t_2)$ and $\eta_3 \in (s_1, s_2)$ such that

$$\begin{aligned} \big(\rho_0(x) + t_2 + s_2\big)^{\gamma_1} - \big(\rho_0(x) + t_1 + s_1\big)^{\gamma_1} &= \gamma_1(t_2 - t_1 + s_2 - s_1)\big(\rho_0(x) + \eta_1\big)^{\gamma_1 - 1}, \\ \big(\rho_0(x) + t_2\big)^{\alpha_1} - \big(\rho_0(x) + t_1\big)^{\alpha_1} &= \alpha_1(t_2 - t_1)\big(\rho_0(x) + \eta_2\big)^{\alpha_1 - 1} \end{aligned}$$

and

$$\frac{1}{\big(\rho_0(x) + s_2\big)^{-\nu_1}} - \frac{1}{\big(\rho_0(x) + s_1\big)^{-\nu_1}} = \nu_1(s_2 - s_1)\frac{1}{\big(\rho_0(x) + \eta_3\big)^{1 - \nu_1}}.$$

So

$$\begin{aligned} &f(x, t_2, s_2) - f(x, t_1, s_1) \\ &\quad = \frac{1}{\big(\rho_0(x)\big)^{\beta_1}} \left[\frac{\big(\rho_0(x) + t_2 + s_2\big)^{\gamma_1} \big(\rho_0(x) + t_2\big)^{\alpha_1}}{\big(\rho_0(x) + s_2\big)^{-\nu_1}} \right. \\ &\qquad\qquad \left. - \frac{\big(\rho_0(x) + t_1 + s_1\big)^{\gamma_1} \big(\rho_0(x) + t_1\big)^{\alpha_1}}{\big(\rho_0(x) + s_1\big)^{-\nu_1}} \right] \\ &\quad = \frac{1}{\big(\rho_0(x)\big)^{\beta_1}} \left[\frac{\Big[\big(\rho_0(x) + t_2 + s_2\big)^{\gamma_1} - \big(\rho_0(x) + t_1 + s_1\big)^{\gamma_1}\Big] \big(\rho_0(x) + t_2\big)^{\alpha_1}}{\big(\rho_0(x) + s_2\big)^{-\nu_1}} \right. \\ &\qquad\qquad \left. + \frac{\big(\rho_0(x) + t_1 + s_1\big)^{\gamma_1} \Big(\big(\rho_0(x) + t_2\big)^{\alpha_1} - \big(\rho_0(x) + t_1\big)^{\alpha_1}\Big)}{\big(\rho_0(x) + s_2\big)^{-\nu_1}} \right] \\ &\qquad + \frac{\big(\rho_0(x) + t_1 + s_1\big)^{\gamma_1} \big(\rho_0(x) + t_1\big)^{\alpha_1}}{\big(\rho_0(x)\big)^{\beta_1}} \left[\frac{1}{\big(\rho_0(x) + s_2\big)^{-\nu_1}} - \frac{1}{\big(\rho_0(x) + s_1\big)^{-\nu_1}} \right] \end{aligned}$$

$$= \frac{1}{\big(\rho_0(x)\big)^{\beta_1}} \Bigg[\frac{\gamma_1(t_2-t_1+s_2-s_1)\big(\rho_0(x)+\eta_1\big)^{\gamma_1-1}\big(\rho_0(x)+t_2\big)^{\alpha_1}}{\big(\rho_0(x)+s_2\big)^{-\nu_1}}$$

$$+ \frac{\alpha_1(t_2-t_1)\big(\rho_0(x)+\eta_2\big)^{\alpha_1-1}\big(\rho_0(x)+t_1+s_1\big)^{\gamma_1}}{\big(\rho_0(x)+s_2\big)^{-\nu_1}}$$

$$+\nu_1(s_2-s_1)\big(\rho_0+t_1+s_1\big)^{\gamma_1}\big(\rho_0(x)+t_1\big)^{\alpha_1}\frac{1}{\big(\rho_0(x)+\eta_3\big)^{1-\nu_1}}\Bigg]$$

$$= \frac{1}{(\rho_0(x))^{\beta_1}}\left[(t_2-t_1)\,T_1+(s_2-s_1)\,T_2\right],$$

where

$$T_1 = \frac{\gamma_1(\rho_0(x)+\eta_1)^{\gamma_1-1}(\rho_0(x)+t_2)^{\alpha_1}+\alpha_1(\rho_0(x)+\eta_2)^{\alpha_1-1}(\rho_0(x)+t_1+s_1)^{\gamma_1}}{(\rho_0(x)+s_2)^{-\nu_1}}$$

and

$$T_2 = \frac{\gamma_1\big(\rho_0(x)+\eta_1\big)^{\gamma_1-1}\big(\rho_0(x)+t_2\big)^{\alpha_1}}{\big(\rho_0(x)+s_2\big)^{-\nu_1}} + \frac{\nu_1\big(\rho_0+t_1+s_1\big)^{\gamma_1}\big(\rho_0(x)+t_1\big)^{\alpha_1}}{\big(\rho_0(x)+\eta_3\big)^{1-\nu_1}}.$$

To estimate T_1 and T_2, we discuss the following cases.
Case 1. $\alpha_1 \geq 1$
In this case we discuss nine subcases.
1) If $\gamma_1 \geq 1$ and $\nu_1 \geq 1$, then $T_1 \leq C$ and $T_2 \leq C$. Hence

$$|f(x,t_2,s_2)-f(x,t_1,s_1)| \leq \frac{C}{(\rho_0(x))^{\beta_1}}\left[(t_2-t_1)+(s_2-s_1)\right].$$

2) If $\gamma_1 \geq 1$ and $0 \leq \nu_1 < 1$, then $T_1 \leq C$ and $T_2 \leq \dfrac{C}{(\rho_0(x))^{1-\nu_1}}$. Hence

$$|f(x,t_2,s_2)-f(x,t_1,s_1)| \leq \frac{C}{(\rho_0(x))^{\beta_1+1-\nu_1}}\left[(t_2-t_1)+(s_2-s_1)\right].$$

3) If $\gamma_1 \geq 1$ and $\nu_1 < 0$, then $T_1 \leq \dfrac{C}{(\rho_0(x))^{-\nu_1}}$ and $T_2 \leq \dfrac{C}{(\rho_0(x))^{1-\nu_1}}$. Hence

$$|f(x,t_2,s_2)-f(x,t_1,s_1)| \leq \frac{C}{(\rho_0(x))^{\beta_1+1-\nu_1}}\left[(t_2-t_1)+(s_2-s_1)\right].$$

4) If $0 \leq \gamma_1 < 1$ and $\nu_1 \geq 1$, then $T_1 \leq \dfrac{C}{(\rho_0(x))^{1-\gamma_1}}$ and $T_2 \leq \dfrac{C}{(\rho_0(x))^{1-\gamma_1}}$. Hence

$$|f(x,t_2,s_2)-f(x,t_1,s_1)| \leq \frac{C}{(\rho_0(x))^{\beta_1+1-\gamma_1}}\left[(t_2-t_1)+(s_2-s_1)\right].$$

5) If $0 \leq \gamma_1 < 1$ and $0 \leq \nu_1 < 1$, then

$$T_1 \leq \frac{C}{(\rho_0(x))^{1-\gamma_1}} \quad \text{and} \quad T_2 \leq \frac{C_1}{(\rho_0(x))^{1-\gamma_1}} + \frac{C_2}{(\rho_0(x))^{1-\nu_1}} \leq \frac{C}{(\rho_0(x))^{\max(1-\gamma_1,1-\nu_1)}}.$$

Hence

$$|f(x,t_2,s_2)-f(x,t_1,s_1)| \le \frac{C}{(\rho_0(x))^{\beta_1-\min(\gamma_1-1,\nu_1-1)}}\left[(t_2-t_1)+(s_2-s_1)\right].$$

6) If $0 \le \gamma_1 < 1$ and $\nu_1 < 0$, then

$$T_1 \le \frac{C}{(\rho_0(x))^{1-\gamma_1-\nu_1}} \quad \text{and} \quad T_2 \le \frac{C_1}{(\rho_0(x))^{1-\gamma_1-\nu_1}} + \frac{C_2}{(\rho_0(x))^{1-\nu_1}} \le \frac{C}{(\rho_0(x))^{1-\nu_1}}.$$

Hence

$$|f(x,t_2,s_2)-f(x,t_1,s_1)| \le \frac{C}{(\rho_0(x))^{\beta_1+1-\nu_1}}\left[(t_2-t_1)+(s_2-s_1)\right].$$

7) If $-\alpha_1 \le \gamma_1 < 0$ and $\nu_1 \ge 1$, then

$$T_1 \le \frac{C_1}{(\rho_0(x))^{1-\gamma_1}} + \frac{C_2}{(\rho_0(x))^{-\gamma_1}} \le \frac{C}{(\rho_0(x))^{1-\gamma_1}} \quad \text{and} \quad T_2 \le \frac{C}{(\rho_0(x))^{1-\gamma_1}}.$$

Hence

$$|f(x,t_2,s_2)-f(x,t_1,s_1)| \le \frac{C}{(\rho_0(x))^{\beta_1+1-\gamma_1}}\left[(t_2-t_1)+(s_2-s_1)\right].$$

8) If $-\alpha_1 \le \gamma_1 < 0$ and $0 \le \nu_1 < 1$, then

$$T_1 \le \frac{C_1}{(\rho_0(x))^{1-\gamma_1}} + \frac{C_2}{(\rho_0(x))^{-\gamma_1}} \le \frac{C}{(\rho_0(x))^{1-\gamma_1}}$$

and

$$T_2 \le \frac{C_1}{(\rho_0(x))^{1-\gamma_1}} + \frac{C_2}{(\rho_0(x))^{1-\gamma_1-\nu_1}} \le \frac{C}{(\rho_0(x))^{1-\gamma_1}}.$$

Hence

$$|f(x,t_2,s_2)-f(x,t_1,s_1)| \le \frac{C}{(\rho_0(x))^{\beta_1+1-\gamma_1}}\left[(t_2-t_1)+(s_2-s_1)\right].$$

9) If $-\alpha_1 \le \gamma_1 < 0$ and $\nu_1 < 0$, then

$$T_1 \le \frac{C_1}{(\rho_0(x))^{1-\gamma_1-\nu_1}} + \frac{C_2}{(\rho_0(x))^{-\gamma_1-\nu_1}} \le \frac{C}{(\rho_0(x))^{1-\gamma_1-\nu_1}} \text{ and } T_2 \le \frac{C}{(\rho_0(x))^{1-\gamma_1-\nu_1}}.$$

Hence

$$|f(x,t_2,s_2)-f(x,t_1,s_1)| \le \frac{C}{(\rho_0(x))^{\beta_1+1-\gamma_1-\nu_1}}\left[(t_2-t_1)+(s_2-s_1)\right].$$

Case 2. $0 \le \alpha_1 < 1$
In this case we discuss also nine subcases.
1) If $\gamma_1 \ge 1$ and $\nu_1 \ge 1$, then

$$T_1 \le C_1 + \frac{C_2}{(\rho_0(x))^{1-\alpha_1}} \le \frac{C}{(\rho_0(x))^{1-\alpha_1}}$$

and $T_2 \le C$.

Hence

$$|f(x,t_2,s_2)-f(x,t_1,s_1)| \le \frac{C}{(\rho_0(x))^{\beta_1+1-\alpha_1}}\left[(t_2-t_1)+(s_2-s_1)\right].$$

2) If $\gamma_1 \ge 1$ and $0 \le \nu_1 < 1$, then $T_1 \le C_1 + \frac{C_2}{(\rho_0(x))^{1-\alpha_1}}$ and $T_2 \le C_1 + \dfrac{C_2}{(\rho_0(x))^{1-\nu_1}}$. Hence

$$|f(x,t_2,s_2)-f(x,t_1,s_1)| \le \frac{C}{(\rho_0(x))^{\beta_1-\min(\alpha_1-1,\nu_1-1)}}\left[(t_2-t_1)+(s_2-s_1)\right].$$

3) If $\gamma_1 \ge 1$ and $\nu_1 < 0$, then

$$T_1 \le \frac{C_1}{(\rho_0(x))^{-\nu_1}} + \frac{C_2}{(\rho_0(x))^{1-\alpha_1-\nu_1}} \le \frac{C}{(\rho_0(x))^{1-\alpha_1-\nu_1}}$$

and

$$T_2 \le \frac{C_1}{(\rho_0(x))^{-\nu_1}} + \frac{C_2}{(\rho_0(x))^{1-\nu_1}} \le \frac{C}{(\rho_0(x))^{1-\nu_1}}.$$

Hence

$$|f(x,t_2,s_2)-f(x,t_1,s_1)| \le \frac{C}{(\rho_0(x))^{\beta_1+1-\nu_1}}\left[(t_2-t_1)+(s_2-s_1)\right].$$

4) If $0 \le \gamma_1 < 1$ and $\nu_1 \ge 1$, then

$$T_1 \le \frac{C_1}{(\rho_0(x))^{1-\gamma_1}} + \frac{C_1}{(\rho_0(x))^{1-\alpha_1}} \le \frac{C}{(\rho_0(x))^{\max(1-\gamma_1,1-\alpha_1)}}$$

and

$$T_2 \le \frac{C_1}{(\rho_0(x))^{1-\gamma_1}} + C_2.$$

Hence

$$|f(x,t_2,s_2)-f(x,t_1,s_1)| \le \frac{C}{(\rho_0(x))^{\beta_1-\min(\gamma_1-1,\alpha_1-1)}}\left[(t_2-t_1)+(s_2-s_1)\right].$$

5) If $0 \le \gamma_1 < 1$ and $0 \le \nu_1 < 1$, then

$$T_1 \le \frac{C_1}{(\rho_0(x))^{1-\gamma_1}} + \frac{C_2}{(\rho_0(x))^{1-\alpha_1}}$$

and

$$T_2 \le \frac{C_1}{(\rho_0(x))^{1-\gamma_1}} + \frac{C_2}{(\rho_0(x))^{1-\nu_1}} \le \frac{C}{(\rho_0(x))^{\max(1-\gamma_1,1-\nu_1)}}.$$

Hence

$$|f(x,t_2,s_2)-f(x,t_1,s_1)| \le \frac{C}{(\rho_0(x))^{\beta_1-\min(\gamma_1-1,\nu_1-1,\alpha_1-1)}}\left[(t_2-t_1)+(s_2-s_1)\right].$$

6) If $0 \leq \gamma_1 < 1$ and $\nu_1 < 0$, then

$$T_1 \leq \frac{C_1}{(\rho_0(x))^{1-\gamma_1-\nu_1}} + \frac{C_2}{(\rho_0(x))^{1-\alpha_1-\nu_1}} \leq \frac{C}{(\rho_0(x))^{\max(1-\gamma_1,1-\alpha_1)}}$$

and

$$T_2 \leq \frac{C_1}{(\rho_0(x))^{1-\gamma_1-\nu_1}} + \frac{C_2}{(\rho_0(x))^{1-\nu_1}} \leq \frac{C}{(\rho_0(x))^{1-\nu_1}}.$$

Hence

$$|f(x,t_2,s_2) - f(x,t_1,s_1)| \leq \frac{C}{(\rho_0(x))^{\beta_1+1-\nu_1}} \left[(t_2-t_1)+(s_2-s_1)\right].$$

7) If $-\alpha_1 \leq \gamma_1 < 0$ and $\nu_1 \geq 1$, then

$$T_1 \leq \frac{C_1}{(\rho_0(x))^{1-\gamma_1}} + \frac{C_2}{(\rho_0(x))^{-\gamma_1}} \leq \frac{C}{(\rho_0(x))^{1-\gamma_1}} \quad \text{and} \quad T_2 \leq \frac{C}{(\rho_0(x))^{1-\gamma_1}}.$$

Hence

$$|f(x,t_2,s_2) - f(x,t_1,s_1)| \leq \frac{C}{(\rho_0(x))^{\beta_1+1-\gamma_1}} \left[(t_2-t_1)+(s_2-s_1)\right].$$

8) If $-\alpha_1 \leq \gamma_1 < 0$ and $0 \leq \nu_1 < 1$, then

$$T_1 \leq \frac{C_1}{(\rho_0(x))^{1-\gamma_1}} + \frac{C_2}{(\rho_0(x))^{-\gamma_1}} \leq \frac{C}{(\rho_0(x))^{1-\gamma_1}}$$

and

$$T_2 \leq \frac{C_1}{(\rho_0(x))^{1-\gamma_1}} + \frac{C_2}{(\rho_0(x))^{1-\gamma_1-\nu_1}} \leq \frac{C}{(\rho_0(x))^{1-\gamma_1}}.$$

Hence

$$|f(x,t_2,s_2) - f(x,t_1,s_1)| \leq \frac{C}{(\rho_0(x))^{\beta_1+1-\gamma_1}} \left[(t_2-t_1)+(s_2-s_1)\right].$$

9) If $-\alpha_1 \leq \gamma_1 < 0$ and $\nu_1 < 0$, then

$$T_1 \leq \frac{C_1}{(\rho_0(x))^{1-\gamma_1-\nu_1}} + \frac{C_2}{(\rho_0(x))^{-\gamma_1-\nu_1}} \leq \frac{C}{(\rho_0(x))^{1-\gamma_1-\nu_1}}$$

and

$$T_2 \leq \frac{C}{(\rho_0(x))^{1-\gamma_1-\nu_1}}.$$

Hence

$$|f(x,t_2,s_2) - f(x,t_1,s_1)| \leq \frac{C}{(\rho_0(x))^{\beta_1+1-\gamma_1-\nu_1}} \left[(t_2-t_1)+(s_2-s_1)\right].$$

By regrouping all these cases we obtain

$$\begin{aligned}&|f(x,t_2,s_2)-f(x,t_1,s_1)|\\&\leq \frac{C}{(\rho_0(x))^{\beta_1-\min(\gamma_1+\nu_1-1,\gamma_1-1,\nu_1-1,\alpha_1-1,0)}}\left[(t_2-t_1)+(s_2-s_1)\right].\end{aligned}$$

This shows that f satisfy (H_6) with

$$q_M(x)=\frac{C}{(\rho_0(x))^{\beta_1-\min(\gamma_1+\nu_1-1,\gamma_1-1,\nu_1-1,\alpha_1-1,0)}}.$$

To guarantee that f satisfy (H_7), we assume the following stronger conditions on β_1. Namely, $\beta_1<1+\min(\gamma_1+\nu_1-1,\gamma_1-1,\nu_1-1,\alpha_1-1,0)$. Indeed using the fact that $\alpha_1\geq 0$, we obtain

$$\min(\gamma_1+\nu_1-1,\alpha_1-1,)=-1+\min(\gamma_1+\nu_1,\alpha_1,)\leq -1+\gamma_1+\nu_1+\alpha_1.$$

Hence

$$\min(\gamma_1+\nu_1-1,\gamma_1-1,\nu_1-1,\alpha_1-1,0)\leq -1+\gamma_1+\nu_1+\alpha_1.$$

Consequently

$$\begin{aligned}\omega_1(x)&=f(x,0,0)+\|\phi_2\|_\infty\, q_M(x)\\&=\frac{1}{(\rho_0(x))^{\beta_1-\gamma_1-\alpha_1-\nu_1}}+\frac{C\|\phi_2\|_\infty}{(\rho_0(x))^{\beta_1-\min(\gamma_1+\nu_1-1,\gamma_1-1,\nu_1-1,\alpha_1-1,0)}}\\&\leq \frac{C\|\phi_2\|_\infty}{(\rho_0(x))^{\beta_1-\min(\gamma_1+\nu_1-1,\gamma_1-1,\nu_1-1,\alpha_1-1,0)}}.\end{aligned}$$

Now, since $\beta_1-\min(\gamma_1+\nu_1-1,\gamma_1-1,\nu_1-1,\alpha_1-1,0)<1$, then we deduce from proposition 5 in [15] that there exists $C_1>0$ such that

$$\frac{1}{C_1}\rho_0(x)\leq {}^DG(\omega_1)(x)\leq C_1\rho_0(x)$$

for every $x\in D$. On the other hand, since D is a bounded $C^{1,1}$-domain and $H_D\phi_1$ is positive and harmonic in D, then it follows from Corollary 6.2 in [1] that there exists $C_2>0$ depending only on ϕ_1 and D such that $C_2\,\rho_0(x)\leq H_D\phi_1(x)$ for every $x\in D$. Consequently for $x\in D$ we have

$$\frac{H_D\phi_1(x)}{{}^DG(\omega_1)(x)}\geq \frac{C_2\,\rho_0(x)}{C_1\,\rho_0(x)}=\frac{C_2}{C_1}>0$$

and then $\sigma_1>0$.

The development of the previous example leads us to the following consequence that we cannot deduce using results of [2, 11] and [19].

Corollary 4.6. *Let* α_1,α_2, β_1, β_2, γ_1, γ_2, $\nu_1, \nu_2 \in \mathbb{R}$ *such that* $\alpha_1 \geq 0$, $\alpha_1 + \gamma_1 \geq 0$, $\nu_2 \geq 0$, $\nu_2 + \gamma_2 \geq 0$, $\beta_1 < 1 + \min(\gamma_1 + \nu_1 - 1, \gamma_1 - 1, \nu_1 - 1, \alpha_1 - 1, 0)$ *and* $\beta_2 < 1 + \min(\gamma_2 + \alpha_2 - 1, \gamma_2 - 1, \alpha_2 - 1, \nu_2 - 1, 0)$. *Let* D *is a bounded* $C^{1,1}$*-domain of* $\mathbb{R}^d$, $d \geq 2$, ϕ_1, ϕ_2 *nontrivial nonnegative continuous functions on* ∂D. *Then there exist* $\lambda_0 > 0$ *and* $\mu_0 > 0$ *such that for every* $\lambda \in [0, \lambda_0)$ *and* $\mu \in [0, \mu_0)$ *the system*

$$\begin{cases} \Delta u = \frac{\lambda}{(\rho_0(x))^{\beta_1}} \big(\rho_0(x) + u(x) + v(x)\big)^{\gamma_1} \big(\rho_0(x) + u(x)\big)^{\alpha_1} \big(\rho_0(x) + v(x)\big)^{\nu_1} \; in\, D, \\ \Delta v = \frac{\mu}{(\rho_0(x))^{\beta_2}} \big(\rho_0(x) + u(x) + v(x)\big)^{\gamma_2} \big(\rho_0(x) + u(x)\big)^{\alpha_2} \big(\rho_0(x) + v(x)\big)^{\nu_2} \; in\, D, \\ u = \phi_1 \quad and \; v = \phi_2 \; on \; \partial D, \end{cases}$$

has a positive continuous weak solution (u, v) *satisfying*

$$\widetilde{c}_{1,\lambda}\, H_D\phi_1 \leq u \leq H_D\phi_1 \quad and \quad \widetilde{c}_{2,\mu}\, H_D\phi_2 \leq v \leq H_D\phi_2 \;\; in \; D,$$

where $\widetilde{c}_{1,\lambda}$, $\widetilde{c}_{2,\mu} \in [0, 1)$.

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