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THE SPECTRUM PROBLEM FOR DIGRAPHS OF ORDER 4 AND SIZE 5

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Abstract. The paw graph consists of a triangle with a pendant edge attached to one of the three vertices. We obtain a multigraph by adding exactly one repeated edge to the paw. Now, let D be a directed graph obtained by orientating the edges of that multigraph. For 12 of the 18 possibilities for D, we establish necessary and sufficient conditions on n for the existence of a (K_n^*, D)-design. Partial results are given for the remaining 6 possibilities for D.

Keywords: spectrum problem, digraph decompositions.

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1. INTRODUCTION

If a and b are integers we denote $\{a, a+1, \ldots, b\}$ by $[a, b]$ (if $a > b$, then $[a, b] = \varnothing$). Let $\mathbb{Z}_m$ denote the set of integers modulo m. For a graph H, let $V(H)$ and $E(H)$ denote the vertex set of H and the edge set of H, respectively. Similarly, for a digraph D, let $V(D)$ and $A(D)$ denote the vertex set of D and the arc set of D, respectively. The *order* and the *size* of a graph H (or digraph D) are $|V(H)|$ and $|E(H)|$ (or $|V(D)|$ and $|A(D)|$), respectively.

We denote the complete multipartite graph with parts of sizes a_i for $1 \leq i \leq m$ by $K_{a_1,a_2,\ldots,a_m}$. If $a_i = a$ for all $i \in [1, m]$, then we use the notation $K_{m\times a}$. Furthermore, let $V(K_{m\times a}) = \mathbb{Z}_{ma}$ with vertex partition $\{V_0, V_1, \ldots, V_{m-1}\}$ where $V_i = \{j \in \mathbb{Z}_{ma} : j \equiv i \pmod{m}\}$. Then $E(K_{m\times a})$ consists of all edges $\{i, j\}$ such that $i \not\equiv j \pmod{m}$.

The *complete graph of order n with a hole of size t*, denoted $K_n \setminus K_t$, is the graph with vertex set V and edge set $\{\{a, b\} : a \in V, b \in V \setminus U, a \neq b\}$ where $|V| = n$, $U \subseteq V$, and $|U| = t$. The vertices in U are said to be the *vertices in the hole*.

Let tG denote the graph consisting of t vertex-disjoint copies of G. The *join* of two vertex-disjoint graphs G and H, denoted $G \vee H$, is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{\{a, b\} : a \in V(G), b \in V(H)\}$. For example,

K_{5x+1} could be described as $(xK_5 \vee K_1) \cup K_{x\times 5}$. Note that, by convention, the union of two graphs implies the graphs are edge-disjoint, but not necessarily vertex-disjoint.

Let H be a graph and let $\mathcal{G}$ be a set of subgraphs of H. We will refer to a graph $G \in \mathcal{G}$ as a *G-block*. A *$\mathcal{G}$-decomposition of H* is a set $\Delta = \{G_1, G_2, \ldots, G_r\}$ of pairwise edge-disjoint subgraphs of H such that for every $i \in [1, r]$, $G_i \cong G$ for some $G \in \mathcal{G}$ and such that $E(H) = \bigcup_{i=1}^{r} E(G_i)$. Of particular importance is when $\mathcal{G} = \{G\}$, in which case we write "G-decomposition of H" instead of "$\{G\}$-decomposition of H". A G-decomposition of K_n is also known as a *(K_n, G)-design*. The set of all n for which K_n admits a G-decomposition is called the *spectrum of G*. The spectrum has been determined for many classes of graphs, including all graphs on at most 4 vertices [3] and all graphs on 5 vertices (see [4] and [11]). We direct the reader to [2] and [5] for recent surveys on graph decompositions.

A vertex-disjoint collection of blocks of a (K_n, G)-design that contains every vertex of K_n is called a *parallel class*. A (K_n, G)-design is called *resolvable* if the blocks of the design can be partitioned into parallel classes. In particular, a resolvable (K_n, K_3)-design is called a *Kirkman Triple System*, or more concisely a KTS(n).

1.1. DEFINITIONS FOR DIGRAPHS

Similar concepts to those defined above for undirected graphs can be defined for digraphs. First, we introduce additional notation. For an undirected graph G, let G^* denote the digraph obtained from G by replacing each edge $\{u, v\} \in E(G)$ with the arcs (u, v) and (v, u). Thus K_n^*, the *complete digraph of order n*, is the digraph on n vertices with the arcs (u, v) and (v, u) between every pair of distinct vertices u and v.

Let H and D be digraphs such that D is a subgraph of H. A *D-decomposition* of H is a set $\Delta = \{D_1, D_2, \ldots, D_r\}$ of pairwise arc-disjoint subgraphs of H each of which is isomorphic to D and such that $A(H) = \bigcup_{i=1}^{r} A(D_i)$. As with the undirected case, a D-decomposition of K_n^* is also known as a *(K_n^*, D)-design*, and the set of all n for which K_n^* admits a D-decomposition is called the *spectrum of D*.

The spectra for several digraphs of small order have been determined. This includes the spectra for all digraphs on at most 3 vertices [13], all bipartite digraphs on 4 vertices with up to 5 arcs [8], and the orientations of a triangle with a pendent edge [6].

The *paw graph* consists of a K_3 with a pendant edge. We obtain a multigraph when we replace one edge in the paw with a double edge (resulting in one of three non-isomorphic multigraphs). The spectra for these multigraphs are found in [7]. In this paper, we extend the known results on small digraphs by determining the spectra for the orientations of two such multigraphs (Figure 1) and by giving partial results on the orientations of the remaining multigraph (Figure 2). The digraphs under investigation are shown in Figures 1 and 2 along with a key that denotes a labeled copy of each of the 18 digraphs. We use the naming convention found in *An Atlas of Graphs* [14] by Read and Wilson. For example, D68$[w, x, y, z]$ refers to the digraph with vertex set $\{w, x, y, z\}$ and arc set $\big\{(x, w), (x, y), (x, z), (z, x), (z, y)\big\}$.

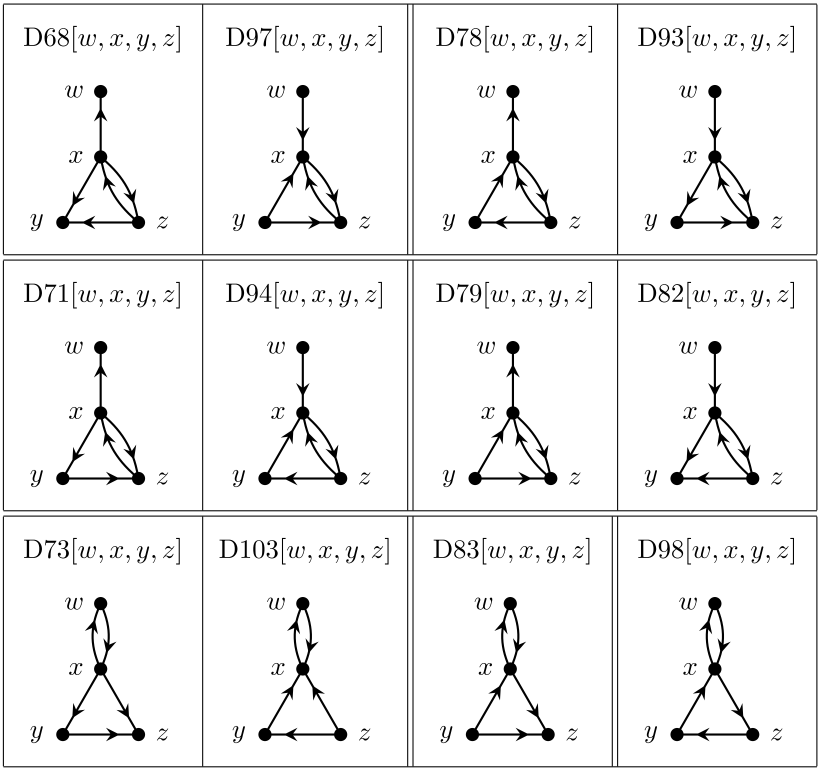

Fig. 1. The 12 digraphs for which we settle the spectrum. Note that these are all possible orientations of the multigraphs obtained from adding a double edge to the paw graph that shares a vertex with the pendent edge

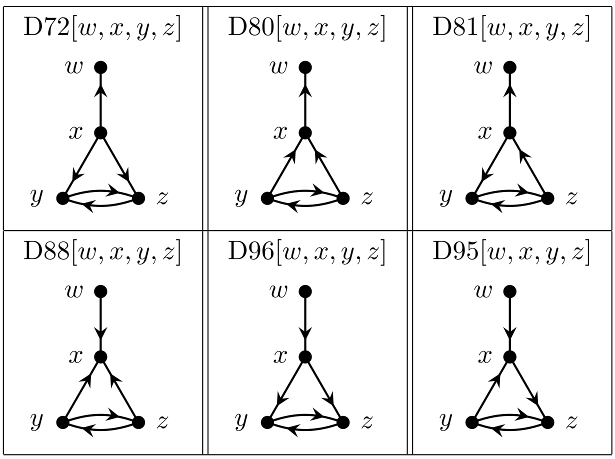

Fig. 2. The 6 digraphs for which we give partial results. Note that these are all possible orientations of the multigraph obtained from adding a double edge to the paw graph that does not share a vertex with the pendent edge

1.2. SOME BASIC RESULTS

The obvious necessary conditions for a digraph D to decompose K_n^* are

(A) $|V(D)| \leq n$,
(B) $|A(D)|$ divides $|A(K_n^*)| = n(n-1)$, and
(C) both $\gcd\{\text{outdegree}(v) : v \in V(D)\}$ and $\gcd\{\text{indegree}(v) : v \in V(D)\}$ divide $n-1$.

Applying these necessary conditions to the 18 digraphs under consideration, we obtain the following necessary conditions:

Lemma 1.1. *For* $D \in \{\text{D68}, \text{D71}, \text{D72}, \text{D73}, \text{D78}, \text{D79}, \text{D80}, \text{D81}, \text{D82}, \text{D83}, \text{D88}, \text{D93}, \text{D94}, \text{D95}, \text{D96}, \text{D97}, \text{D98}, \text{D103}\}$, *a* (K_n^*, D)*-design exists only if* $n \geq 5$ *and* $n \equiv 0$ *or* $1 \pmod 5$.

Given a digraph D, the *reverse orientation of* D, denoted $\text{Rev}(D)$, is the digraph with vertex set $V(D)$ and arc set $\{(v,u) : (u,v) \in A(D)\}$. We make use of the following fact:

Observation 1.2. *Let* D *and* H *be digraphs. a* D*-decomposition of* H *exists if and only if a* $(\text{Rev}(D))$*-decomposition of* $\text{Rev}(H)$ *exists.*

The fact that $K_n^* \cong \text{Rev}(K_n^*)$ leads to the following corollary:

Corollary 1.3. *Let* D *be a digraph. a* (K_n^*, D)*-design exists if and only if a* $(K_n^*, \text{Rev}(D))$*-design exists.*

Note that 16 of the 18 digraphs of interest in this paper occur in pairs with respect to their reverse orientations. Namely,

$$\begin{array}{lll} \text{D68} \cong \text{Rev(D97)}, & \text{D71} \cong \text{Rev(D94)}, & \text{D72} \cong \text{Rev(D88)}, \\ \text{D73} \cong \text{Rev(D103)}, & \text{D78} \cong \text{Rev(D93)}, & \text{D79} \cong \text{Rev(D82)}, \\ \text{D80} \cong \text{Rev(D96)}, & \text{D81} \cong \text{Rev(D95)}. & \end{array}$$

The other 2 digraphs in this paper are reverse orientations of themselves. Namely, $\text{D83} \cong \text{Rev(D83)}$ and $\text{D98} \cong \text{Rev(D98)}$.

The following theorems on decompositions of complete graphs and complete multipartite graphs are used extensively in proving our main results. Note that these background results concern graphs, as opposed to digraphs. All of these results can be found in the *Handbook of Combinatorial Designs* [9] (see for example [1] and [10]).

Theorem 1.4. *If* $n \equiv 3 \pmod 6$, *then a* $KTS(n)$ *exists.*

Theorem 1.5. *If* $n \equiv 0 \pmod 6$, *then a* $(K_n - I, K_3)$*-design exists, where* I *is a 1-factor in* K_n.

Theorem 1.6. *a* $\{K_3, K_4\}$*-decomposition of* K_n *exists if and only if* $n \equiv 0$ *or* $1 \pmod 3$ *and* $n \neq 6$.

Theorem 1.7. *If* n *is an odd positive integer, then there exists a* $\{K_3, K_5\}$*-decomposition of* K_n.

Theorem 1.8. *The necessary and sufficient conditions for the existence of a K_3-decomposition of $K_{u\times m}$ are*

(i) $u \geq 3$,
(ii) $(u-1)m \equiv 0 \pmod 2$, *and*
(iii) $u(u-1)m^2 \equiv 0 \pmod 6$.

The following is a well-known result that is a special case of Wilson's Fundamental Construction (see [12]).

Theorem 1.9. *If there exists a $(K_{u\times m}, K_r)$-design, then there exists a $(K_{u\times mt}, K_{r\times t})$--design for any positive integer t.*

2. EXAMPLES OF SMALL DESIGNS

We now turn our attention to the designs of small order which will be used for the general constructions.

Given a digraph represented by the notation $D[a,b,c,d]$ and some $i \in \mathbb{Z}_n$, we define

$$D[a,b,c,d] + i = D[a+i, b+i, c+i, d+i]$$

where all addition is performed in $\mathbb{Z}_n$. By convention, define $\infty + 1 = \infty$.

Example 2.1. There exists a (K_5^*, D)-design for $D \in \{\mathrm{D71}, \mathrm{D73}, \mathrm{D78}, \mathrm{D79}, \mathrm{D82}, \mathrm{D83}, \mathrm{D93}, \mathrm{D94}, \mathrm{D98}, \mathrm{D103}\}$.

Let $V(K_5^*) = \mathbb{Z}_4 \cup \{\infty\}$.
A $(K_5^*, \mathrm{D71})$-design is given by $\{\mathrm{D71}[2,0,\infty,1] + i : i \in \mathbb{Z}_4\}$.
A $(K_5^*, \mathrm{D73})$-design is given by $\{\mathrm{D73}[1,0,\infty,2] + i : i \in \mathbb{Z}_4\}$.
A $(K_5^*, \mathrm{D78})$-design is given by $\{\mathrm{D78}[2,0,\infty,1] + i : i \in \mathbb{Z}_4\}$.
A $(K_5^*, \mathrm{D79})$-design is given by

$$\{\mathrm{D79}[3,1,\infty,0], \mathrm{D79}[3,2,1,\infty], \mathrm{D79}[1,3,0,\infty], \mathrm{D79}[1,2,3,0]\}.$$

A $(K_5^*, \mathrm{D83})$-design is given by $\{\mathrm{D83}[\infty,1,0,3] + i : i \in \mathbb{Z}_4\}$.
A $(K_5^*, \mathrm{D98})$-design is given by

$$\{\mathrm{D98}[2,3,\infty,0], \mathrm{D98}[1,3,0,\infty], \mathrm{D98}[0,2,1,\infty], \mathrm{D98}[0,1,2,\infty]\}.$$

Applying Corollary 1.3, we obtain the remaining designs.

Example 2.2. There exists a (K_6^*, D)-design for $D \in \{\mathrm{D68}, \mathrm{D71}, \mathrm{D72}, \mathrm{D73}, \mathrm{D78}, \mathrm{D79}, \mathrm{D80}, \mathrm{D81}, \mathrm{D82}, \mathrm{D83}, \mathrm{D88}, \mathrm{D93}, \mathrm{D94}, \mathrm{D95}, \mathrm{D96}, \mathrm{D97}, \mathrm{D98}, \mathrm{D103}\}$.

Let $V(K_6^*) = \mathbb{Z}_6$.
A $(K_6^*, \mathrm{D68})$-design is given by $\{\mathrm{D68}[5,0,3,2] + i : i \in \mathbb{Z}_6\}$.
A $(K_6^*, \mathrm{D72})$-design is given by $\{\mathrm{D72}[4,0,2,3] + i : i \in \mathbb{Z}_6\}$.
A $(K_6^*, \mathrm{D71})$-design is given by $\{\mathrm{D71}[1,0,5,2] + i : i \in \mathbb{Z}_6\}$.
A $(K_6^*, \mathrm{D73})$-design is given by

$$\{\mathrm{D73}[5,2,0,1], \mathrm{D73}[5,1,0,2], \mathrm{D73}[3,0,4,5], \mathrm{D73}[3,5,4,0], \mathrm{D73}[1,4,2,3], \mathrm{D73}[1,3,2,4]\}.$$

A $(K_6^*, \mathrm{D78})$-design is given by $\{\mathrm{D78}[4,0,3,1]+i : i \in \mathbb{Z}_6\}$.
A $(K_6^*, \mathrm{D79})$-design is given by $\{\mathrm{D79}[3,0,1,2]+i : i \in \mathbb{Z}_6\}$.
A $(K_6^*, \mathrm{D80})$-design is given by $\{\mathrm{D80}[3,0,1,5]+i : i \in \mathbb{Z}_6\}$.
A $(K_6^*, \mathrm{D81})$-design is given by $\{\mathrm{D81}[1,0,5,3]+i : i \in \mathbb{Z}_6\}$.
A $(K_6^*, \mathrm{D83})$-design is given by

$$\{\mathrm{D83}[3,2,4,1], \mathrm{D83}[5,2,1,4], \mathrm{D83}[4,5,3,1],\\ \mathrm{D83}[0,5,1,3], \mathrm{D83}[2,0,4,3], \mathrm{D83}[1,0,3,4]\}.$$

A $(K_6^*, \mathrm{D98})$-design is given by

$$\{\mathrm{D98}[0,5,4,1], \mathrm{D98}[3,1,4,5], \mathrm{D98}[0,3,2,5],\\ \mathrm{D98}[1,2,3,5], \mathrm{D98}[1,0,4,2], \mathrm{D98}[3,4,0,2]\}.$$

Applying Corollary 1.3, we obtain the remaining designs.

Example 2.3. There exists a (K_{10}^*, D)-design for $D \in \{\mathrm{D68}, \mathrm{D71}, \mathrm{D73}, \mathrm{D78}, \mathrm{D79}, \mathrm{D82}, \mathrm{D83}, \mathrm{D93}, \mathrm{D94}, \mathrm{D97}, \mathrm{D98}, \mathrm{D103}\}$.

Let $V(K_{10}^*) = \mathbb{Z}_9 \cup \{\infty\}$.
A $(K_{10}^*, \mathrm{D68})$-design is given by

$$\{\mathrm{D68}[3,2,4,0]+3i : i \in \mathbb{Z}_3\} \cup \{\mathrm{D68}[6,0,8,5]+3i : i \in \mathbb{Z}_3\}\\ \cup \{\mathrm{D68}[1,4,0,6]+3i : i \in \mathbb{Z}_3\} \cup \{\mathrm{D68}[7,4,5,8]+3i : i \in \mathbb{Z}_3\}\\ \cup \{\mathrm{D68}[5,7,\infty,6]+3i : i \in \mathbb{Z}_3\} \cup \{\mathrm{D68}[0,\infty,7,8]+3i : i \in \mathbb{Z}_3\}.$$

A $(K_{10}^*, \mathrm{D71})$-design is given by

$$\{\mathrm{D71}[4,0,\infty,7]+i : i \in \mathbb{Z}_9\} \cup \{\mathrm{D71}[6,0,3,8]+i : i \in \mathbb{Z}_9\}.$$

A $(K_{10}^*, \mathrm{D73})$-design is given by

$$\{\mathrm{D73}[6,0,\infty,7]+i : i \in \mathbb{Z}_9\} \cup \{\mathrm{D73}[4,0,8,1]+i : i \in \mathbb{Z}_9\}.$$

A $(K_{10}^*, \mathrm{D78})$-design is given by

$$\{\mathrm{D78}[4,0,\infty,2]+i : i \in \mathbb{Z}_9\} \cup \{\mathrm{D78}[8,0,4,3]+i : i \in \mathbb{Z}_9\}.$$

A $(K_{10}^*, \mathrm{D79})$-design is given by

$$\{\mathrm{D79}[8,0,4,2]+3i : i \in \mathbb{Z}_3\} \cup \{\mathrm{D79}[7,4,5,8]+3i : i \in \mathbb{Z}_3\}\\ \cup \{\mathrm{D79}[1,4,0,6]+3i : i \in \mathbb{Z}_3\} \cup \{\mathrm{D79}[1,8,7,\infty]+3i : i \in \mathbb{Z}_3\}\\ \cup \{\mathrm{D79}[0,6,\infty,7]+3i : i \in \mathbb{Z}_3\} \cup \{\mathrm{D79}[\infty,0,8,5]+3i : i \in \mathbb{Z}_3\}.$$

A $(K_{10}^*, \mathrm{D83})$-design is given by

$$\{\mathrm{D83}[\infty,0,5,2]+i : i \in \mathbb{Z}_9\} \cup \{\mathrm{D83}[1,0,2,5]+i : i \in \mathbb{Z}_9\}.$$

A $(K_{10}^*, \mathrm{D98})$-design is given by

$$\{\mathrm{D98}[\infty,0,5,2]+i : i \in \mathbb{Z}_9\} \cup \{\mathrm{D98}[1,0,2,5]+i : i \in \mathbb{Z}_9\}.$$

Applying Corollary 1.3, we obtain the remaining designs.

Example 2.4. There exists a (K_{11}^*, D)-design for $D \in \{\mathrm{D68}, \mathrm{D71}, \mathrm{D72}, \mathrm{D73}, \mathrm{D78}, \mathrm{D79}, \mathrm{D80}, \mathrm{D81}, \mathrm{D82}, \mathrm{D83}, \mathrm{D88}, \mathrm{D93}, \mathrm{D94}, \mathrm{D95}, \mathrm{D96}, \mathrm{D97}, \mathrm{D98}, \mathrm{D103}\}$.

Let $V(K_{11}^*) = \mathbb{Z}_{11}$.
A $(K_{11}^*, \mathrm{D68})$-design is given by
$$\{\mathrm{D68}[2,0,9,5]+i : i \in \mathbb{Z}_{11}\} \cup \{\mathrm{D68}[1,0,10,3]+i : i \in \mathbb{Z}_{11}\}.$$
A $(K_{11}^*, \mathrm{D71})$-design is given by
$$\{\mathrm{D71}[6,0,2,7]+i : i \in \mathbb{Z}_{11}\} \cup \{\mathrm{D71}[1,0,10,8]+i : i \in \mathbb{Z}_{11}\}.$$
A $(K_{11}^*, \mathrm{D72})$-design is given by
$$\{\mathrm{D72}[3,0,5,6]+i : i \in \mathbb{Z}_{11}\} \cup \{\mathrm{D72}[8,0,2,9]+i : i \in \mathbb{Z}_{11}\}.$$
A $(K_{11}^*, \mathrm{D73})$-design is given by
$$\{\mathrm{D73}[4,0,3,2]+i : i \in \mathbb{Z}_{11}\} \cup \{\mathrm{D73}[5,0,8,9]+i : i \in \mathbb{Z}_{11}\}.$$
A $(K_{11}^*, \mathrm{D78})$-design is given by
$$\{\mathrm{D78}[3,0,1,5]+i : i \in \mathbb{Z}_{11}\} \cup \{\mathrm{D78}[4,0,3,2]+i : i \in \mathbb{Z}_{11}\}.$$
A $(K_{11}^*, \mathrm{D79})$-design is given by
$$\{\mathrm{D79}[3,0,2,4]+i : i \in \mathbb{Z}_{11}\} \cup \{\mathrm{D79}[8,0,6,1]+i : i \in \mathbb{Z}_{11}\}.$$
A $(K_{11}^*, \mathrm{D80})$-design is given by
$$\{\mathrm{D80}[8,0,5,6]+i : i \in \mathbb{Z}_{11}\} \cup \{\mathrm{D80}[3,0,2,9]+i : i \in \mathbb{Z}_{11}\}.$$
A $(K_{11}^*, \mathrm{D81})$-design is given by
$$\{\mathrm{D81}[6,0,1,4]+i : i \in \mathbb{Z}_{11}\} \cup \{\mathrm{D81}[10,0,4,6]+i : i \in \mathbb{Z}_{11}\}.$$
A $(K_{11}^*, \mathrm{D83})$-design is given by
$$\{\mathrm{D83}[5,0,3,1]+i : i \in \mathbb{Z}_{11}\} \cup \{\mathrm{D83}[4,0,1,3]+i : i \in \mathbb{Z}_{11}\}.$$
A $(K_{11}^*, \mathrm{D98})$-design is given by
$$\{\mathrm{D98}[5,0,3,1]+i : i \in \mathbb{Z}_{11}\} \cup \{\mathrm{D98}[4,0,1,3]+i : i \in \mathbb{Z}_{11}\}.$$
Applying Corollary 1.3, we obtain the remaining designs.

Example 2.5. There exists a $(K_{3\times5}^*, D)$-design for $D \in \{\mathrm{D68}, \mathrm{D71}, \mathrm{D72}, \mathrm{D73}, \mathrm{D78}, \mathrm{D79}, \mathrm{D80}, \mathrm{D81}, \mathrm{D82}, \mathrm{D83}, \mathrm{D88}, \mathrm{D93}, \mathrm{D94}, \mathrm{D95}, \mathrm{D96}, \mathrm{D97}, \mathrm{D98}, \mathrm{D103}\}$.

Let $V(K_{3\times5}^*) = \mathbb{Z}_{15}$ with vertex partition $\{V_i : i \in \mathbb{Z}_3\}$, where
$$V_i = \{j \in \mathbb{Z}_{15} : j \equiv i \pmod 3\}.$$
A $(K_{3\times5}^*, \mathrm{D68})$-design is given by
$$\{\mathrm{D68}[1,0,7,2]+i : i \in \mathbb{Z}_{15}\} \cup \{\mathrm{D68}[8,0,14,4]+i : i \in \mathbb{Z}_{15}\}.$$
A $(K_{3\times5}^*, \mathrm{D71})$-design is given by
$$\{\mathrm{D71}[7,0,10,11]+i : i \in \mathbb{Z}_{15}\} \cup \{\mathrm{D71}[14,0,5,13]+i : i \in \mathbb{Z}_{15}\}.$$
A $(K_{3\times5}^*, \mathrm{D72})$-design is given by
$$\{\mathrm{D72}[11,0,4,5]+i : i \in \mathbb{Z}_{15}\} \cup \{\mathrm{D72}[7,0,8,10]+i : i \in \mathbb{Z}_{15}\}.$$
A $(K_{3\times5}^*, \mathrm{D73})$-design is given by
$$\{\mathrm{D73}[2,0,11,10]+i : i \in \mathbb{Z}_{15}\} \cup \{\mathrm{D73}[7,0,1,5]+i : i \in \mathbb{Z}_{15}\}.$$

A $(K^*_{3\times 5}, \mathrm{D78})$-design is given by

$$\{\mathrm{D78}[14,0,5,4]+i : i\in\mathbb{Z}_{15}\}\cup\{\mathrm{D78}[7,0,10,2]+i : i\in\mathbb{Z}_{15}\}.$$

A $(K^*_{3\times 5}, \mathrm{D79})$-design is given by

$$\{\mathrm{D79}[5,0,4,8]+i : i\in\mathbb{Z}_{15}\}\cup\{\mathrm{D79}[10,0,1,2]+i : i\in\mathbb{Z}_{15}\}.$$

A $(K^*_{3\times 5}, \mathrm{D80})$-design is given by

$$\{\mathrm{D80}[8,0,4,5]+i : i\in\mathbb{Z}_{15}\}\cup\{\mathrm{D80}[4,0,8,10]+i : i\in\mathbb{Z}_{15}\}.$$

A $(K^*_{3\times 5}, \mathrm{D81})$-design is given by

$$\{\mathrm{D81}[11,0,5,13]+i : i\in\mathbb{Z}_{15}\}\cup\{\mathrm{D81}[13,0,10,11]+i : i\in\mathbb{Z}_{15}\}.$$

A $(K^*_{3\times 5}, \mathrm{D83})$-design is given by

$$\{\mathrm{D83}[2,0,5,1]+i : i\in\mathbb{Z}_{15}\}\cup\{\mathrm{D83}[7,0,1,5]+i : i\in\mathbb{Z}_{15}\}.$$

A $(K^*_{3\times 5}, \mathrm{D98})$-design is given by

$$\{\mathrm{D98}[2,0,5,1]+i : i\in\mathbb{Z}_{15}\}\cup\{\mathrm{D98}[7,0,1,5]+i : i\in\mathbb{Z}_{15}\}.$$

Applying Corollary 1.3, we obtain the remaining designs.

Example 2.6. There exists a $(K^*_{4\times 5}, D)$-design for $D \in \{\mathrm{D68}, \mathrm{D71}, \mathrm{D72}, \mathrm{D73}, \mathrm{D78}, \mathrm{D79}, \mathrm{D80}, \mathrm{D81}, \mathrm{D82}, \mathrm{D83}, \mathrm{D88}, \mathrm{D93}, \mathrm{D94}, \mathrm{D95}, \mathrm{D96}, \mathrm{D97}, \mathrm{D98}, \mathrm{D103}\}$.

Let $V(K^*_{4\times 5}) = \mathbb{Z}_{20}$ with vertex partition $\{V_i : i \in \mathbb{Z}_4\}$, where

$$V_i = \{j \in \mathbb{Z}_{20} : j \equiv i \pmod 4\}.$$

A $(K^*_{4\times 5}, \mathrm{D68})$-design is given by

$$\{\mathrm{D68}[13,0,18,1]+i : i\in\mathbb{Z}_{20}\}\cup\{\mathrm{D68}[11,0,7,5]+i : i\in\mathbb{Z}_{20}\}$$
$$\cup\{\mathrm{D68}[10,0,9,6]+i : i\in\mathbb{Z}_{20}\}.$$

A $(K^*_{4\times 5}, \mathrm{D71})$-design is given by

$$\{\mathrm{D71}[3,0,6,19]+i : i\in\mathbb{Z}_{20}\}\cup\{\mathrm{D71}[10,0,2,9]+i : i\in\mathbb{Z}_{20}\}$$
$$\cup\{\mathrm{D71}[14,0,17,15]+i : i\in\mathbb{Z}_{20}\}.$$

A $(K^*_{4\times 5}, \mathrm{D72})$-design is given by

$$\{\mathrm{D72}[18,0,5,14]+i : i\in\mathbb{Z}_{20}\}\cup\{\mathrm{D72}[6,0,2,3]+i : i\in\mathbb{Z}_{20}\}$$
$$\cup\{\mathrm{D72}[15,0,10,17]+i : i\in\mathbb{Z}_{20}\}.$$

A $(K^*_{4\times 5}, \mathrm{D73})$-design is given by

$$\{\mathrm{D73}[3,0,10,5]+i : i\in\mathbb{Z}_{20}\}\cup\{\mathrm{D73}[6,0,11,9]+i : i\in\mathbb{Z}_{20}\}$$
$$\cup\{\mathrm{D73}[7,0,19,1]+i : i\in\mathbb{Z}_{20}\}.$$

A $(K^*_{4\times 5}, \mathrm{D78})$-design is given by

$$\{\mathrm{D78}[9,0,3,1]+i : i\in\mathbb{Z}_{20}\}\cup\{\mathrm{D78}[6,0,10,7]+i : i\in\mathbb{Z}_{20}\}$$
$$\cup\{\mathrm{D78}[18,0,6,15]+i : i\in\mathbb{Z}_{20}\}.$$

A $(K^*_{4\times 5}, \mathrm{D79})$-design is given by

$$\{\mathrm{D79}[15,0,9,3]+i : i\in\mathbb{Z}_{20}\}\cup\{\mathrm{D79}[10,0,13,2]+i : i\in\mathbb{Z}_{20}\}$$
$$\cup\{\mathrm{D79}[13,0,15,1]+i : i\in\mathbb{Z}_{20}\}.$$

A $(K^*_{4\times 5}, \mathrm{D80})$-design is given by

$$\{\mathrm{D80}[2,0,5,14]+i : i \in \mathbb{Z}_{20}\} \cup \{\mathrm{D80}[14,0,2,3]+i : i \in \mathbb{Z}_{20}\}$$
$$\cup\,\{\mathrm{D80}[5,0,10,17]+i : i \in \mathbb{Z}_{20}\}.$$

A $(K^*_{4\times 5}, \mathrm{D81})$-design is given by

$$\{\mathrm{D81}[11,0,1,3]+i : i \in \mathbb{Z}_{20}\} \cup \{\mathrm{D81}[15,0,3,10]+i : i \in \mathbb{Z}_{20}\}$$
$$\cup\,\{\mathrm{D81}[19,0,9,15]+i : i \in \mathbb{Z}_{20}\}.$$

A $(K^*_{4\times 5}, \mathrm{D83})$-design is given by

$$\{\mathrm{D83}[1,0,10,5]+i : i \in \mathbb{Z}_{20}\} \cup \{\mathrm{D83}[2,0,17,6]+i : i \in \mathbb{Z}_{20}\}$$
$$\cup\,\{\mathrm{D83}[7,0,3,14]+i : i \in \mathbb{Z}_{20}\}.$$

A $(K^*_{4\times 5}, \mathrm{D98})$-design is given by

$$\{\mathrm{D98}[3,1,7,6]+4i : i \in \mathbb{Z}_{5}\} \cup \{\mathrm{D98}[19,1,6,7]+4i : i \in \mathbb{Z}_{5}\}$$
$$\cup\,\{\mathrm{D98}[3,8,9,6]+4i : i \in \mathbb{Z}_{5}\} \cup \{\mathrm{D98}[15,8,6,9]+4i : i \in \mathbb{Z}_{5}\}$$
$$\cup\,\{\mathrm{D98}[13,4,15,1]+4i : i \in \mathbb{Z}_{5}\} \cup \{\mathrm{D98}[17,4,1,15]+4i : i \in \mathbb{Z}_{5}\}$$
$$\cup\,\{\mathrm{D98}[17,18,5,15]+4i : i \in \mathbb{Z}_{5}\} \cup \{\mathrm{D98}[3,18,15,5]+4i : i \in \mathbb{Z}_{5}\}$$
$$\cup\,\{\mathrm{D98}[14,12,6,17]+4i : i \in \mathbb{Z}_{5}\} \cup \{\mathrm{D98}[2,12,17,6]+4i : i \in \mathbb{Z}_{5}\}$$
$$\cup\,\{\mathrm{D98}[2,11,18,12]+4i : i \in \mathbb{Z}_{5}\} \cup \{\mathrm{D98}[8,11,12,18]+4i : i \in \mathbb{Z}_{5}\}.$$

Applying Corollary 1.3, we obtain the remaining designs.

Example 2.7. There exists a $(K^*_{5\times 5}, D)$-design for $D \in \{\mathrm{D68}, \mathrm{D71}, \mathrm{D72}, \mathrm{D73}, \mathrm{D78}, \mathrm{D79}, \mathrm{D80}, \mathrm{D81}, \mathrm{D82}, \mathrm{D83}, \mathrm{D88}, \mathrm{D93}, \mathrm{D94}, \mathrm{D95}, \mathrm{D96}, \mathrm{D97}, \mathrm{D98}, \mathrm{D103}\}$.

Let $V(K^*_{5\times 5}) = \mathbb{Z}_{25}$ with vertex partition $\{V_i : i \in \mathbb{Z}_5\}$, where

$$V_i = \{j \in \mathbb{Z}_{25} : j \equiv i \pmod 5\}.$$

A $(K^*_{5\times 5}, \mathrm{D68})$-design is given by

$$\{\mathrm{D68}[6,0,9,2]+i : i \in \mathbb{Z}_{25}\} \cup \{\mathrm{D68}[19,0,13,1]+i : i \in \mathbb{Z}_{25}\}$$
$$\cup\,\{\mathrm{D68}[16,0,14,3]+i : i \in \mathbb{Z}_{25}\} \cup \{\mathrm{D68}[18,0,21,17]+i : i \in \mathbb{Z}_{25}\}.$$

A $(K^*_{5\times 5}, \mathrm{D71})$-design is given by

$$\{\mathrm{D71}[24,0,6,9]+i : i \in \mathbb{Z}_{25}\} \cup \{\mathrm{D71}[19,0,7,11]+i : i \in \mathbb{Z}_{25}\}$$
$$\cup\,\{\mathrm{D71}[18,0,22,23]+i : i \in \mathbb{Z}_{25}\} \cup \{\mathrm{D71}[8,0,21,13]+i : i \in \mathbb{Z}_{25}\}.$$

A $(K^*_{5\times 5}, \mathrm{D72})$-design is given by

$$\{\mathrm{D72}[1,0,2,23]+i : i \in \mathbb{Z}_{25}\} \cup \{\mathrm{D72}[24,0,9,16]+i : i \in \mathbb{Z}_{25}\}$$
$$\cup\,\{\mathrm{D72}[8,0,11,14]+i : i \in \mathbb{Z}_{25}\} \cup \{\mathrm{D72}[17,0,6,19]+i : i \in \mathbb{Z}_{25}\}.$$

A $(K^*_{5\times 5}, \mathrm{D73})$-design is given by

$$\{\mathrm{D73}[7,0,22,21]+i : i \in \mathbb{Z}_{25}\} \cup \{\mathrm{D73}[9,0,1,4]+i : i \in \mathbb{Z}_{25}\}$$
$$\cup\,\{\mathrm{D73}[11,0,19,17]+i : i \in \mathbb{Z}_{25}\} \cup \{\mathrm{D73}[12,0,6,8]+i : i \in \mathbb{Z}_{25}\}.$$

A $(K^*_{5\times 5}, \mathrm{D78})$-design is given by

$$\{\mathrm{D78}[4,0,3,9]+i : i\in\mathbb{Z}_{25}\}\cup\{\mathrm{D78}[6,0,4,11]+i : i\in\mathbb{Z}_{25}\}$$
$$\cup\{\mathrm{D78}[12,0,18,17]+i : i\in\mathbb{Z}_{25}\}\cup\{\mathrm{D78}[13,0,22,23]+i : i\in\mathbb{Z}_{25}\}.$$

A $(K^*_{5\times 5}, \mathrm{D79})$-design is given by

$$\{\mathrm{D79}[24,0,6,13]+i : i\in\mathbb{Z}_{25}\}\cup\{\mathrm{D79}[9,0,8,14]+i : i\in\mathbb{Z}_{25}\}$$
$$\cup\{\mathrm{D79}[16,0,3,21]+i : i\in\mathbb{Z}_{25}\}\cup\{\mathrm{D79}[8,0,22,23]+i : i\in\mathbb{Z}_{25}\}.$$

A $(K^*_{5\times 5}, \mathrm{D80})$-design is given by

$$\{\mathrm{D80}[24,0,2,23]+i : i\in\mathbb{Z}_{25}\}\cup\{\mathrm{D80}[1,0,9,16]+i : i\in\mathbb{Z}_{25}\}$$
$$\cup\{\mathrm{D80}[17,0,11,14]+i : i\in\mathbb{Z}_{25}\}\cup\{\mathrm{D80}[8,0,6,19]+i : i\in\mathbb{Z}_{25}\}.$$

A $(K^*_{5\times 5}, \mathrm{D81})$-design is given by

$$\{\mathrm{D81}[18,0,9,12]+i : i\in\mathbb{Z}_{25}\}\cup\{\mathrm{D81}[12,0,6,8]+i : i\in\mathbb{Z}_{25}\}$$
$$\cup\{\mathrm{D81}[19,0,7,11]+i : i\in\mathbb{Z}_{25}\}\cup\{\mathrm{D81}[11,0,8,9]+i : i\in\mathbb{Z}_{25}\}.$$

A $(K^*_{5\times 5}, \mathrm{D83})$-design is given by

$$\{\mathrm{D83}[9,0,4,1]+i : i\in\mathbb{Z}_{25}\}\cup\{\mathrm{D83}[7,0,1,4]+i : i\in\mathbb{Z}_{25}\}$$
$$\cup\{\mathrm{D83}[12,0,8,2]+i : i\in\mathbb{Z}_{25}\}\cup\{\mathrm{D83}[11,0,2,8]+i : i\in\mathbb{Z}_{25}\}.$$

A $(K^*_{5\times 5}, \mathrm{D98})$-design is given by

$$\{\mathrm{D98}[9,0,4,1]+i : i\in\mathbb{Z}_{25}\}\cup\{\mathrm{D98}[7,0,1,4]+i : i\in\mathbb{Z}_{25}\}$$
$$\cup\{\mathrm{D98}[12,0,8,2]+i : i\in\mathbb{Z}_{25}\}\cup\{\mathrm{D98}[11,0,2,8]+i : i\in\mathbb{Z}_{25}\}.$$

Applying Corollary 1.3, we obtain the remaining designs.

Example 2.8. There exists a $(K^*_{6\times 5}, D)$-design for $D\in\{\mathrm{D68},\mathrm{D97}\}$.

Let $V(K^*_{6\times 5})=\mathbb{Z}_{30}$ with vertex partition $\{V_i : i\in\mathbb{Z}_6\}$, where

$$V_i=\{j\in\mathbb{Z}_{30} : j\equiv i \pmod 6\}.$$

A $(K^*_{6\times 5}, \mathrm{D68})$-design is given by

$$\{\mathrm{D68}[3,0,25,20]+i : i\in\mathbb{Z}_{30}\}\cup\{\mathrm{D68}[27,0,23,16]+i : i\in\mathbb{Z}_{30}\}$$
$$\cup\{\mathrm{D68}[9,0,29,28]+i : i\in\mathbb{Z}_{30}\}\cup\{\mathrm{D68}[21,0,17,4]+i : i\in\mathbb{Z}_{30}\}$$
$$\cup\{\mathrm{D68}[15,0,19,8]+i : i\in\mathbb{Z}_{30}\}.$$

Applying Corollary 1.3, we obtain a $(K^*_{6\times 5}, \mathrm{D97})$-design.

Example 2.9. There exists a $(K^*_{3\times 10}, D)$-design for $D\in\{\mathrm{D68},\mathrm{D71},\mathrm{D73},\mathrm{D78},\mathrm{D79},\mathrm{D82},\mathrm{D83},\mathrm{D93},\mathrm{D94},\mathrm{D97},\mathrm{D98},\mathrm{D103}\}$.

First, let $D\in\{\mathrm{D68},\mathrm{D71},\mathrm{D73},\mathrm{D78},\mathrm{D79},\mathrm{D82},\mathrm{D83},\mathrm{D93},\mathrm{D94},\mathrm{D97},\mathrm{D98},\mathrm{D103}\}$. By Theorem 1.8, there exists a $(K_{3\times 2}, K_3)$-design. Furthermore, by Theorem 1.9, there exists a $(K_{3\times 10}, K_{3\times 5})$-design. Thus, a $(K^*_{3\times 10}, K^*_{3\times 5})$-design exists. Since a $(K^*_{3\times 5}, D)$-design exists by Example 2.5, the desired $(K^*_{3\times 10}, D)$-design exists.

Example 2.10. There exists a $((K_{10} \setminus K_5)^*, D)$-design for $D \in \{\text{D68}, \text{D97}\}$.

Let $V((K_{10} \setminus K_5)^*) = \mathbb{Z}_5 \cup \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\}$, where the vertices in the hole are ∞_1, ∞_2, ∞_3, ∞_4, and ∞_5.
A $((K_{10} \setminus K_5)^*, \text{D68})$-design is given by

$$\begin{aligned}\{&\text{D68}[\infty_1, 0, 1, \infty_3], \text{D68}[\infty_3, 2, \infty_4, 0], \text{D68}[2, 3, \infty_5, 0], \text{D68}[2, \infty_2, 4, 0],\\ &\text{D68}[\infty_3, 1, 3, \infty_2], \text{D68}[\infty_4, 1, 0, \infty_5], \text{D68}[2, 1, \infty_1, 4], \text{D68}[0, \infty_1, 1, 2],\\ &\text{D68}[\infty_2, 3, 4, \infty_1], \text{D68}[\infty_5, 4, \infty_2, 2], \text{D68}[4, \infty_5, 3, 2], \text{D68}[2, \infty_4, 0, 4]\\ &\text{D68}[\infty_3, 3, 1, \infty_4], \text{D68}[2, \infty_3, 3, 4]\}.\end{aligned}$$

Applying Corollary 1.3, we obtain a $((K_{10} \setminus K_5)^*, \text{D97})$-design.

Example 2.11. There exists a (K_{15}^*, D)-design for $D \in \{\text{D68}, \text{D97}\}$.

First, partition $V(K_{15}^*)$ into three sets V_0, V_1, and V_2 each of cardinality 5. Next for each $i \in \mathbb{Z}_3$, let G_i be the graph with vertex set $V_i \cup V_{i+1}$ and edge set $\big\{\{u, v\} : u, v \in V_i\big\} \cup \big\{\{u, w\} : u \in V_i, w \in V_{i+1}\big\}$, where subscript addition is done modulo 3. Then $\{G_0, G_1, G_2\}$ is a $(K_{15}^*, (K_{10} \setminus K_5)^*)$-design. Since a $((K_{10} \setminus K_5)^*, D)$-design exists by Example 2.10, the desired (K_{15}^*, D)-design exists.

Example 2.12. There exists a (K_{30}^*, D)-design for $D \in \{\text{D68}, \text{D71}, \text{D73}, \text{D78}, \text{D79}, \text{D82}, \text{D83}, \text{D93}, \text{D94}, \text{D97}, \text{D98}, \text{D103}\}$.

Observe that $K_{30}^* = 3K_{10}^* \cup K_{3\times 10}^*$. Since a (K_{10}^*, D)-design exists by Example 2.3 and a $(K_{3\times 10}^*, D)$-design exists by Example 2.9, the desired (K_{30}^*, D)-design exists.

Example 2.13. There exists a (K_{31}^*, D)-design for $D \in \{\text{D68}, \text{D71}, \text{D72}, \text{D73}, \text{D78}, \text{D79}, \text{D80}, \text{D81}, \text{D82}, \text{D83}, \text{D88}, \text{D93}, \text{D94}, \text{D95}, \text{D96}, \text{D97}, \text{D98}, \text{D103}\}$.

A (K_{31}, K_6)-design can be obtained from a projective plane of order 5. Thus, there exists a (K_{31}^*, K_6^*)-design. Since a (K_6^*, D)-design exists by Example 2.2, the desired (K_{31}^*, D)-design exists.

Example 2.14. There exists a (K_{35}^*, D)-design for $D \in \{\text{D68}, \text{D97}\}$.

Observe that

$$K_{35}^* = (6K_5^* \vee K_5^*) \cup K_{6\times 5}^* = (K_{10}^* \cup \bigcup_{i=1}^{5} (K_{10} \setminus K_5)^*) \cup K_{6\times 5}^*.$$

Since a (K_{10}^*, D)-design exists by Example 2.3, a $((K_{10} \setminus K_5)^*, D)$-design exists by Example 2.10, and a $(K_{6\times 5}^*, D)$-design exists by Example 2.8, the desired (K_{35}^*, D)-design exists.

3. MAIN RESULTS

We finally address the general constructions needed to piece together the small designs mentioned previously to prove (near) sufficiency of the necessary conditions.

Theorem 3.1. *If* $n \equiv 1 \pmod 5$ *and* $n \geq 6$, *then a* (K_n^*, D)*-design exists for* $D \in \{\mathrm{D68}, \mathrm{D71}, \mathrm{D72}, \mathrm{D73}, \mathrm{D78}, \mathrm{D79}, \mathrm{D80}, \mathrm{D81}, \mathrm{D82}, \mathrm{D83}, \mathrm{D88}, \mathrm{D93}, \mathrm{D94}, \mathrm{D95}, \mathrm{D96}, \mathrm{D97}, \mathrm{D98}, \mathrm{D103}\}$.

Proof. Let $D \in \{\mathrm{D68}, \mathrm{D71}, \mathrm{D72}, \mathrm{D73}, \mathrm{D78}, \mathrm{D79}, \mathrm{D80}, \mathrm{D81}, \mathrm{D82}, \mathrm{D83}, \mathrm{D88}, \mathrm{D93}, \mathrm{D94}, \mathrm{D95}, \mathrm{D96}, \mathrm{D97}, \mathrm{D98}, \mathrm{D103}\}$ and let $n = 5x+1$ for some positive integer x. When x is 1, 2, or 6, the result follows from Examples 2.2, 2.4, and 2.13 respectively. The remainder of the proof breaks into three cases.

Case 1. $x \equiv 0$ or $1 \pmod 3$ with $x \geq 3$ and $x \neq 6$.
By Theorem 1.6 there exists a $\{K_3, K_4\}$-decomposition of K_x. Thus, by Theorem 1.9, there exists a $\{K_{3\times 5}, K_{4\times 5}\}$-decomposition of $K_{x\times 5}$. Note that

$$K_{5x+1} = (xK_5 \vee K_1) \cup K_{x\times 5} = K_{x\times 5} \cup \bigcup_{i=1}^{x} K_6.$$

Thus,

$$K_n^* = K_{x\times 5}^* \cup \bigcup_{i=1}^{x} K_6^*.$$

Since there exists a $(K_{3\times 5}^*, D)$-design (by Example 2.5) and there exists a $(K_{4\times 5}^*, D)$-design (by Example 2.6), there exists a $(K_{x\times 5}^*, D)$-design. Since there also exists a (K_6^*, D)-design (by Example 2.2), there exists a (K_n^*, D)-design.

Case 2. $x \equiv 2 \pmod 6$.
Let $x = 6y+2$ for some integer $y \geq 1$. Hence, $n = 10(3y+1)+1$. By Theorem 1.8 there exists a K_3-decomposition of $K_{(3y+1)\times 2}$. Thus, by Theorem 1.9, there exists a $K_{3\times 5}$-decomposition of $K_{(3y+1)\times 10}$. Note that

$$K_{30y+11} = ((3y+1)K_{10} \vee K_1) \cup K_{(3y+1)\times 10} = K_{(3y+1)\times 10} \cup \bigcup_{i=1}^{3y+1} K_{11}.$$

Thus,

$$K_n^* = K_{(3y+1)\times 10}^* \cup \bigcup_{i=1}^{3y+1} K_{11}^*.$$

Since there exists a $(K_{3\times 5}^*, D)$-design (by Example 2.5), there exists a $(K_{(3y+1)\times 10}^*, D)$--design. Since there also exists a (K_{11}^*, D)-design (by Example 2.4), there exists a (K_n^*, D)-design.

Case 3. $x \equiv 5 \pmod 6$.
Let $x = 6y+5$ for some integer $y \geq 0$. Hence, $n = 5(6y+5)+1$. By Theorem 1.7 there exists a $\{K_3, K_5\}$-decomposition of K_{6y+5}. Thus, by Theorem 1.9, there exists a $\{K_{3\times 5}, K_{5\times 5}\}$-decomposition of $K_{(6y+5)\times 5}$. Note that

$$K_{30y+26} = ((6y+5)K_5 \vee K_1) \cup K_{(6y+5)\times 5} = K_{(6y+5)\times 5} \cup \bigcup_{i=1}^{6y+5} K_6.$$

Thus,

$$K_n^* = K_{(6y+5)\times 5}^* \cup \bigcup_{i=1}^{6y+5} K_6^*.$$

Since there exists a $(K_{3\times 5}^*, D)$-design (by Example 2.5) and there exists a $(K_{5\times 5}^*, D)$-design (by Example 2.7), there exists a $(K_{(6y+5)\times 5}^*, D)$-design. Since there also exists a (K_6^*, D)-design (by Example 2.2), there exists a (K_n^*, D)-design. □

Theorem 3.2. *If* $n \equiv 0 \pmod 5$ *with* $n \geq 5$*, then a* (K_n^*, D)*-design exists for* $D \in \{\mathrm{D71}, \mathrm{D73}, \mathrm{D78}, \mathrm{D79}, \mathrm{D82}, \mathrm{D83}, \mathrm{D93}, \mathrm{D94}, \mathrm{D98}, \mathrm{D103}\}$.

Proof. Let $D \in \{\mathrm{D71}, \mathrm{D73}, \mathrm{D78}, \mathrm{D79}, \mathrm{D82}, \mathrm{D83}, \mathrm{D93}, \mathrm{D94}, \mathrm{D98}, \mathrm{D103}\}$ and let $n = 5x$ for some positive integer x. When x is 1, 2, or 6, the result follows from Examples 2.1, 2.3, and 2.12, respectively. The remainder of the proof breaks into three cases similar to those of the proof of Theorem 3.1. We proceed in a similar fashion to the proof of Theorem 3.1 with the exceptions that in Case 1 we now have

$$K_n^* = K_{x\times 5}^* \cup \bigcup_{i=1}^{x} K_5^*,$$

in Case 2 we have

$$K_n^* = K_{(3y+1)\times 10}^* \cup \bigcup_{i=1}^{3y+1} K_{10}^*,$$

and in Case 3 we have

$$K_n^* = K_{(6y+5)\times 5}^* \cup \bigcup_{i=1}^{6y+5} K_5^*.$$

□

Theorem 3.3. *If* $n \equiv 0 \pmod 5$ *with* $n \geq 10$*, then a* (K_n^*, D)*-design exists for* $D \in \{\mathrm{D68}, \mathrm{D97}\}$.

Proof. It must first be proven that no $(K_5^*, \mathrm{D68})$- or $(K_5^*, \mathrm{D97})$-design exists. Let $V(K_5^*) = \{a, b, c, d, e\}$. Suppose for contradiction that there exists a $(K_5^*, \mathrm{D68})$-design with $\mathrm{D68}[a, b, c, d]$ as one of the blocks. Note that the outdegree of each vertex in K_5^* is 4. Furthermore, since outdegree$(b) = 3$ in $\mathrm{D68}[a, b, c, d]$, there must exist a copy of D68 in the design with a vertex whose outdegree is 1. However, this is a contradiction since there are no vertices in D68 with an outdegree of 1. Therefore, a $(K_5^*, \mathrm{D68})$-design does not exist, and by Corollary 1.3, a $(K_5^*, \mathrm{D97})$-design does not exist.

Let $n = 5x$ for some integer $x \geq 2$. When x is 2, 3, 6, or 7, the result follows from Examples 2.3, 2.11, 2.12, and 2.14, respectively. The remainder of the proof breaks into two cases.

Case 1. $x \equiv 1$ or $2 \pmod 3$ with $x \geq 4$ and $x \neq 7$.
By Theorem 1.6, there exists a $\{K_3, K_4\}$-decomposition of K_{x-1}. Thus, by Theorem 1.9, there exists a $\{K_{3\times 5}, K_{4\times 5}\}$-decomposition of $K_{(x-1)\times 5}$. Observe that

$$K_n^* = ((x-1)K_5^* \vee K_5^*) \cup K_{(x-1)\times 5}^* = K_{(x-1)\times 5}^* \cup K_{10}^* \cup \bigcup_{i=2}^{x-1} (K_{10} \setminus K_5)^*.$$

Since there exists a $(K^*_{3\times 5}, D)$-design (by Example 2.5) and there exists a $(K^*_{4\times 5}, D)$-design (by Example 2.6), there exists a $(K^*_{(x-1)\times 5}, D)$-design. Since there also exists a (K^*_{10}, D)-design (by Example 2.3) and a $((K_{10} \setminus K_5)^*, D)$-design (by Example 2.10), there exists a (K^*_n, D)-design.

Case 2. $x \equiv 0 \pmod 3$.
We consider two subcases. Suppose that $x \equiv 3 \pmod 6$. By Theorem 1.4, a KTS(x), i.e. a resolvable (K_x, K_3)-design, exists. After the removal of a parallel class, this becomes a $(K_{\frac{x}{3}\times 3}, K_3)$-design. Thus, by Theorem 1.9, there exists a $(K_{\frac{x}{3}\times 15}, K_{3\times 5})$-design. Note that

$$K^*_n = \frac{x}{3} K^*_{15} \cup K^*_{\frac{x}{3}\times 15}.$$

Since there exists a $(K^*_{3\times 5}, D)$-design (by Example 2.5), there exists a $(K^*_{\frac{x}{3}\times 15}, D)$-design. Since there also exists a (K^*_{15}, D)-design (by Example 2.11), there exists a (K^*_n, D)-design.

Finally, suppose that $x \equiv 0 \pmod 6$. By Theorem 1.5 a $(K_x - I, K_3)$-design exists, where I is a 1-factor in K_x. Equivalently, this is a $(K_{\frac{x}{2}\times 2}, K_3)$-design. Thus, by Theorem 1.9, there exists a $(K_{\frac{x}{2}\times 10}, K_{3\times 5})$-design. Note that

$$K^*_n = \frac{x}{2} K^*_{10} \cup K^*_{\frac{x}{2}\times 10}.$$

Since there exists a $(K^*_{3\times 5}, D)$-design (by Example 2.5), there exists a $(K^*_{\frac{x}{2}\times 10}, D)$-design. Since there also exists a (K^*_{10}, D)-design (by Example 2.3), there exists a (K^*_n, D)-design. □

We combine the results from the previous 3 theorems to show that the necessary conditions in Lemma 1.1 are sufficient for $D \in$ {D68, D71, D73, D78, D79, D82, D83, D93, D94, D97, D98, D103} with the exceptions that neither a $(K^*_5, \text{D68})$-design nor a $(K^*_5, \text{D97})$-design exists. Hence, our main result can be summarized as Theorem 3.4.

Theorem 3.4. *For* $D \in$ {D68, D71, D73, D78, D79, D82, D83, D93, D94, D97, D98, D103}*, there exists a* (K^*_n, D)*-design if and only if* $n \equiv 0$ *or* $1 \pmod 5$ *and* $n \geq 5$ *with the exceptions that neither a* $(K^*_5, \text{D68})$*- nor a* $(K^*_5, \text{D97})$*-design exists.*

For $D \in \{\text{D72}, \text{D80}, \text{D81}, \text{D88}, \text{D95}, \text{D96}\}$, we have shown that a (K^*_n, D)-design exists when $n \equiv 1 \pmod 5$ and $n \geq 6$. As for the case $n \equiv 0 \pmod 5$, the existence of a (K^*_n, D)-design appears to be difficult to determine at this time. Note that the underlying undirected multigraph G for such a digraph D is a K_3 with a pendent double-edge. In [7], it shown that G does not decompose ${}^2K_5$, the complete multigraph of order 5 and edge multiplicity 2. Thus, a (K^*_5, D)-design cannot exist. However, a result of Wilson's [15] ensures that if $n \equiv 0 \pmod 5$ is sufficiently large, then a (K^*_n, D)-design exists.

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