Opuscula Math. 36, no. 6 (2016), 807–818
http://dx.doi.org/10.7494/OpMath.2016.36.6.807 Opuscula Mathematica

# ON THE SPECTRUM OF PERIODIC PERTURBATIONS OF CERTAIN UNBOUNDED JACOBI OPERATORS

Jaouad Sahbani

*Communicated by P.A. Cojuhari*

**Abstract.** It is known that a purely off-diagonal Jacobi operator with coefficients $a_n = n^\alpha$, $\alpha \in (0,1]$, has a purely absolutely continuous spectrum filling the whole real axis. We show that a 2-periodic perturbation of these operators creates a non trivial gap in the spectrum.

**Keywords:** essential spectrum, spectral gap, periodic perturbation.

**Mathematics Subject Classification:** 47A10, 47B36, 39A70.

## 1. INTRODUCTION

In this note we consider Jacobi matrices of the form

$$\mathcal{J}(a_n, b_n) = \begin{pmatrix} b_1 & a_1 & 0 & \cdots \\ a_1 & b_2 & a_2 & \ddots \\ 0 & a_2 & b_3 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}, \quad \text{with} \quad a_n > 0 \quad \text{and} \quad b_n \in \mathbb{R}. \tag{1.1}$$

We will especially focus on the case where the matrix elements $a_n$ and $b_n$ are of the form

$$a_n = n^\alpha + \eta_n \quad \text{for all} \quad n \geq 1 \tag{1.2}$$

$$\eta_{n+2} = \eta_n \quad \text{and} \quad b_{n+2} = b_n \quad \text{for all} \quad n \geq 1, \tag{1.3}$$

where $\alpha \in (0,1]$. Of course up to a translation one may assume without loss of generality that

$$b_1 = -b_2 = b \quad \text{for some} \quad b \in \mathbb{R}. \tag{1.4}$$

 

According to Carleman condition, see [1], $\mathcal{J}(a_n, b_n)$ defines an essentially self-adjoint operator on $l^2 = l^2(\mathbb{N})$. Moreover, it is well known that if $\eta_n = b_n = 0$, then $\mathcal{J}(a_n, b_n) = \mathcal{J}(n^\alpha, 0)$ has a purely absolutely continuous spectrum filling the whole real axis, see for example [6,13] and references therein. Our main result is the following.

**Theorem 1.1.** *The operator $\mathcal{J} = \mathcal{J}(n^\alpha + \eta_n, b_n)$ is essentially self-adjoint and has no essential spectrum in the interval $(-\sqrt{b^2 + (\eta_2 - \eta_1)^2}, \sqrt{b^2 + (\eta_2 - \eta_1)^2})$.*

In particular, if $b = 0$, then we get the following known result obtained for example in [2,4,8,10].

**Corollary 1.2.** *If $b = 0$, then $\sigma_{ess}(\mathcal{J}) = \mathbb{R} \setminus (-|\eta_2 - \eta_1|, |\eta_2 - \eta_1|)$.*

Similarly, a purely diagonal periodic perturbation creates also a non trivial gap:

**Corollary 1.3.** *If $\eta_1 = \eta_2 = 0$, then $\sigma_{ess}(\mathcal{J}) = \mathbb{R} \setminus (-|b|, |b|)$.*

To show these results we transform $\mathcal{J}(a_n, b_n)$ in a Block Jacobi matrix $\mathcal{J}(A_n, B_n)$ acting in the Hilbert space $l^2(\mathbb{N}, \mathbb{C}^2)$, where $A_n$ is a nilpotent matrix. In particular, squaring $\mathcal{J}(A_n, B_n)$ allows one to diagonalize it. This is related and in fact explains deeply the trick used in [2,4]. Notice that no asymptotic analysis of the generalized eigenfunctions is needed here.

## 2. PRELIMINARIES

Recall that the Jacobi matrix (1.1) induces an operator acting in $l^2$ by the difference expression

$$(\tau\psi)_n = a_{n-1}\psi_{n-1} + b_n\psi_n + a_n\psi_{n+1} \quad \text{for all } n \geq 1, \tag{2.1}$$

with $a_n > 0$, $b_n \in \mathbb{R}$ and $\psi_0 = 0$. Let $\mathcal{J}_{min}$ be the restriction of $\tau$ to the subspace $l_0^2$ of sequences with only finitely many non zero coordinates. It is easy to verify that $\mathcal{J}_{min}^* = \mathcal{J}_{max}$, where $\mathcal{J}_{max}$ is the restriction of $\tau$ to $D(\mathcal{J}_{max}) = \{\psi \in l^2 \;/\; \tau\psi \in l^2\}$. Let $\mathcal{J} := \overline{\mathcal{J}_{min}} = \mathcal{J}_{max}^*$ be the closure of $\mathcal{J}_{min}$. Clearly, $\mathcal{J}_{min}$ is essentially self-adjoint on $l_0^2$ if and only if $\mathcal{J}_{max}$ is symmetric. In such a case, $\mathcal{J} = \mathcal{J}_{max}$ will be said essentially self-adjoint on $l_0^2$ and denoted by $\mathcal{J}(a_n, b_n)$ when the coefficients dependence should be stressed. If $\mathcal{J}$ is not essentially self-adjoint on $l_0^2$, then $\mathcal{J}$ has uncountably many self-adjoint extensions and each one has a purely discrete spectrum. For a deeper discussion of the self-adjointness question of Jacobi matrices we refer the reader to [1].

The coming theorem follows from different known results, see [1,3,5–7,11,13].

**Theorem 2.1.** *Let $\alpha > 0$ and assume that $a_n = n^\alpha$ and $b_n = \lambda(a_n + a_{n-1})$. Then the following assertions hold.*

(a) *If $|\lambda| > 1$, then $\mathcal{J}$ is essentially self-adjoint and has no essential spectrum for all $\alpha > 0$.*
(b) *If $|\lambda| < 1$, then $\alpha$ comes in the game as follows:*
   (b1) *if $\alpha > 1$, then $\mathcal{J}$ is not essentially self-adjoint on $l_0^2$;*
   (b2) *if $0 < \alpha \leq 1$, then $\mathcal{J}$ is essentially self-adjoint on $l_0^2$ and has a purely absolutely continuous spectrum filling the whole real axis $\mathbb{R}$.*

(c) *If $\lambda = \pm 1$, then $\mathcal{J}$ is essentially self-adjoint on $l_0^2$ regardless of $\alpha$, but:*
  (c1) *if $\alpha > 2$, then the essential spectrum of $\mathcal{J}$ is empty;*
  (c2) *if $0 < \alpha < 2$, then the spectrum of $\mathcal{J}$ is absolutely continuous and fills the semi-axis $[0, \pm\infty)$ (i.e. $(-\infty, 0]$ if $\lambda = -1$ and $[0, +\infty)$ if $\lambda = 1$);*
  (c3) *if $\alpha = 2$, then the spectrum of $\mathcal{J}$ is purely absolutely continuous and fills the semi-axis $[1/4, \pm\infty)$.*

**Remark 2.2.**

(i) The point (a) is a particular case of Theorem 4.1 of [7] ensuring that $\mathcal{J}(a_n, b_n)$ *is essentially self-adjoint on $l_0^2$ and has no essential spectrum if*

$$\lim_{n\to\infty} a_n = +\infty \quad \text{and} \quad \liminf_{n\to\infty} \frac{b_n^2}{a_n^2 + a_{n-1}^2} > 2. \tag{2.2}$$

(ii) The assertion (b1) extends Theorem 1.5 of page 507 of [1] which only covers the case where $\lambda = 0$, for more details see [13].
(iii) The assertion (b2) can be deduced from [6, 7], see also [13].
(iv) The point (c1) follows from Theorem 8 of [5]. Here we give a quite elementary proof of this assertion based on an explicit computation of the resolvent that we will use elsewhere in this note. More specifically, *assume that*

$$\lim_{n\to\infty} n^2/a_n = 0$$

*and*

$$\beta_n^- = b_n + (a_n + a_{n-1}) \ (\textit{respectively, } \beta_n^+ = b_n - (a_n + a_{n-1})) \quad \textit{is bounded}$$

*then $\mathcal{J}(a_n, b_n)$ is essentially self-adjoint on $l_0^2$ and has no essential spectrum.* Indeed, by using the unitary operator given by $(U\psi)_n = (-1)^n \psi_n$, it is enough to study the minus case. Moreover, by [1, Theorem 1.4, p. 505] and [1, the Corollary, p. 506], the operator $\mathcal{J}$ is self-adjoint and semi-bounded. Recall that, according to Hardy's inequality, the Carleman operator defined by

$$(Cf)_n = \frac{1}{n} \sum_{j=1}^{j=n} f_j \tag{2.3}$$

is bounded in $\mathcal{H}$. Thus the operator defined by

$$(Tf)_n = \frac{1}{\sqrt{a_n}} \sum_{j=1}^{j=n} f_j \tag{2.4}$$

is compact in $\mathcal{H}$ since $n/\sqrt{a_n}$ tends to zero at infinity. Now it is easy to verify that $\mathcal{J} = \mathcal{J}(a_n, b_n)$ is invertible and $\mathcal{J}^{-1} = -T^*T$ which is clearly compact. Hence $\sigma_{ess}(\mathcal{J}) = \emptyset$.
(v) Notice that one may deduce also that if $\{n^2/a_n\}_{n\geq 1}$ is bounded, then $\mathcal{J}^{\pm} = \mathcal{J}(a_n, \pm(a_n + a_{n-1}))$ is invertible so that $0 \notin \sigma(\mathcal{J}^{\pm})$.

The following lemma comes from [13].

**Lemma 2.3.** *The operator $\mathcal{J}(\eta_n, \beta_n)$ is $\mathcal{J}(a_n, b_n)$-compact (i.e. relatively compact with respect to $\mathcal{J}(a_n, b_n)$) provided that*

$$\lim_{n\to\infty} \left|\frac{\eta_n}{a_n}\right| + \left|\eta_n \frac{a_{n-1}}{a_n} - \eta_{n-1}\right| + \left|\eta_n \frac{b_n}{a_n} - \beta_n\right| = 0. \tag{2.5}$$

**Remark 2.4.** If $a_n = n^\alpha$ with $0 < \alpha \le 1$ and $b_n = 0$, then conditions of Lemma 2.3 are equivalent to

$$\lim_{n\to\infty} \left(\left|\frac{\eta_n}{n^\alpha}\right| + |\partial\eta_n| + |\beta_n|\right) = 0. \tag{2.6}$$

In contrast, according to Theorem 1.1, if $\eta_n$ or $\beta_n$ is 2-periodic, then $\mathcal{J}(a_n+\eta_n, b_n+\beta_n)$ has a spectral gap in its essential spectrum while the spectrum of $\mathcal{J}(a_n, b_n)$ fills the whole real axis. This illustrates the sharpness of our Lemma 2.3 .

**Corollary 2.5.** *For $\alpha \in (0,1)$, $\mathcal{J}(n^{2\alpha-1}, \pm 2n^{2\alpha-1})$ is $\mathcal{J}(n^{2\alpha}, \pm(n^{2\alpha} + (n-1)^{2\alpha}))$--compact operator.*

*Proof.* Here $\eta_n = n^{2\alpha-1}$ and $\beta_n = n^{2\alpha-1} + (n-1)^{2\alpha-1}$. Then it is clear that $\frac{\eta_n}{n^{2\alpha}} = n^{2\alpha-2} \to 0$ at infinity. Similarly,

$$\eta_n \frac{(n-1)^\alpha}{n^\alpha} - \eta_{n-1} = n^{\alpha-1}(n-1)^\alpha - (n-1)^{2\alpha-1} = O(n^{2\alpha-2})$$

which tends to zero at infinity. Finally,

$$\eta_n \frac{n^{2\alpha} + (n-1)^{2\alpha}}{n^{2\alpha}} - \beta_n = O(n^{2\alpha-2}).$$

The proof is finished. The minus case is similar. □

**Corollary 2.6.** *Assume that $\lim_{n\to\infty} a_n = +\infty$, $\lim_{n\to\infty} a_{n+1}/a_n = 1$ and $\lim_{n\to\infty} b_n/a_n = 2\lambda$. Then $\mathcal{J}(1, 2\lambda)$ is $\mathcal{J}(a_n, b_n)$-compact.*

The following Proposition does not follow from Lemma 2.3.

**Proposition 2.7.** *The operator $\mathcal{J}(n, \pm(2n-1))$ is $\mathcal{J}(n^2, \pm(n^2 + (n-1)^2))$-compact.*

*Proof.* It is enough to treat the minus case, the plus case is similar. Put $\mathcal{J} = \mathcal{J}(n^2, -(n^2 + (n-1)^2))$. We know that $0 \notin \sigma(J)$ and

$$(\mathcal{J}^{-1}f)_n = -\sum_{k=n}^{\infty} \frac{1}{k^2} \sum_{j=1}^{k} f_j \quad \text{for all} \quad f \in \mathcal{H}.$$

Let us set $V = \mathcal{J}(n, -(2n-1))$ and $f \in l^2(\mathbb{N})$. One has

$$\begin{aligned} -(V\mathcal{J}^{-1}f)_n &= n \sum_{k=n+1}^{\infty} \frac{1}{k^2} \sum_{j=1}^{k} f_j - (2n-1) \sum_{k=n}^{\infty} \frac{1}{k^2} \sum_{j=1}^{k} f_j + (n-1) \sum_{k=n-1}^{\infty} \frac{1}{k^2} \sum_{j=1}^{k} f_j \\ &= \frac{1}{(n-1)n} \sum_{j=1}^{n} f_j - \frac{1}{n-1} f_n \\ &= (DCf)_n - (Df)_n, \end{aligned}$$

where $C$ in the Carleman operator defined by (2.3) and $D$ the diagonal operator defined by $(Df)_1 = 0$ and $(Df)_n = \frac{1}{n-1} f_n$ for all $n \geq 2$. Since $C$ is a bounded operator in $l^2(\mathbb{N})$ and $D$ is a compact one, we immediately deduce that $V\mathcal{J}^{-1} = D(C - I)$ is compact. □

## 3. BLOCK JACOBI MATRICES

Let $\mathcal{H} = l^2(\mathbb{N}, \mathbb{C}^2)$ be the Hilbert space of square summable vector-valued sequences $(\psi_n)_{n\geq 1}$ endowed with the scalar product

$$\langle \phi, \psi \rangle = \sum_{n\in\mathbb{Z}} \langle \phi_n, \psi_n \rangle_{\mathbb{C}^2},$$

where $\langle \cdot, \cdot \rangle_{\mathbb{C}^2}$ is the usual scalar product of $\mathbb{C}^2$. Let $A_n$ and $B_n$ be two sequences of $2 \times 2$ matrices such that $B_n = B_n^*$ for all $n \in \mathbb{Z}$. Here we denote by $T^*$ the adjoint matrix of a given matrix $T$. Let us consider the block Jacobi operator $\mathcal{J} = \mathcal{J}(A_n, B_n)$ acting in $\mathcal{H}$ by

$$(\mathcal{J}\psi)_n = A_{n-1}^* \psi_{n-1} + B_n \psi_n + A_n \psi_{n+1} \quad \text{for all} \quad n \geq 1 \tag{3.1}$$

with $\psi_0 = 0$. In [14] we studied different classes of bounded self-adjoint block Jacobi operators given by (3.1) with applications to some concrete models. In this section we will focus on special unbounded cases. More specifically, we assume that

$$A_n = \begin{pmatrix} 0 & 0 \\ \alpha_n & 0 \end{pmatrix}, \quad B_n = \begin{pmatrix} b & \beta_n \\ \beta_n & -b \end{pmatrix}, \quad n \geq 1, \tag{3.2}$$

where $b \in \mathbb{R}, \alpha_n > 0, \beta_n > 0$ for all $n \geq 1$, and

$$\lim_{n\to+\infty} \alpha_n = \lim_{n\to+\infty} \beta_n = +\infty \quad \text{and} \quad \sum_{n\geq 1} \frac{1}{\alpha_n} = +\infty. \tag{3.3}$$

According to Carleman condition, see [1], $\mathcal{J} = \mathcal{J}(A_n, B_n)$ is an essentially self-adjoint in $\mathcal{H}$.

**Example 3.1.** Let $\mathcal{J}(a_n, b_n)$ be the Jacobi operator defined by (1.1) and $\mathcal{U} : l^2(\mathbb{N}) \to l^2(\mathbb{N}, \mathbb{C}^2)$ defined by $(\mathcal{U}\psi)_n = (\psi_{2n-1}, \psi_{2n})$. It is clear that $\mathcal{U}$ is a unitary operator and

$$\mathcal{U}\mathcal{J}(a_n, b_n)\mathcal{U}^{-1} = \mathcal{J}(A_n, B_n)$$

$$\text{with} \quad A_n = \begin{pmatrix} 0 & 0 \\ a_{2n} & 0 \end{pmatrix}, \quad B_n = \begin{pmatrix} b_{2n-1} & a_{2n-1} \\ a_{2n-1} & b_{2n} \end{pmatrix}, \quad n \geq 1.$$

This explains partially our motivation to study block Jacobi matrices given by (3.1) and (3.2). For example, if $a_n$ and $b_n$ are given by (1.2) and (1.3), then it is enough to study the case where

$$\alpha_n = (2n)^\alpha + \eta_2 \quad \text{and} \quad \beta_n = (2n-1)^\alpha + \eta_1 \quad \text{and} \quad b \in \mathbb{R}.$$

**Proposition 3.2.** *Let*

$$A'_n = \begin{pmatrix} \alpha_n\beta_n & 0 \\ 0 & \alpha_n\beta_{n+1} \end{pmatrix}, \quad B'_n = \begin{pmatrix} b^2 + \alpha_{n-1}^2 + \beta_n^2 & 0 \\ 0 & b^2 + \alpha_n^2 + \beta_n^2 \end{pmatrix}, \qquad n \geq 1$$

*with* $\alpha_0 = 0$. *The operator* $\mathcal{J} = \mathcal{J}(A_n, B_n)$ *is essentially self-adjoint and* $\mathcal{J}^2 = \mathcal{J}(A'_n, B'_n)$.

*Proof.* By direct computations based on the special form of the matrices $A_n$. □

**Corollary 3.3.** *In the representation* $l^2(\mathbb{N}, \mathbb{C}^2) = l^2(\mathbb{N}) \oplus l^2(\mathbb{N})$ *one has*

$$\mathcal{J}^2 = \mathcal{J}_1 \oplus \mathcal{J}_2,$$

*where* (*with* $\alpha_0 = 0$)

$$\mathcal{J}_1 = \mathcal{J}(\alpha_n\beta_n, b^2 + \alpha_{n-1}^2 + \beta_n^2) \quad \text{and} \quad \mathcal{J}_2 = \mathcal{J}(\alpha_n\beta_{n+1}, b^2 + \alpha_n^2 + \beta_n^2).$$

**Example 3.4.** Assume that $\alpha_n = n^\alpha$ and $\beta_n = (n - \frac{1}{2})^\alpha$ for some $\alpha \in (0, 1)$. Then

$$\begin{cases} \alpha_n\beta_n = n^{2\alpha} - \frac{\alpha}{2} n^{2\alpha-1} + \varepsilon_n, \\ \alpha_{n-1}^2 + \beta_n^2 = n^{2\alpha} + (n-1)^{2\alpha} - \alpha n^{2\alpha-1} + \kappa_n, \end{cases}$$

where $\varepsilon_n, \kappa_n \to 0$ at infinity. Hence for some compact operator $K$ one has

$$\mathcal{J}_1 = \mathcal{J}(\alpha_n\beta_n, \alpha_{n-1}^2 + \beta_n^2) = \mathcal{J}(n^{2\alpha}, n^{2\alpha} + (n-1)^{2\alpha}) - \frac{\alpha}{2}\mathcal{J}(n^{2\alpha-1}, 2n^{2\alpha-1}) + K.$$

Since $\mathcal{J}(n^{2\alpha-1}, 2n^{2\alpha-1})$ is $\mathcal{J}(n^{2\alpha}, n^{2\alpha} + (n-1)^{2\alpha})$-compact we deduce that

$$\sigma_{ess}(\mathcal{J}_1) = \sigma_{ess}(\mathcal{J}(n^{2\alpha}, n^{2\alpha} + (n-1)^{2\alpha})).$$

Similarly, we prove that for some compact operator $K'$ one has

$$\mathcal{J}_2 = \mathcal{J}(n^{2\alpha}, n^{2\alpha} + (n-1)^{2\alpha}) + \frac{\alpha}{2}\mathcal{J}(n^{2\alpha-1}, 2n^{2\alpha-1}) + K'$$

so that

$$\sigma_{ess}(\mathcal{J}_2) = \sigma_{ess}(\mathcal{J}(n^{2\alpha}, n^{2\alpha} + (n-1)^{2\alpha})).$$

According to Example 3.1, $\mathcal{J}(A_n, B_n)$ is unitarily equivalent to $2^{-\alpha}\mathcal{J}(n^\alpha, 0)$, so that $\sigma_{ess}(\mathcal{J}(A_n, B_n)) = \mathbb{R}$. Hence,

$$\sigma_{ess}(\mathcal{J}(n^{2\alpha}, n^{2\alpha} + (n-1)^{2\alpha})) = [0, +\infty).$$

This represents a simple proof of the point (c2) of Theorem 2.1 which is related to the trick used by Dombrowski and Pedersen in [3, 4, 11].

**Example 3.5.** Assume that $\alpha_n = n$ and $\beta_n = n - \frac{1}{2}$. Then

$$\begin{cases} \alpha_n\beta_n = n^2 - \frac{1}{2}n, \\ \alpha_{n-1}^2 + \beta_n^2 = n^2 + (n-1)^2 - n + \frac{1}{4}. \end{cases}$$

Hence

$$\mathcal{J}_1 = \mathcal{J}(n^2, n^2 + (n-1)^2) - \frac{1}{2}\mathcal{J}(n, 2n-1) - \frac{1}{4}.$$

Since $\mathcal{J}(n, 2n-1)$ is $\mathcal{J}(n^2, n^2 + (n-1)^2)$-compact, we deduce that

$$\sigma_{ess}(\mathcal{J}_1) = \sigma_{ess}(\mathcal{J}(n^2, n^2 + (n-1)^2) - \frac{1}{4}.$$

Similarly, we prove that

$$\mathcal{J}_2 = \mathcal{J}(n^2, n^2 + (n-1)^2) + \frac{1}{2}\mathcal{J}(n, 2n-1) - \frac{1}{4}$$

so that

$$\sigma_{ess}(\mathcal{J}_2) = \sigma_{ess}(\mathcal{J}(n^2, n^2 + (n-1)^2)) - \frac{1}{4}.$$

As $\sigma_{ess}(\mathcal{J}(A_n, B_n) = \mathbb{R}$, we deduce that

$$\sigma_{ess}(\mathcal{J}(n^2, n^2 + (n-1)^2)) = [1/4\ , +\infty).$$

This represents a simple proof of the point (c2) of Theorem 2.1 which is related also to Dombrowski-Pedersen method's, see [3,4,11].

**Remark 3.6.** Notice that in [13] we studied operators of the form $\mathcal{J}(n^\alpha, n^\alpha + (n-1)^\alpha))$ with $\alpha > 0$. In particular, we proved Mourre estimates for these operators with $\alpha \in (0,1]$. The case where $\alpha \in (1,2)$ is not covered by [13]. Indeed, in this case we were not able to complete our proof for a lack of information on the asymptotic behavior of the Green function of $\mathcal{J}$ that are now available in [9]. In other words, combining [13] and [9] we get a Mourre estimate for $\mathcal{J}(n^\alpha, n^\alpha + (n-1)^\alpha))$ with $\alpha \in (0,2)$. For $\alpha = 2$, no Mourre estimate is known to our knowledge.

**Theorem 3.7.** *Assume that* $\alpha_n = n^\alpha + \eta_2, \beta_n = (n - \frac{1}{2})^\alpha + \eta_1$ *with* $\alpha \in (0,1]$. *Then*

$$\sigma_{ess}(\mathcal{J}^2(A_n, B_n) = [b^2 + (\eta_2 - \eta_1)^2, +\infty).$$

*Proof.* (i) Let us start with case $\alpha = 1$ so that $\alpha_n = n + \eta_2$ and $\beta_n = n - \frac{1}{2} + \eta_1$. Then

$$\begin{cases} \alpha_n\beta_n = n^2 + (\eta_1 + \eta_2 - \frac{1}{2})n + \eta_2(\eta_1 - \frac{1}{2}), \\ \alpha_{n-1}^2 + \beta_n^2 = n^2 + (n-1)^2 + (\eta_1 + \eta_2 - \frac{1}{2})(2n-1) + 2\eta_2(\eta_1 - \frac{1}{2}) + (\eta_1 - \eta_2)^2 - \frac{1}{4} \end{cases}$$

with $\alpha_0 = 0$. Hence $\mathcal{J}_1 = \mathcal{J}_1' - \eta_2^2\langle\cdot, \delta_1\rangle\delta_1$, where $\delta_1 = (1, 0, \cdots) \in l^2$ and

$$\begin{aligned} \mathcal{J}_1' = \mathcal{J}(n^2, n^2 + (n-1)^2) &+ \Big(\eta_1 + \eta_2 - \frac{1}{2}\Big)\mathcal{J}(n, 2n-1) \\ &+ \eta_2\Big(\eta_1 - \frac{1}{2}\Big)\mathcal{J}(1,2) - \frac{1}{4} + b^2 + (\eta_2 - \eta_1)^2. \end{aligned}$$

Since $\mathcal{J}(1,2)$ and $\mathcal{J}(n, 2n-1)$ are $\mathcal{J}(n^2, n^2 + (n-1)^2)$-compact and

$$\sigma_{ess}(\mathcal{J}(n^2, n^2 + (n-1)^2)) = [1/4\ , +\infty),$$

we deduce that
$$\sigma_{ess}(\mathcal{J}_1) = [b^2 + (\eta_2 - \eta_1)^2, +\infty).$$
Similar calculation shows that
$$\mathcal{J}_2 = \mathcal{J}_1 + \mathcal{J}(n, 2n-1) + \eta_2 \mathcal{J}(1,2) + \eta_2^2 \langle \cdot, \delta_1 \rangle \delta_1.$$
It follows that
$$\sigma_{ess}(\mathcal{J}_2) = [b^2 + (\eta_2 - \eta_1)^2, +\infty),$$
which finishes the proof of the desired assertion in this case.

(ii) Assume now that $\alpha \in (0,1)$. We have then
$$\begin{cases} \alpha_n \beta_n = n^{2\alpha} - \frac{\alpha}{2} n^{2\alpha-1} + (\eta_1 + \eta_2) n^\alpha + \eta_1 \eta_2 + \varepsilon_n, \\ \alpha_{n-1}^2 + \beta_n^2 = n^{2\alpha} + (n-1)^{2\alpha} - \alpha n^{2\alpha-1} + 2(\eta_1+\eta_2) n^\alpha + \kappa_n, \end{cases}$$
where $\varepsilon_n, \kappa_n \to 0$ at infinity. Hence for some compact operator $K$ one has
$$\begin{aligned} \mathcal{J}_1 &= \mathcal{J}(n^{2\alpha}, n^{2\alpha} + (n-1)^{2\alpha}) \\ &\quad - \frac{\alpha}{2} \mathcal{J}(n^{2\alpha-1}, 2n^{2\alpha-1}) + (\eta_1+\eta_2) \mathcal{J}(n^\alpha, 2n^\alpha) + b^2 + (\eta_2 - \eta_1)^2 + K. \end{aligned}$$
But $\mathcal{J}(n^\alpha, 2n^\alpha)$ and $\mathcal{J}(n^{2\alpha-1}, 2n^{2\alpha-1})$ are $\mathcal{J}(n^{2\alpha}, n^{2\alpha} + (n-1)^{2\alpha})$-compact we deduce that
$$\sigma_{ess}(\mathcal{J}_1) = [b^2 + (\eta_2 - \eta_1)^2, +\infty).$$
Similarly, we prove that $\sigma_{ess}(\mathcal{J}_2) = [b^2 + (\eta_2 - \eta_1)^2, +\infty)$. The proof is finished. □

## 4. PROOF OF THEOREM 1.1

Since $a_n$ and $b_n$ are given by (1.2) and (1.3), then according to Example 3.1, $\mathcal{J}(a_n, b_n)$ is unitarily equivalent to $2^\alpha \mathcal{J}(A_n, B_n)$, where the coefficients of $A_n$ and $B_n$ are given by
$$\alpha_n = n^\alpha + 2^{-\alpha}\eta_2 \ , \quad \beta_n = \left(n - \frac{1}{2}\right)^\alpha + 2^{-\alpha}\eta_1 \quad \text{and} \quad 2^{-\alpha} b \in \mathbb{R}.$$
The proof can be completed by a direct application of Theorem 3.7.

## 5. PROOF OF COROLLARY 1.2

Since $b = 0$, then according to Theorem 1.1 one has
$$\sigma_{ess}(\mathcal{J}^2(a_n, 0)) = \left[(\eta_2 - \eta_1)^2, +\infty\right).$$
But direct computation shows that $U\mathcal{J}(a_n,0)U^{-1} = -\mathcal{J}(a_n, 0)$, where $U$ is the unitary operator on $l^2$ defined by $(U\psi)_n = (-1)^n \psi_n$. Hence the spectrum of $\mathcal{J}(a_n, 0)$ is symmetric with respect to the origin and the desired equality follows.

## 6. PROOF OF COROLLARY 1.3

Since $\eta_1 = \eta_2 = 0$, then according to Theorem 1.1, one has

$$\sigma_{ess}(\mathcal{J}^2(a_n, b_n)) = \left[b^2, +\infty\right).$$

In this case the spectrum of $\mathcal{J}(a_n, b_n))$ is, in general, not symmetric with respect to the origin, see next Proposition. Nevertheless, we will show that the essential spectrum of $\mathcal{J}(a_n, b_n)$ is symmetric with respect to the origin which is enough for us. Indeed, direct computation shows that the $U\mathcal{J}(a_n, b_n)U^{-1} = -\mathcal{J}(a_n, -b_n)$. In particular,

$$\sigma_{ess}(\mathcal{J}(a_n, -b_n)) = -\sigma_{ess}(\mathcal{J}(a_n, b_n))$$

But

$$\mathcal{J}(a_n, -b_n) = \begin{pmatrix} -b & a_1 & 0 & \ddots \\ a_1 & b & a_2 & \ddots \\ 0 & a_2 & -b & \ddots \\ \ddots & \ddots & \ddots & \ddots \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \ddots \\ 0 & b & a_2 & \ddots \\ 0 & a_2 & -b & \ddots \\ \ddots & \ddots & \ddots & \ddots \end{pmatrix} + \text{finite rank operator.}$$

It follows that

$$\sigma_{ess}(\mathcal{J}(a_n, -b_n)) = \sigma_{ess}(\mathcal{J}(a_{n+1}, b_n)).$$

If $a_n = n^\alpha, \alpha \in (0, 1)$, then $a_{n+1} - a_n \to 0$ at infinity. In particular, the difference $\mathcal{J}(a_{n+1}, b_n) - \mathcal{J}(a_n, b_n)$ is a compact operator. In particular, $\sigma_{ess}(\mathcal{J}(a_n, -b_n)) = \sigma_{ess}(\mathcal{J}(a_n, b_n))$. Hence

$$\sigma_{ess}(\mathcal{J}(a_n, b_n)) = -\sigma_{ess}(\mathcal{J}(a_n, b_n)),$$

which is the desired property.

If $a_n = n$, then $\mathcal{J}(a_{n+1}, b_n) = \mathcal{J}(a_n, b_n) + \mathcal{J}(1, 0)$. But, according to Lemma 2.3, $\mathcal{J}(1, 0)$ is $\mathcal{J}(a_n, b_n)$-compact. In particular,

$$\sigma_{ess}(\mathcal{J}(a_n, -b_n)) = \sigma_{ess}(\mathcal{J}(a_{n+1}, b_n)) = \sigma_{ess}(\mathcal{J}(a_n, b_n).$$

The proof of of Corollary 1.2 is complete.

**Proposition 6.1.** *Assume that* $\alpha_n = n + \eta_2$ *and* $\beta_n = n - \frac{1}{2} + \eta_1$.

(i) *If* $\eta_1 - \frac{1}{2} \geq 0$ *and* $\eta_2 \geq 0$, *then* $\mathcal{J}(A_n, B_n)$ *has at most two eigenvalues in the spectral gap* $G = (-\sqrt{b^2 + (\eta_1 - \eta_2)^2}, \sqrt{b^2 + (\eta_1 - \eta_2)^2})$ *and a purely absolutely continuous spectrum filling* $\mathbb{R} \setminus G$.

(ii) $b$ *is an eigenvalue of* $\mathcal{J}(A_n, B_n)$ *if and only if* $\eta_2 > \eta_1$. *In this case,* $-b$ *is an eigenvalue of* $\mathcal{J}(A_n, B_n)$ *if and only if* $b = 0$.

*Proof.* (i) Recall that in this case $\mathcal{J}_1 = \mathcal{J}_1' - \eta_2^2\langle\cdot,\delta_1\rangle\delta_1$, where

$$\mathcal{J}_1' = \mathcal{J}(n^2, n^2+(n-1)^2) + \Big(\eta_1+\eta_2-\frac{1}{2}\Big)\mathcal{J}(n,2n-1) \\ + \eta_2\Big(\eta_1-\frac{1}{2}\Big)\mathcal{J}(1,2) - \frac{1}{4} + b^2 + (\eta_2-\eta_1)^2.$$

Consider the self-adjoint operator defined on $l^2$ by

$$(iA\psi)_n = n\psi_{n+1} - (n-1)\psi_{n-1}.$$

According to [13], $A$ is $\mathcal{J}(n^2, n^2+(n-1)^2)$-bounded and the commutator

$$[\mathcal{J}_1', i\mathbf{A}] = \mathcal{J}^2(n,2n-1) + 2\Big(\eta_1+\eta_2-\frac{1}{2}\Big)\mathcal{J}(n,2n-1) + \eta_2\Big(\eta_1-\frac{1}{2}\Big)\left(4-\mathcal{J}^2(1,0)\right).$$

But $\mathcal{J}(n,2n-1)$ and $4-\mathcal{J}^2(1,0)$ are positive operators. Hence, according to the Putnam-Kato theorem, $\mathcal{J}_1'$ has a purely absolutely continuous spectrum if $\eta_1 - \frac{1}{2} \geq 0$ and $\eta_2 \geq 0$. In particular, $\mathcal{J}_1$ has a purely absolutely continuous spectrum on $[b^2+(\eta_1-\eta_2)^2, +\infty)$ plus at most a simple eigenvalue in the spectral gap $(-\infty, b^2+(\eta_1-\eta_2)^2)$.

Similarly, we show that $\mathcal{J}_2$ has a purely absolutely continuous spectrum if $\eta_1+\frac{1}{2}\geq 0$ and $\eta_2\geq 0$. Finally, if $\eta_1-\frac{1}{2}\geq 0$ and $\eta_2 \geq 0$, then $\mathcal{J}(A_n,B_n)$ is purely absolutely continuous on $\mathbb{R}\setminus(-\sqrt{b^2+(\eta_1-\eta_2)^2}, \sqrt{b^2+(\eta_1-\eta_2)^2})$ and has at most two eigenvalues in the spectral gap $(-\sqrt{b^2+(\eta_1-\eta_2)^2}, \sqrt{b^2+(\eta_1-\eta_2)^2})$.

In contrast, $b$ is an eigenvalue of $\mathcal{J}(A_n,B_n)$ if and only if there exists an non zero $\psi$ defined by $\psi_n = \begin{pmatrix} x_n \\ y_n \end{pmatrix}$ such that, for all $n \geq 1$

$$\begin{cases} \beta_n y_n & +\alpha_{n-1}y_{n-1} = 0, \\ \beta_n x_n - 2by_n & +\alpha_n x_{n+1} \quad = 0 \end{cases}$$

with $\alpha_0 = 0$. By induction we show that $y_n = 0$ and $x_{n+1} = -(\beta_n/\alpha_n)x_n$, for all $n \geq 1$. Hence, $b$ is an eigenvalue of $\mathcal{J}(A_n,B_n)$ if and only if the sequence defined by $x_1 = 1$ and

$$x_{n+1} = (-1)^n \prod_{i=1}^{n} \frac{\beta_i}{\alpha_i}, \quad n \geq 1, \tag{6.1}$$

is square summable. But according to Gauss's test this is equivalent to $\eta_2 > \eta_1$, since

$$\frac{x_n^2}{x_{n+1}^2} = \frac{\alpha_n^2}{\beta_n^2} = 1 + \frac{2(\eta_2-\eta_1+\frac{1}{2})}{n} + \frac{g_n}{n^2}, \quad \text{as } n\to\infty,$$

for some bounded sequence $g_n$. Notice that in this case the associated eigenvectors are proportional to the vector $\psi$ given by $\psi_n = (x_n, 0)$, for all $n \geq 1$.

Similarly, one may prove that if $-b$ is an eigenvalue of $\mathcal{J}(A_n,B_n)$, then $\eta_2 > \eta_1$ and associated eigenvectors are of the form $\varphi_n = (cx_n, y_n)$, where $x_n$ is given by (6.1). Hence if $b \neq 0$, then $b$ and $-b$ are two distinct eigenvalues of $\mathcal{J}$ so that $\langle\psi,\varphi\rangle = 0$ which implies that $c = 0$ so that $y_n = 0$ and $-b$ is not an eigenvalue of $\mathcal{J}$. □

**Remark 6.2.** Notice that the same argument shows that if $\alpha \in (0,1)$, then $b$ is an eigenvalue of $\mathcal{J}(A_n, B_n)$ if and only if the sequence $x_n$ defined by (6.1) is square summable. Here again one may show that this is equivalent to $\eta_2 > \eta_1$.

**Proposition 6.3.** *Assume that $\alpha_n = \beta_n$, the sequence $\alpha_n(2\alpha_n - \alpha_{n+1} - \alpha_{n-1})$ is bounded below and $\lim_{n\to\infty} n/\alpha_n = 0$. Then $\sigma_{ess}(\mathcal{J}_1) = \sigma_{ess}(\mathcal{J}_2) = \emptyset$. In particular, $\mathcal{J}(a_n, 0)$ has no essential spectrum if $a_{2n-1} = a_{2n} = \alpha_n$.*

*Proof.* Recall that $\mathcal{J}_1 = \mathcal{J}(\alpha_n^2, \alpha_n^2 + \alpha_{n-1}^2)$ and $\mathcal{J}_2 = \mathcal{J}(\alpha_n\alpha_{n+1}, 2\alpha_n^2)$. Moreover, the sequence

$$2\alpha_n^2 - \alpha_n\alpha_{n+1} - \alpha_n\alpha_{n-1} = \alpha_n(2\alpha_n - \alpha_{n+1} - \alpha_{n-1})$$

is bounded from below. A direct application of Theorem 8 of [5] to $\mathcal{J}_1, \mathcal{J}_2$ finishes the proof. □

**Example 6.4.** The last proposition applies to $\alpha_n = n \ln n$. In particular, $\mathcal{J}(n \ln n, 0)$ has a purely absolutely continuous spectrum filling the whole real axis, while $\mathcal{J}(a_n, 0)$ has no essential spectrum if $a_{2n-1} = a_{2n} = n \ln n$.

**Proposition 6.5.** *Assume that $\beta_n = c\alpha_n$ for some positive constant $c \neq 1$ and $\alpha_n/\alpha_{n+1} \to 1$ a $n \to \infty$. Then $\mathcal{J}(A_n, B_n)$ has no essential spectrum. In particular, the operator $\mathcal{J}(a_n, 0)$ has no essential spectrum if $a_{2n-1} = ca_{2n} = cn^\alpha$ with $\alpha \in (0,1)$.*

*Proof.* Since $c \neq 1$, one has

$$\lim_{n\to\infty} \frac{(\alpha_{n-1}^2 + \beta_n^2)^2}{\alpha_n^2\beta_n^2 + \alpha_{n-1}^2\beta_{n-1}^2} = \frac{(c^2+1)^2}{2c^2} > 2.$$

Hence, according to (i) of the remark following Theorem 2.1, $\mathcal{J}_1$ has purely discrete spectrum. Similar argument works for $\mathcal{J}_2$. The proof is complete. □

## REFERENCES


[1] Yu.M. Berezanskii, *Expansions in Eigenfunctions of Selfadjoint Operators*, vol. 17, Translation of Mathematical Monographs, Amer. Math. Soc., Providence, RI, 1968.

[2] J. Dombrowski, J. Janas, M. Moszyński, S. Pedersen, *Spectral gaps resulting from periodic perturbations of a class of Jacobi operators*, Constr. Approx. **20** (2004) 4, 585–601.

[3] J. Dombrowski, S. Pedersen, *Absolute continuity for Jacobi matrices with constant row sums*, J. Math. Anal. Appl. **267** (2002), 695–713.

[4] J. Dombrowski, S. Pedersen, *Spectral transition parameters for a class of Jacobi matrices*, Studia Math. **152** (2002), 217–229.

[5] D.B. Hinton, R.T. Lewis, *Spectral analysis of second-order difference equations*, J. Math. Anal. Appl. **63** (1978), 421–438.

[6] J. Janas, S.N. Naboko, *Jacobi matrices with power-like weight grouping in blocks approach*, J. Func. Anal. **166** (1999), 218–243.

[7] J. Janas, S.N. Naboko, *Multi-threshold spectral phase transitions for a class of Jacobi matrices*, Operator Theory: Adv. Appl. **124** (2001), 267–285.

[8] J. Janas, S. Naboko, G. Stolz, *Spectral theory for a class of periodically perturbed unbounded Jacobi matrices: elementary methods*, J. Comput. Appl. Math. **171** (2004) 1–2, 265–276.

[9] J. Janas, S. Naboko, G. Stolz, *Decay bounds on eigenfunctions and the singular spectrum of unbounded Jacobi matrices*, Int. Math. Res. Not. IMRN (2009), no. 4, 736–764.

[10] M. Moszynski, *Spectral properties of some Jacobi matrices with double weights*, J. Math. Anal. Appl. **280** (2003), 400–412.

[11] S. Pedersen, *Absolutely continuous Jacobi operators*, Proc. Amer. Math. Soc. **130** (2002), 2369–2376.

[12] J. Sahbani, *Spectral properties of Jacobi matrices of certain birth and death processes*, J. Operator Theory **56** (2006) 2, 377–390.

[13] J. Sahbani, *Spectral theory of certain unbounded Jacobi matrices*, J. Math. Anal. Appl. **342** (2008) 1, 663–681.

[14] J. Sahbani, *Spectral theory of a class of block Jacobi matrices and applications*, J. Math. Anal. Appl. **438** (2016) 1, 93–118.

Jaouad Sahbani
jaouad.sahbani@imj-prg.fr

Université Paris Diderot
Institut de Mathématiques de Jussieu-Paris Rive Gauche-UMR7586
Bâtiment Sophie Germain-case 7012
5 rue Thomas Mann, 75205 Paris Cedex 13, France

*Received: July 20, 2016.*
*Accepted: July 28, 2016.*