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Time-dependant modelling of systems: some open questions

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system, reliability, failure, state, critical, transition, Boolean, graph

Abstract

Time dependent modelling of complex system is one of the important topics in system reliability engineering. Although system complexity is increasing, the existing models are numerically satisfactory. However, some formal development is still lacking in reliability theory. A full system time-dependency modelling and analysis is not possible without some formal answers on critical transitions and related issues. This is still one of the open questions in system reliability theory. Some promising development relative to critical states is given in this paper with an application case.

1. Introduction

To carry out system time-dependence analyses, one should be able to describe the states and the transitions of the system. When system analysts are more interested in describing system's failure/success, they will naturally be oriented to the use of the fault tree / reliability block type of analysis. They will certainly make use of the Boolean techniques and produce cuts (minimal or not) describing the system failure/success. I would call that *Boolean Based Models* (BBM). For a dynamic analysis needs based on fault tree analysis, analysts may use some dynamic fault tree analysis or dynamic reliability block diagram. This is the case of some models / algorithms given in [2], [5]. A complete work on the Boolean Techniques in Reliability Theory is given in [6]. The BBMs provide a complete description of all system failure/success combinatory cuts.

On the other hand, if the Analyst wants to analyze system transitions, he will be oriented à priori towards the use of *State Based Models* (SBM) such as Markov, semi-Markov, or the use of *Simulation Based Techniques* (SBT) such as Monte-Carlo, Petri-networks, Stochastic Petri Networks, SBMs provide a complete description of the system possible states including critical states and transitions. This is necessary in order to proceed to dynamic analysis.

Models and algorithms that link between BBM and SMB are lacking. However, we may find some preliminary and promising works in [3], [1] but not exclusively.

The paper presents some original ideas towards establishing formal links between these BBMs and SBMs. It is a partial answer on the following question:

Does a Boolean expression of a system failure/success contain information about the system critical states and transitions?

If yes, how can it be extracted?

Critical states are, by definition, those states where only one elementary transition (off $\Leftrightarrow$ on) may result in a system transition (off $\Leftrightarrow$ on). Elementary transitions are associated to the elementary components of the system. Elementary transitions are, by definition, binary (off/on) and independent. That implies we have enough feedback experience about these "elementary components" such that we may fully describe them by a failure rate (λ), a repair rate (μ) and an initial unavailability (γ). Many experts use the term "failure to start up probability" for γ, as well.

2. System description

A system's failure may be fully described by a logical expression of the following type:

$$F = E_1 + E_2 + \ldots + E_{n-1} + E_n \qquad (1)$$

Where, F is the set of all failure states and $[E_i, i = 1,2,...,n]$ are given subsets of failure states. These subsets may be minimal cuts or not, joint or disjoint. They could also be any different kind of sets. They may be directly deduced from a Fault Tree representation of the system failure.
Many logical expressions could equivalently describe the system failure F.
The knowledge of the elementary states (failure/success) $[e_i, i = 1,2,...,k]$ allows the construction of the failure subsets $[E_i, i = 1,2,...,n]$, which describe the system failure F. Elementary states are, by definition, described by binary functions (0/1=off/on) and defined by transition rates (out/in $\equiv \tau_{e_i \to \overline{e_i}} / \tau_{\overline{e_i} \to e_i}$). If e_i is defined as an elementary failure state, then, the failure rate will be defined as the transition rate to state e_i ($\lambda = \tau_{\overline{e_i} \to e_i}$), the repair rate will be defined as the transition rate from state e_i ($\mu = \tau_{e_i \to \overline{e_i}}$) and $\overline{e_i}$ will be defined as the complementary state of e_i (the success state), see *Figure 1*.

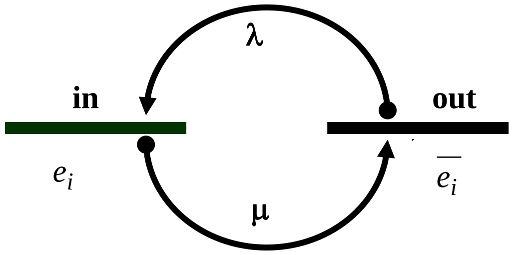

Figure 1. An elementary e_i state defined by its elementary transition rates into/out of its complementary state

The state F may also be fully described by its transition rates into/out of the complementary state $\overline{F}$ ($\tau_{F \to \overline{F}} / \tau_{\overline{F} \to F}$). If the F describes the system's failure, then, the system (equivalent) failure rate will be defined as ($\lambda = \tau_{\overline{F} \to F}$) and its (equivalent) repair rate will be defined as ($\mu = \tau_{F \to \overline{F}}$). Subsequently, the system itself becomes binary and could be considered as merely an elementary state belonging to another more complicated (macro) system. This point of view is somehow close to the "Modular Approach" of Gulati in [5] and others. The system's kinetic could then be schematically described as in *Figure 1* after replacing e_i by F and $\overline{e_i}$ by $\overline{F}$.

Consequently, the system time-behaviour will be fully governed by the following differential-integral equations system

$$\frac{d}{dt} A(t) = -\lambda(t).A(t) + \mu(t).U(t), \qquad \text{(2-a)}$$

$$R(t,\tau) = e^{-\int_t^{t+\tau} \lambda(\xi).d\xi}, \qquad \text{(2-b)}$$

$$S(t,\tau) = e^{-\int_t^{t+\tau} \mu(\xi).d\xi}. \qquad \text{(2-c)}$$

Where, U is the system's probability to be in the given state F (set of failure states) and A is the probability to be in the complementary (macro) state $\overline{F}$ (success) at a given instant 't'. R is the probability to last in the state $\overline{F}$ for the time interval $[t, t+\tau]$ and S is the probability to last in the complementary state F for the time interval $[t, t+\tau]$. λ / μ are the (equivalent) transition rates into/out of the failure state F, respectively.
A, U, R, S, λ and μ could, then, be the availability, the unavailability, the reliability, the reparability, the failure rate and the repair rate of the system, respectively.
Generally, the problem is how to determine the system transition rates (λ, μ), knowing the elementary states $[e_i, i = 1,2,...,k]$ and the definition of the system failures $[E_i, i = 1,2,...,n]$ given in equation (1).
Currently, the solution subsists very often in constructing a graph of states and hopping that:

- The number of states involved in the problem is limited,
- The transition rates between these states are constant (Markovian) or at least slowly varying with the time (semi-Markovian).

Generally, reliability engineers and analysts succeed to find out handsome algorithms to come up with satisfactory numerical answers.
However, formal links between the logical expression of a set of states and the corresponding transition rates (in/out) are lacking.
The link between the logical description of the system failure, $[E_i, i = 1,2,...,n]$ given in equation (1), and the system transition rates given in equations (2), may only be established through the determination of the corresponding critical states.

3. Critical states

The transition of a system, from success ($\overline{F}$) to failure (F) or vice-versa, occurs through some given and well-defined critical states. The determination of these critical states permits the determination of the failure and repair rates of the system, *Figure (2).*
A critical state is a state in which the system may switch on (/off) by switching only one elementary state on (/off). Then, we have as many sets of critical states as elementary events.

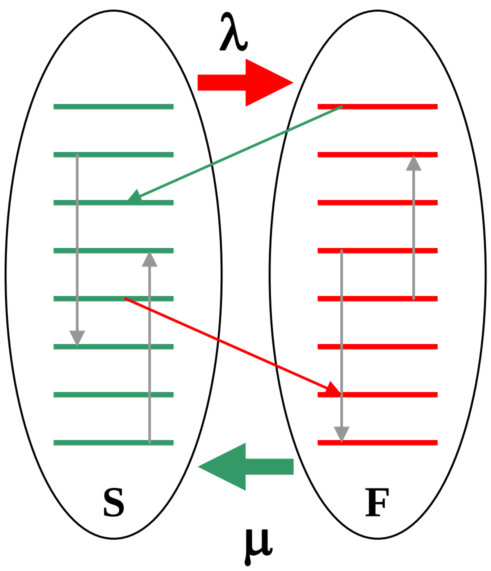

Figure 2. Schematic representation of critical transitions between the space S and F (S=Success, F=Failure)

In the paper, a method is given in order to determine the critical states starting from the logical description of the failure (/success) of a given system. The demonstration of the method is not given in the paper and will be published independently.
Let, [$e_i, i = 1,2,...,k$] be the elementary failure events related to a given system. An elementary event has only two states (off/on) and the transitions between these binary states are full determined by [(λ_i, μ_i), $i = 1,2,...,k$] which are statistically measured.
If the system failure (F) is described by a logical expression as given in equation (1), the Modulus M_i of the critical states associated to the elementary event (e_i) is determined by the intersection of two subsets

$$M_i = X_i \bullet \overline{Y_i} . \qquad (3)$$

Where X_i and Y_i are determined, respectively, by identification, using the following logical expression

$$F \bullet e_i = X_i \bullet e_i , \qquad (4\text{-a})$$

$$F \bullet \overline{e_i} = Y_i \bullet \overline{e_i} . \qquad (4\text{-b})$$

The method shows that the set of critical failure states $E(\overline{e_i})$, will be given by

$$E(\overline{e_i}) = \overline{e_i} \bullet M_i . \qquad (5\text{-a})$$

And the critical repair states $E(e_i)$, will be given by

$$E(e_i) = e_i \bullet M_i . \qquad (5\text{-b})$$

Once the sets of critical events (failure/repair) have been determined for the given system, we can then write:

$$\lambda_S.A_S = \sum_{i=1}^{k} \lambda_i.P\left[E(\overline{e_i})\right] \qquad (6\text{-a})$$

and

$$\mu_S.U_S = \sum_{i=1}^{k} \mu_i.P\left[E(e_i)\right]. \qquad (6\text{-a})$$

Where A_S and U_S are the system availability and unavailability, λ_i and μ_i are the transitions rates of the elementary event (i), while $P\left[E(e_i)\right]$ and $P\left[E(\overline{e_i})\right]$ are the probability of being in the set of critical repair states and in the set of critical failure states, respectively.
Because of the binary aspect of the method, the system is either available (in operation) unavailable (in reparation). The equivalence "Availability = Operation" and "Unavailability = Reparation" may be at disturbing, at the first glance, and giving the impression of something missing regarding engineering systems real life. That may result in confusion in some situations.
One of these potentially confusing situations can be the one when the system is in a standby phase. In this case, the system is available but not in operation. It is not operating but not in reparation, neither.
However, this still could be treated by distinguishing different phases in real operating life of a given system. In a passive standby phase, when a given system does not fail and is not in reparation, the system may still be defined in this phase by its failure and repair rates such that

$$\lambda_s = \mu_s = 0$$

Where, the logical description of the critical states will still be valid.
Our main objective is rather to apply the method than to demonstrate it. We have chosen an application case whose results could be obtained by other methods (graph state). That would allow better appreciating the original added value of the method and its real potentials for complex system analysis.
Before leaving this section, it is worth to underline that the method to determine the critical states is independent on the logical expression used to describe the system. Analysts may indifferently use the logical expression of success as well as the logical expression of failure.
An active redundancy of the type (N-1)/N has been selected. A specific attention is paid to the system time-dependency analysis.

4. (N-1)/N active redundancy

In the following sections, we will be interested in making a study case in order to illustrate some time-dependant characteristics in a relatively complex system.
We propose an (N-1)/N active redundancy type of systems.
The generalization of the method to other types of complex systems is straightforward. As it has been mentioned above, the most original part in the method is the one about the determination of the set of critical states per each elementary failure event, using the formal mathematical tools of the Boolean algebra.

4.1. System state logical expression

Both systems' Failure (F) and system success (S) can be equally described. F and S are complementary sets in the Boolean sense.
This is true as long as F and S hold for the following properties

$$F \bullet S = \Theta,$$

$$F + S = I$$

Where "$\bullet$" and "$+$" are the logical operators intersection and union, respectively. Θ and I are the empty and universal sets, respectively.
We will use the system success expression S. The success of a system in a configuration of (n-1)/n active redundancy may be expressed by

$$S = [e_2 \bullet e_3 \bullet e_4 \bullet e_5 \bullet \ldots \bullet e_{n-1} \bullet e_n]$$

$$+ [e_1 \bullet e_3 \bullet e_4 \bullet e_5 \bullet \ldots \bullet e_{n-1} \bullet e_n]$$

$$+ [e_1 \bullet e_2 \bullet e_4 \bullet e_5 \bullet \ldots \bullet e_{n-1} \bullet e_n]$$

$$+ [e_1 \bullet e_2 \bullet e_3 \bullet e_5 \bullet \ldots \bullet e_{n-1} \bullet e_n]$$

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$$+ [e_1 \bullet e_2 \bullet e_3 \bullet e_4 \bullet e_5 \bullet \ldots \bullet e_n]$$

$$+ [e_1 \bullet e_2 \bullet e_3 \bullet e_4 \bullet e_5 \bullet \ldots \bullet e_{n-1}]. \quad (7)$$

Where e_i is an elementary event describing the success of the elementary component belonging to the system. Elementary events are independent, by definition.
The system's success is logically described in equation (7) using the success minimal cuts. Instead, we prefer to put it in the form of disjoint cut sets, regarding our immediate need to calculate the system availability 'A'. To carry out the transformation of the minimal (joint) cuts to disjoint (but not minimal) cuts, we may proceed in the following way. Using a reduced expression of the system success, equation (7) may be written as

$$S = E_1 + E_2 + E_3 + \ldots + E_{n-1} + E_n. \quad (8)$$

Where $\{E_i$, i=1,2, … , n$\}$ are any type of cut sets. In order to construct an expression of the system success using disjoint cut sets, one may then rearrange the above expression in the following manner

$$S = E_1 + \overline{E_1} \bullet E_2 + \overline{E_1} \bullet \overline{E_2} \bullet E_3 + \ldots$$

$$+ \left(\prod_{j=1}^{n-2} \overline{E_j} \right) \bullet E_{n-1} + \left(\prod_{j=1}^{n-1} \overline{E_j} \right) \bullet E_n. \quad (9)$$

(For full description on Boolean expressions and function handling see [2])
Expressions (8) and (9) are equivalent. Moreover, there exists n! equivalent possible expressions.
The expressing in Equation (7) could, then, be written as following:

$$S = [e_2 \bullet e_3 \bullet e_4 \bullet e_5 \bullet \ldots \bullet e_{n-1} \bullet e_n]$$

$$+ [e_1 \bullet \overline{e_2} \bullet e_3 \bullet e_4 \bullet e_5 \bullet \ldots \bullet e_{n-1} \bullet e_n]$$

$$+ [e_1 \bullet e_2 \bullet \overline{e_3} \bullet e_4 \bullet e_5 \bullet ... \bullet e_{n-1} \bullet e_n]$$

$$+ [e_1 \bullet e_2 \bullet e_3 \bullet \overline{e_4} \bullet e_5 \bullet ... \bullet e_{n-1} \bullet e_n]$$

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$$+ [e_1 \bullet e_2 \bullet e_3 \bullet e_4 \bullet e_5 \bullet ... \bullet \overline{e_{n-1}} \bullet e_n]$$

$$+ [e_1 \bullet e_2 \bullet e_3 \bullet e_4 \bullet e_5 \bullet ... \bullet e_{n-1} \bullet \overline{e_n}]. \quad (10)$$

That allows immediately describing the system availability A_s by

$$A_s = [\prod_{i=1}^{n-1} A_i + \sum_{l=1}^{n-1} U_l . \left(\prod_{\substack{j=1 \\ j \neq l}}^{n} A_j \right)]. \quad (11)$$

Where A_i and U_i are the availability and the unavailability of the elementary components, respectively. They obey the differential equation system given in (2). If all elementary events are identical (same transition rates and initial condition), one may replace A_i and U_i by A and U. Equation (11) will become

$$A_s = A^{n-1}[1 + (n-1).U]. \quad (12)$$

4.2. Critical states

Our objective now is to determine the n sets of critical states corresponding to the n elementary events. These sets are necessary to determine the system transition rates. Following the proposed method, lets determine the critical set corresponding to the elementary event e_1, as following

$$S \bullet e_1 = [e_1 \bullet e_2 \bullet e_3 \bullet e_4 \bullet e_5 \bullet ... \bullet e_{n-1} \bullet e_n]$$

$$+ [e_1 \bullet \overline{e_2} \bullet e_3 \bullet e_4 \bullet e_5 \bullet ... \bullet e_{n-1} \bullet e_n]$$

$$+ [e_1 \bullet e_2 \bullet \overline{e_3} \bullet e_4 \bullet e_5 \bullet ... \bullet e_{n-1} \bullet e_n]$$

$$+ [e_1 \bullet e_2 \bullet e_3 \bullet \overline{e_4} \bullet e_5 \bullet ... \bullet e_{n-1} \bullet e_n]$$

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$$+ [e_1 \bullet e_2 \bullet e_3 \bullet e_4 \bullet e_5 \bullet ... \bullet \overline{e_{n-1}} \bullet e_n]$$

$$+ [e_1 \bullet e_2 \bullet e_3 \bullet e_4 \bullet e_5 \bullet ... \bullet e_{n-1} \bullet \overline{e_n}]$$

$$S \bullet e_1 = e_1 \bullet \{[e_2 \bullet e_3 \bullet e_4 \bullet e_5 \bullet ... \bullet e_{n-1} \bullet e_n]$$

$$+ [\overline{e_2} \bullet e_3 \bullet e_4 \bullet e_5 \bullet ... \bullet e_{n-1} \bullet e_n]$$

$$+ [e_2 \bullet \overline{e_3} \bullet e_4 \bullet e_5 \bullet ... \bullet e_{n-1} \bullet e_n]$$

$$+ [e_2 \bullet e_3 \bullet \overline{e_4} \bullet e_5 \bullet ... \bullet e_{n-1} \bullet e_n]$$

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$$+ [e_2 \bullet e_3 \bullet e_4 \bullet e_5 \bullet ... \bullet \overline{e_{n-1}} \bullet e_n]$$

$$+ [e_2 \bullet e_3 \bullet e_4 \bullet e_5 \bullet ... \bullet e_{n-1} \bullet \overline{e_n}]\}$$

$$= e_1 \bullet X_1 .$$

By identification, X_1 can be defined as

$$X_1 = [e_2 \bullet e_3 \bullet e_4 \bullet e_5 \bullet ... \bullet e_{n-1} \bullet e_n]$$

$$+ [\overline{e_2} \bullet e_3 \bullet e_4 \bullet e_5 \bullet ... \bullet e_{n-1} \bullet e_n]$$

$$+ [e_2 \bullet \overline{e_3} \bullet e_4 \bullet e_5 \bullet ... \bullet e_{n-1} \bullet e_n]$$

$$+ [e_2 \bullet e_3 \bullet \overline{e_4} \bullet e_5 \bullet ... \bullet e_{n-1} \bullet e_n]$$

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$$+ [e_2 \bullet e_3 \bullet e_4 \bullet e_5 \bullet ... \bullet \overline{e_{n-1}} \bullet e_n]$$

$$+ [e_2 \bullet e_3 \bullet e_4 \bullet e_5 \bullet ... \bullet e_{n-1} \bullet \overline{e_n}]. \quad (13)$$

Secondly, one should calculate the modulus relative to the event $\overline{e_1}$ as following:

$$S \bullet \overline{e_1} = \overline{e_1} \bullet \{[e_2 \bullet e_3 \bullet e_4 \bullet e_5 \bullet ... \bullet e_{n-1} \bullet e_n]$$

$$+ [e_1 \bullet \overline{e_2} \bullet e_3 \bullet e_4 \bullet e_5 \bullet ... \bullet e_{n-1} \bullet e_n]$$

$$+ [e_1 \bullet e_2 \bullet \overline{e_3} \bullet e_4 \bullet e_5 \bullet ... \bullet e_{n-1} \bullet e_n]$$

$$+ [e_1 \bullet e_2 \bullet e_3 \bullet \overline{e_4} \bullet e_5 \bullet ... \bullet e_{n-1} \bullet e_n]$$

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$$+ [e_1 \bullet e_2 \bullet e_3 \bullet e_4 \bullet e_5 \bullet \ldots \bullet \overline{e_{n-1}} \bullet e_n]$$

$$+ [e_1 \bullet e_2 \bullet e_3 \bullet e_4 \bullet e_5 \bullet \ldots \bullet e_{n-1} \bullet \overline{e_n}]\}$$

$$= \overline{e_1} \bullet [e_2 \bullet e_3 \bullet e_4 \bullet e_5 \bullet \ldots \bullet e_{n-2} \bullet e_{n-1} \bullet e_n]$$

$$= \overline{e_1} \bullet Y_1 .$$

By identification, Y_1 may be defined as

$$Y_1 = [e_2 \bullet e_3 \bullet e_4 \bullet e_5 \bullet \ldots \bullet e_{n-2} \bullet e_{n-1} \bullet e_n]. \quad (14)$$

The modulus M_1 of the critical sets corresponding to the elementary event $\overline{e_1}$ is then

$$M_1 = X_1 \bullet \overline{Y_1}$$

$$= \{[e_2 \bullet e_3 \bullet e_4 \bullet e_5 \bullet \ldots \bullet e_{n-1} \bullet e_n]$$

$$+ [\overline{e_2} \bullet e_3 \bullet e_4 \bullet e_5 \bullet \ldots \bullet e_{n-1} \bullet e_n]$$

$$+ [e_2 \bullet \overline{e_3} \bullet e_4 \bullet e_5 \bullet \ldots \bullet e_{n-1} \bullet e_n]$$

$$+ [e_2 \bullet e_3 \bullet \overline{e_4} \bullet e_5 \bullet \ldots \bullet e_{n-1} \bullet e_n]$$

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$$+ [e_2 \bullet e_3 \bullet e_4 \bullet e_5 \bullet \ldots \bullet \overline{e_{n-1}} \bullet e_n]$$

$$+ [e_2 \bullet e_3 \bullet e_4 \bullet e_5 \bullet \ldots \bullet e_{n-1} \bullet \overline{e_n}]\}$$

$$\bullet [\overline{e_2 \bullet e_3 \bullet e_4 \bullet e_5 \bullet \ldots \bullet e_{n-2} \bullet e_{n-1} \bullet e_n}].$$

Finally,

$$M_1 = [\overline{e_2} \bullet e_3 \bullet e_4 \bullet e_5 \bullet \ldots \bullet e_{n-1} \bullet e_n]$$

$$+ [e_2 \bullet \overline{e_3} \bullet e_4 \bullet e_5 \bullet \ldots \bullet e_{n-1} \bullet e_n]$$

$$+ [e_2 \bullet e_3 \bullet \overline{e_4} \bullet e_5 \bullet \ldots \bullet e_{n-1} \bullet e_n]$$

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$$+ [e_2 \bullet e_3 \bullet e_4 \bullet e_5 \bullet \ldots \bullet \overline{e_{n-1}} \bullet e_n]$$

$$+ [e_2 \bullet e_3 \bullet e_4 \bullet e_5 \bullet \ldots \bullet e_{n-1} \bullet \overline{e_n}]. \quad (15)$$

M_1 is the modulus of the set of the critical states associated to elementary event e_1. One should repeat the same procedure (equations 13,14,15) to obtain the other modulus $\{M_i, i = 2,3,\ldots,n\}$ associated to all other elementary events $\{e_i, i = 2,3,\ldots,n\}$.

4.3. System transition rate

According to equation (5-a), the set of the critical failure states associated to the elementary event (e_1) is then

$$E(e_1) = e_1 \bullet M_1 \quad (16)$$

And the probability of this set of events is equal to

$$P[E(e_1)] = P[e_1 \bullet M_1] = P[e_1] . P[M_1] \quad (17)$$

As, the elementary events were supposed all identical (transition rates and initial condition), thus

$$P[E(e_1)] = A.[(n-1).U.A^{n-2}]. \quad (18)$$

So, we can determine the second term in equation (6-a):

$$\sum_{i=1}^{k} \lambda_i . P[E(e_i)] = \sum_{i=1}^{k} \lambda_i . P[E(e_i)]$$

$$= [\lambda . n(n-1).U.A^{n-1}]. \quad (19)$$

Considering equations (6-a), (10) and (17), we may determine the overall failure rate of the system as following

$$\lambda_s . A_s = \lambda_s . A^{n-1} [1 + (n-1).U]$$

$$= [\lambda . n(n-1).U.A^{n-1}].$$

That gives

$$\lambda_s = \left[\lambda . \frac{n(n-1).U}{[1 + (n-1)U]} \right]. \quad (20)$$

Where λ is the failure rate of the elementary component, and U is its unavailability. U obeys equation (2-a) whose solution gives (for constant transition rates)

$$U = \frac{\lambda}{\lambda+\mu}\left[1-e^{-(\lambda+\mu)t}\right]+\gamma e^{-(\lambda+\mu)t} \qquad (21)$$

Where, λ, μ and γ are a constant failure rate, a constant repair rate and an initial condition ($\gamma = U(0)$).

Considering equations (21) and (20) leads to the conclusion that: "Although the elementary failure rates λ's do not depend on time, the overall system failure rate λ_S shows time dependency."

Secondly, the system overall failure rate λ_S depends not only on the elementary failure rates λ, but also on the elementary repair rates μ, via the unavailability U, as well as the initial conditions via the failure to start-up probabilities γ.

The same procedure can also be used to determine a system overall repair rate. The demarche is identical and straightforward starting from equation (16) after having replaced the critical failure states by the critical repair states associated to the elementary events (e_i).

An exhaustive analysis of the method is out of the scope of this paper. The author limits the presentation to the use of the system overall failure rate and the system overall unavailability in order to demonstrate some interesting aspects related to the system time-dependency.

We may conceive a wide range of indicators that should allow us judging and assessing the attractiveness of a given (n-1)/n redundancy. We will introduce only two indicators and examine with some details their time behaviour.

5. Redundancy benefits

Different measures of benefits can be conceived in order to assess the real interest of a given redundancy.

If there were no redundancy, the system could have been composed of only (n-1) identical elementary items. These elementary items will be connected by an OR gate. That means that without this type of redundancy, the system's availability (A_S) and failure rate (λ_S) would have been, respectively, equal to

$$A_{s,o} = A^{n-1}, \qquad (22)$$

$$\lambda_{s,o} = (n-1).\lambda . \qquad (23)$$

Where A and λ are those of the elementary failure events. The elementary failure rate λ may have any time-dependant form.

Equations (22) and (23) provide us with reference values for the system availability and the system overall failure rate.

Thus, we may conceive some indicators to allow assessing the attractiveness of a given redundancy. We will limit our analysis in this paper to two indicators:

- Availability Gain indicator, and
- Failure Reduction Factor indicator

One recalls that our main objective is not to develop indicators measuring the interest of a redundancy but to assess the inter-dependency of these indicators and their time-dependency.

5.1. Availability gain

One way to evaluate the interest of using a given [(n-1)/n] active redundancy may be to calculate the absolute gain increase (G) in system availability, such as

$$G(t) = A_{s,w}(t) - A_{s,o}(t). \qquad (24)$$

Where $A_{s,w}(t)$ and $A_{s,o}(t)$ are the system availability with and without the [(n-1)/n] redundancy. Substituting equations (12) and (22) in equation (24), we get

$$G(t) = A^{n-1}\left[1+(n-1).U\right] - A^{n-1}$$

$$= (n-1).U(t).A^{n-1}(t). \qquad (25)$$

Figure 3 illustrates how this gain in system availability varies with the elementary component unavailability and with the degree of the redundancy of the system.

Many interesting aspects deserve to be underlined. First of all, it is worthy observing that the gain factor (G) is always positive and becomes null if and only if the elementary unavailability is null or equal to unity:

$$\lim_{\substack{U\to 0\\ U\to 1}} G(t) = \lim_{\substack{U\to 0\\ U\to 1}} (n-1).U(t).A^{n-1}(t) \;\to 0 \qquad (26)$$

Thus, it is not possible to loose in system availability if one uses an active redundancy with independent elementary events. Active redundancy is useless (G=0) if the elementary unavailability is null or if it is equal to unity.

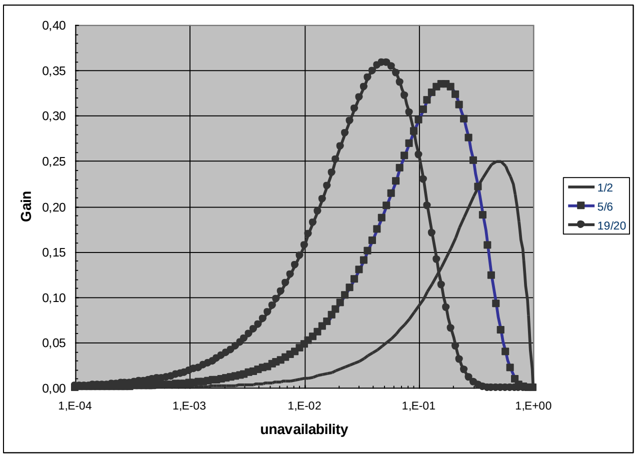

Figure 3. Increase in system overall availability versus elementary component unavailability

The second remark is that the absolute gain in system availability depends on the elementary unavailability. This dependency has maximum values. Higher is the order of the redundancy, higher is the maximum gain in system availability. In the case of a 1/2 active redundancy, the system attends its maximum gain (0.25) when the elementary unavailability is around 0.50. While, at 19/20 redundancy, the maximum gain (0.36) is obtained at 0.05 elementary unavailability.
Third, systems with highly probable elementary failures (> 0.5) would show higher gain factor if they have lower order active redundancy. The gain factor is 0.25 in the case of 1/2 redundancy and 0.06 in the case of 5/6 redundancies, both at elementary failure probability as high as 0.52.
While the systems, with very low elementary unavailability, would show higher gain factor at higher order of redundancy. For an elementary unavailability of the order of 0.01, 19/20 active redundancies shows a gain of the order of 0.16 while 1/2 redundancy shows a gain of only 0.01.
Forth, the active redundancy looses interest at extreme values of elementary unavailability ($U \to 0$, and $U \to 1$). This could be the case of industrial systems containing components whose failure rates are very small while repair rates are high. That results in very low elementary-failure probabilities. Or, that could also the case of industrial systems with components having very high failure rates and very small repair rates. That would result in very high elementary-failure probabilities.
Finally, it is important to underline that whatever the exact values of failure and repair rates of elementary components in a given systems, the elementary-failure (/elementary-success) probabilities evolve in time with accordance to equations (2). That means that the gain in availability evolves with the time, for a given system.

5.2. System failure rate reduction factor

Another type of analyses can also be carried on based on the notion of the gain in reliability.
How much does one gain in reliability using an [(n-1)/n] redundancy, with independent components?
One way to answer is to consider that one gains in reliability as much as one decreases the elementary-failure rate? That is certainly correct even if the correspondence is not directly proportional.
We may define a failure rate reduction factor (f) as follows

$$f = \frac{\lambda_s}{(n-1)\lambda} = \frac{nU}{\left[1+(n-1)U\right]}. \tag{27}$$

Where λ_s and $\left[(n-1)\lambda\right]$ are the system failure rates with and without the redundancy, respectively.
Figure 4 illustrates the variation of the reduction factor (f) as a function of the elementary unavailability and the degree of the system redundancy. Many aspects may be underlined.
At very high order of redundancy, the reduction factor is almost equal to 1, which is not attractive in terms of gain in reliability:

$$f_{n\to\infty} = \lim_{n\to\infty} \frac{nU}{\left[1+(n-1)U\right]} \to 1 \tag{28}$$

The best reduction factor (f = 0) can only be obtained for systems containing perfectly available elementary components [U =0, $\forall n; (n \in \{2,3,...\})$], independently from the degree of redundancy. It means that redundancy degree does not impact on the system failure (reliability). Practically, if elementary components are always available, then no-need to use redundancy.

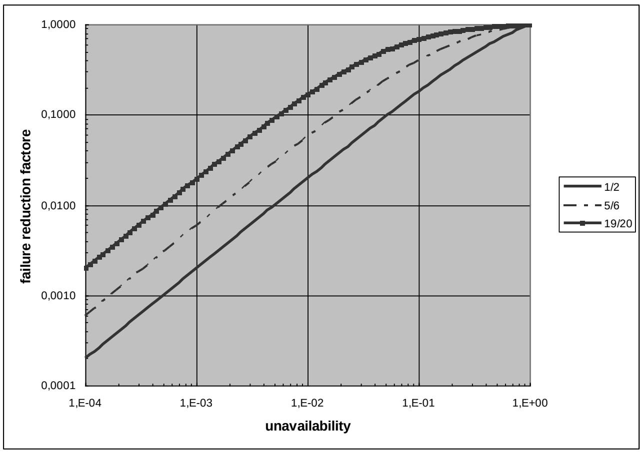

Figure 4. Failure rate reduction facture versus elementary unavailability

One may thus describe (f = 0) as the strongest reduction factor and (f = 1) as the weakest reduction factor.
One observes also that for the same level of elementary unavailability, *Figure 3*, one has stronger reduction factor at lower redundancy. It is very important to underline this aspect that higher is the redundancy degree lower is the reduction factor, independently of the elementary unavailability.

6. System ageing

Examining equation (20) results in the conclusion that the system overall failure rate is time-dependent even if the elementary-failure rates are not. This is mainly due to the fact that the system overall failure rate depends on the elementary unavailability (U), as well.
We know that U obeys the differential equation (2) and subsequently, is dependent on time. We may recall that

- the system is ageing if, $\frac{d}{dt}\lambda_S > 0$,
- the system is time-independent if, $\frac{d}{dt}\lambda_S = 0$,
- the system is regenerating if $\frac{d}{dt}\lambda_S < 0$.

One should understand this (systemic) behaviour in its functional sense.
To illustrate this effect, we recall equation (18)

$$\lambda_S = \left[\lambda . \frac{n(n-1).U}{\left[1+(n-1)U\right]} \right].$$

And consider the case where a given system composed of [(n-1)/n] identical time-independent components. Let the elementary unavailability $U(t)$ varies in such a way that

$$U(0) = 0,$$

and

$$U(t \to \infty) \to 1.$$

That gives the following results:

$$\lambda_S \underset{t=0}{} = 0,$$

and

$$\lambda_S \underset{t\to\infty}{\to} (n-1)\lambda \qquad (29)$$

Under the above hypothesis, the system behaves as if its failure rate evolves from zero to $[(n-1)\lambda]$, while λ is time-independent. What is even more interesting, is that $\lambda_S \underset{t\to\infty}{}$ is proportional to n, when n is relatively high. That is to say, highly redundant systems age faster.
That would equally means that, in practice and in a given complex system, if elementary components do not show ageing, it is not enough to conclude that the system itself does not age.
Complex systems should be analysed and observed through functional specifications not only physical ones.
One more interesting case could be for a system with [(n-1)/n] actively redundant elementary non-reparable components with Weibull-type elementary failure rate.
In that case, the elementary unavailability U and failure rate λ will be given by

$$U = 1 - e^{-\left(\frac{t}{\eta}\right)^{\beta}},$$

$$\lambda = \frac{\beta}{\eta}\left(\frac{t}{\eta}\right)^{\beta-1}.$$

Substituting in equation (18), the system overall failure rate will then be equal to

$$\lambda_S(t) = \left[\lambda . \frac{n(n-1).U}{\left[1+(n-1)U\right]} \right]$$

$$= \frac{\beta}{\eta}\left(\frac{t}{\eta}\right)^{\beta-1} \frac{n(n-1).(1-e^{-\left(\frac{t}{\eta}\right)^{\beta}})}{\left[1+(n-1)(1-e^{-\left(\frac{t}{\eta}\right)^{\beta}})\right]} \qquad (30)$$

Equation (30) shows the time dependency of the system overall failure rate in the case of non-reparable elementary element with Weibull-like failure rate.
Again, we observe that redundancy slow efficiently the system aging for small ‘t’ and small ‘n’. However, it looses its efficiency when ‘n’ and ‘t’ increases.

5.3. Maintenance and ageing

We have examined the impact of the elementary-failure rate (λ) and the degree of redundancy (n) on the system failure (λ_S) time-dependency.

In this section, the impact of the elementary-failure probability (U) will be examined. U obeys the differential equations given in (2)

$$\frac{d}{dt}A = -\lambda.A + \mu.U .$$

If λ (elementary failure rate) and μ (elementary repair rate) are time-independent, then

$$U(t) = \frac{\lambda}{\lambda+\mu}(1+e^{-(\lambda+\mu)t}) + \gamma.e^{-(\lambda+\mu)t},$$

where

$$\underset{t\to 0}{U(t)} \to \gamma,$$

$$\underset{t\to\infty}{U(t)} \to \frac{\lambda}{\lambda+\mu}. \qquad (31)$$

Accordingly and considering equation (20), one may identify three possible situations

- $\gamma < \frac{\lambda}{\lambda+\mu}$,

which means that $U(t)$ increases with time and λ_S increases with time, as well. Subsequently the system ages.

- $\gamma \approx \frac{\lambda}{\lambda+\mu}$,

which means that $U(t)$ is constant with time and subsequently the system is time-independent. Subsequently, λ_S is constant

- $\gamma > \frac{\lambda}{\lambda+\mu}$,

which means that $U(t)$ decreases with time and λ_S decreases with time. Subsequently the system generates.

There are different ways to read the above three situations from maintenance point of view, under the light of equation (20).

The 1st situation reflects the case where

$$(\frac{\mu}{\lambda} < \frac{1-\gamma}{\gamma}),$$

i.e. the elementary components are under-maintained or non-reparable. The elementary-repair rates are small compared to the elementary-failure rates. Improvement in maintenance would slowdown the system ageing.

The 2nd situation reflects the case where

$$(\frac{\mu}{\lambda} \approx \frac{1-\gamma}{\gamma}),$$

i.e. the elementary components are enough maintained. The elementary-repair rate is proportional to the elementary-success probability at $t = 0$. That is to say, regarding the elementary component, more successfully it starts up lower the maintenance it receives, and vice-versa. In all cases, maintenance is efficient enough to rend the system time-independent.

The 3rd situation reflects the case where

$$(\frac{\mu}{\lambda} > \frac{1-\gamma}{\gamma}),$$

i.e. the elementary components are over-maintained. The elementary-repair rate is either too high compared to the elementary failure rate or one maintains a component that never shows failure ($\lambda \to 0$). We have a good maintenance margin.

In these three situations, we are guided by equation (20) that describes the system overall failure rate. In the all situations we considered the elementary failure rate as intrinsic to the elementary components and used the maintenance (μ) as a means to improve elementary component unavailability and consequently the system overall failure rate. Increasing (μ) obviously improves system reliability.

One may also reproduce the same effect on the system overall failure rate through decreasing the elementary failure rate (λ). As the elementary failure rate is an intrinsic property of the elementary component, so the only way to decrease it is to replace periodically the elementary component by a new one.

This is exactly what operators do in order to improve the reliability of their complex systems. They elaborate their maintenance strategy upon 2 basic actions.

Improving elementary component's reparability, which is translated by increasing the elementary component repair rate (μ) through periodic maintenance, and/or

Improving elementary component's failure, which is translated by decreasing the elementary component failure rate (λ) through regular standard replacement of the elementary component(s).

6. Conclusion

In spite of the obvious progress in the system reliability theory during the past decades, some open questions do still need development. One of these open questions concerns the determination of complex systems critical states and transitions.

Very often, analysts and reliability engineers use graphical tools when they are interested in analysing system states and transitions. Once the graph of states in constructed and the critical transitions are identified, they use Markovian (/semi-Markovian) tool to carry out the assessment.

However, they will react differently, if they would like to analyse system failure modes and events taking into account interdependencies between different sets of elementary components failures. They will rather use fault tree analyses and associated Boolean tools.

Our question was:

"Do Boolean expressions of system failure contain any information about systems critical states and transitions?" If yes, "How can it be extracted?"

The author reports on a small but promising progress towards an answer. The author believes that establishing a link between states graph-type of presentations and Boolean failure cut sets would open a promising prospective in the system reliability theory, especially, dynamic reliability.

A very rapid presentation of the method has been done. An application on [(n-1)/n] active-redundancy system is carried out in order to allow the appreciation of the method.

The author has stressed on the behaviour of the system overall failure rate with the time.

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