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OPTIMAL SYSTEM OF 1-D SUBALGEBRAS AND CONSERVED QUANTITIES OF A NONLINEAR WAVE EQUATION IN THREE DIMENSIONS ARISING IN ENGINEERING PHYSICS

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Abstract: The construction of explicit structures of conserved vectors plays diverse crucial roles in the study of nonlinear science inclusive of the fact that they are invoked in developing appropriate numerical schemes and for other mathematical analyses. Therefore, in this paper, we examine the conserved quantities of a nonlinear wave equation, existing in three dimensions, and highlight their applications in physical sciences. The robust technique of the Lie group theory of differential equations (DEs) is invoked to achieve analytic solutions to the equation. This technique is used in a systematic way to generate the Lie point symmetries of the equation under study. Consequently, an optimal system of one-dimensional (1-D) Lie subalgebras related to the equation is obtained. Thereafter, we engage the formal Lagrangian of the nonlinear wave equation in conjunction with various gained subalgebras to construct conservation laws of the equation under study using Ibragimov's theorem for conserved vectors.

Key words: A nonlinear wave equation in three dimensions, Lie group theory of differential equations, 1-D optimal system of subalgebras, formal Lagrangian, conserved vectors

1. INTRODUCTION

In recent years, intensive investigations have been diverted into nonlinear partial differential equations (NLPDEs) alongside their exact travelling wave solutions since many physical phenomena are represented using these NLPDEs. Nonlinearity is a captivating and enthralling element of nature. Scientists in their numbers have deemed it fit to contemplate nonlinear science as the most significant frontier for the fundamental comprehension of nature. Some of these models include a generalised system of three-dimensional variable-coefficient modified Kadomtsev–Petviashvili-Burgers-type equations in Gao's research [1], which has been examined in the present study. Adeyemo et al.'s research [2] studies the generalised advection-diffusion equation, which is a NLPDE in fluid mechanics, characterising the motion of buoyancy-propelled plumes in a bent-on absorptive medium. Moreover, in the study of Khalique and Adeyemo [3], a generalised Korteweg-de Vries-Zakharov-Kuznetsov equation was studied. This equation delineates mixtures of warm adiabatic fluid and hot isothermal fluid, as well as cold immobile background species applicable in fluid dynamics. The modified and generalised Zakharov-Kuznetsov model, which recounts the ion-acoustic drift solitary waves existing in a magnetoplasma with electron-positron-ion that are found in the primordial universe, was investigated in Du et al.'s study [4]. This equation was engaged in modelling ion-acoustic, dust-magneto-acoustic, quantum-dust-ion-acoustic and/or dust-ion-acoustic waves in one of the cosmic or laboratory dusty plasmas. Moreover, in Zhang et al.'s study [5], the vector bright solitons, as well as their interaction characteristics in the coupled Fokas-Lenells system modelling the femtosecond optical pulses in a birefringent optical fibre, were examined. In addition, the Boussinesq-Burgers-type system of equations that delineates shallow-water waves appearing close to lakes or ocean beaches was studied in Gao et al.'s research [6]; the list goes on and on, as can be seen in the literature [7–20].

It is revealed that no general and well-structured technique existed in gaining various exact travelling wave solutions of NLPDEs. However, various sound and efficient techniques have arisen lately to secure a lasting solution to this seemingly neverending problem.

In consequence, we present some of these techniques as $exp(-\Phi(\eta))$-expansion technique [18], Painlev´e expansion [19], bifurcation technique [20], ansatz technique [21], homotopy perturbation technique [22], extended homoclinic test approach [23], mapping and extended mapping technique [17], tanh-coth approach [24], generalised unified technique [25], Adomian decomposition approach [26], Cole-Hopf transformation approach [27], B¨acklund transformation [28], F-expansion technique [29], rational expansion technique [30], tan-cot method [31], Lie symmetry analysis [32, 33], Hirota technique [34], extended simplest equation method [35], Darboux transformation [36], the G'-expansion technique [37], tanh-function technique [38], Kudryashov technique [39], sine-Gordon equation expansion technique [40] and exponential function technique [41], to name a few.

There are a vast number of partial differential equations (DEs) that explicate nonlinear wave motion. One of them is the Kadomtsev–Petviashvili equation (KP) (usually abbreviated as KP, although we use the acronym KPSH for Kadomtsev–Petviashvili equation in the present case) existing in the field of mathematical physics. Moreover, it has been well-established that investigating diverse nonlinear phenomena in studying nonlinear waves is significant.

There are various versions of KPSH studied by many researchers. One among these is the Kadomtsev–Petviashvili equation in three dimensions, given as [42]:

$$(u_t + 6uu_x + u_{xxx})_x \pm 3u_{yy} \pm 3u_{zz} = 0 \quad (1.1)$$

This equation can be traced back to the classic work carried out in 1970 by the Soviet physicists Kadomtsev and Petviashvili [43]. Eq. (1.1) models waves involved in a situation whereby water wavelength and water depth ratio is very small in the context of nonlinear restoring forces that are insubstantial. Besides, it is ascertained to be a generalised version of the Korteweg-de Vries (KdV) – an equation that was named after two Dutch mathematicians, namely Korteweg and De Vries [44]. On the analysis of Eq. (1.1), uux delineates the nonlinearity part of the wave equation and the last two terms portray diffractive divergence, while the highest term depicts weak dispersion [45]. The "±" sign attached to the last two terms relates to the positive or negative magnitude of dispersion. Since then, a good number of researchers have investigated Eq. (1.1) [e.g. 46–51]. Diverse studies carried out on the equation ranged from the achievement of Painlev´e analysis [48], construction of their closed-form multiple wave solutions [49] and the establishment of the stability property of their soliton [50] to the decision of their integrability features [51].

The (3+1)-dimensional generalised KPSH equation that reads [52]:

$$u_{tx} + u_{ty} + 3u_x u_{xy} + 3u_{xx} u_y + u_{xxxy} - u_{zz} = 0, \quad (1.2)$$

was examined by the authors using Plu¨cker relation for determinants to gain one Wronskian solution to it. In addition, the Jacobi identity for determinants in the same work was implored to secure a Grammian solution. Moreover, multiple solitons as well as multiple singular solitons were also achieved for Eq. (1.2) in Wazwaz's research [53] by utilising a simplified structure of Hirota's technique. The investigations conducted in Wazwaz's research [53] revealed the prevalence of a contrast between their results and those gained in Ma et al.'s study [52] with regard to spatial variable z. Later, in the research of Wazwaz and El-Tantawy [54], a new form of the (3 + 1)-dimensional generalised KPSH Eq. (1.2) was introduced with the addition of utz, thus yielding

$$u_{tx} + u_{ty} + u_{tz} + 3u_x u_{xy} + 3u_{xx} u_y + u_{xxxy} - u_{zz} = 0. \quad (1.3)$$

In their investigation, Wazwaz and El-Tantawy [54] made known the fact that the new term added significantly impacted the dispersion relations. Further, they implored Hirota's direct approach to decide multiple soliton solutions of the equation. Besides, Eq. (1.3) was studied in Liu et al.'s research [55], where its analytic solutions were achieved via Hirota's bilinear technique as well as an extended homoclinic test approach.

We contemplate a two-dimensional Kadomtsev–Petviashvili equation, expressed as [56, 57]:

$$u_{tx} - 6u^2 - 6uu_{xx} + u_{xxxx} + 3u_{yy} = 0. \quad (1.4)$$

This version of the Kadomtsev–Petviashvili equation delineates the evolution of nonlinear together with long waves, possessive of small amplitude that slowly depends on its transverse coordinate. In deriving a completely integrable Eq. (1.4), the restriction that the waves be strictly one-dimensional was relaxed by Kadomtsev and Petviashvili [56, 58]. Besides, Eq. (1.4) delineates the evolution of shallow-water waves given in terms of a quasi-one-dimensional, especially in the presence of negligible surface tension and viscose effect.

Eq. (1.4) has been invoked in modelling diverse natural occurrences ranging from tsunami wave that travels in a non-homogeneous domain, emerging at the ocean base, to the study of water waves [59]. Moreover, it also surfaced in the analysis of nonlinear ion-acoustic waves existent in a magnetised dusty plasma [60]. A variety of research output for Eq. (1.4) has been achieved over the past few years. Resultantly, travelling wave solutions in the studies of Borhanifar et al. [61] and Khan and Akbar [62] and rogue wave, as well as a pair of resonance stripe solitons in Zhang et al.'s study [63], have been achieved for Eq. (1.4). Khalique [64] carried out symmetry reductions of the equation and also derived its conserved currents. In Zhao and Ma [65], the Hirota bilinear structure of Eq. (1.4) was implored in gaining mixed lump-kink solutions via Maple software. Further, rational lump solutions alongside line soliton pairs were established for Eq. (1.4) by engaging exponential alongside positive quadratic functions [66].

In this work, we seek to investigate conservation laws of the Kadomtsev–Petviashvili-like equation in three dimensions, which we abbreviated as 3D-extKPLEq and expressed as [67].

$$u_{tx} + \frac{3}{2}u^3 u_x + \frac{3}{2}u^2 u_{xx} + 3uu_x^2 + 3u_x u_{xx} + u_{yy} + u_{zz} = 0. \quad (1.5)$$

Eq. (1.5) was introduced as an extended Kadomtsev–Petviashvili-like equation through the use of a generalised bilinear DE. In their investigation, the bilinear representations in the Hirota sense were engaged by the authors alongside a certain transformation to derive Eq. (1.5). Moreover, based on generalised bilinear equation alongside Bell polynomial theories [68], the authors achieved 18 classes of rational solutions Eq. (1.5) through symbolic computation. Besides, Adeyemo and Khalique [69], in consonance with Lie symmetry reduction, utilised Kudryashov's technique to secure some closed-form solutions of Eq. (1.5). They also achieved a power series solution of the equation. Furthermore, Adeyemo and Khalique [70] went ahead to carry out a more detailed reduction process where various group-invariant solutions associated with the equation were calculated through the use of an optimal system of subalgebras. In consequence, various solutions that are noteworthy were obtained in the process.

However, this research work examines 3D-extKPLEq (1.5) explicitly through the use of the optimal system of subalgebras obtained from the Lie group technique to achieve various new conserved quantities of the equation. In plain terms, we state categorically that the work explicated in this study reveals a copious Lie group analysis of Eq. (1.5) where an optimal system is computed to obtain results, and for this reason the approach adopted in the present study may be adjudged novel. Thus, we organise the rest of this article in the following manner. In Section 2, we outline the methodical way of securing the Lie point symmetries via Lie group analysis of Eq. (1.5). We further calculate the Lie group transformation associated with the Lie point symmetries. Section 3 purveys the detailed steps taken in the construction of an optimal system of subalgebras using the computed symmetries. Section 4 furnishes us with the conservation laws of 3D-extKPLEq (1.5) by invoking Ibragimov's conserved vectors theorem via the associated formal Lagrangian. In addition, the applications of the obtained results in physical sciences are outlined. Later, the concluding remarks follow.

2. LIE GROUP ANALYSIS OF 3D-extKPLEq (1.5)

This section presents a systematic computation of the classical Lie symmetries of 3D-extKPLEq (1.5), which shall be used in the construction of an optimal system of one-dimensional Lie subalgebra. This, in turn, occasions the copious construction of conserved vectors related to the equation that is under study.

2.1 Determination of Lie point symmetries of Eq. (1.5)

The crucial step involved in the Lie symmetry technique [32, 33] is to determine the symmetry of NLPDEs. Thus, this subsection gives a detailed way of obtaining the infinitesimal generators of symmetries as well as geometric vector fields for the 3D-extKPLEq (1.5). Now, we examine an infinitesimal generator of Eq. (1.5) structured as

$$Q = \xi^1\frac{\partial}{\partial t}+\xi^2\frac{\partial}{\partial x}+\xi^3\frac{\partial}{\partial y}+\xi^4\frac{\partial}{\partial z}+\vartheta\frac{\partial}{\partial u},$$

with the coefficient functions (ξ1, ξ2, ξ3, ξ4, ϑ), all depending on (t, x, y, z, u).

Theorem 2.1 Let vector Q be the infinitesimal generators of the classical Lie point symmetry group of 3D-extKPLEq (1.5) where $\xi^i, i=1,2,3,4$ and ϑ are smooth functions of (t, x, y, z, u); then, we generate solutions that are of the form [69, 70]:

$$\left.\begin{aligned}&\xi^1=c_1+\frac{3c_6t}{2},\xi^2=\frac{1}{2}(2c_2+c_6x-c_4y-c_8z),\\&\xi^3=c_3+c_4t+c_6y-c_7z,\\&\xi^4=c_5+c_8t+c_7y+c_6z,\vartheta=-\frac{c_6u}{2}.\end{aligned}\right\}\quad(2.6)$$

with $c_is, \forall i=1,2,3,4,5,7$ regarded as arbitrary constants of solution.

Proof Vector Q generates all the classical Lie point symmetries of Eq. (1.5) if the symmetry invariant condition given as

$$pr^{(2)}Q\begin{bmatrix}u_{tx}+\frac{3}{2}u^3u_x+\frac{3}{2}u^2u_{xx}\\+3uu_x^2+3u_xu_{xx}+u_{yy}+u_{zz}\end{bmatrix}=0,\quad(2.7)$$

anytime.

$$u_{tx}+\frac{3}{2}u^3u_x+\frac{3}{2}u^2u_{xx}+3uu_x^2+3u_xu_{xx}+u_{yy}+u_{zz}=0$$

holds in which $pr^{(2)}Q$ stands for the second prolongation of vector Q, and it is expressed as

$$\begin{aligned}&pr^{(2)}Q=Q+\eta^t\partial_{u_t}+\eta^x\partial_{u_x}+\eta^{tx}\partial_{u_{tx}}\\&+\eta^{xx}\partial_{u_{xx}}+\eta^{yy}\partial_{u_{yy}}+\eta^{zz}\partial_{u_{zz}},\end{aligned}\quad(2.8)$$

representing the coefficient functions in pr(2)Q and the total derivatives Di in Eq. (2.9), given too in general form as

$$\begin{aligned}&\eta^i=D_i(\vartheta)-u_jD_i(\xi^j),\\&\eta^{ij}=D_j(\eta^i)-u_{ik}D_j(\xi^k),ijk=1,2,3,4,\end{aligned}\quad(2.9)$$

representing the coefficient functions in $pr^{(2)}Q$ and the total derivatives D_i in Eq. (2.9), given too in general form as

$$D_i=\frac{\partial}{\partial x^i}+u_i\frac{\partial}{\partial u}+u_{ij}\frac{\partial}{\partial u_j}+\cdots,$$

which can also be used for D_j. Thus, inserting the expanded form of relations Eq. (2.9) together with Eq. (2.8) into Eq. (2.7) gives a polynomial equation comprising diverse derivatives of *u*(*t, x, y, z*) whose coefficients also include certain derivatives with regards to ξ^1, ξ^2, ξ^3, ξ^4 and ϑ. Equating the individual coefficients to zero, one can secure the complete set of determining equations:

$$\begin{aligned}&\vartheta_{xu}=0,\ \ \xi_x^4=0,\ \ \xi_x^3=0,\ \ \xi_x^1=0,\ \ \xi_y^1=0,\\&\xi_z^1=0,\ \ \vartheta_{uu}=0,\xi_u^4=0,n\xi_u^3=0,\\&\xi_u^2=0,\ \ \xi_u^1=0,\ \ \xi_z^3+\xi_y^4=0,\\&2\xi_z^2+\xi_t^4=0,\ \ 2\xi_y^2+\xi_t^3=0,\\&\xi_{xx}^2-u\xi_x^2-\vartheta=0,\ \ \vartheta_u-2\xi_x^2+\xi_t^1=0,\\&2\vartheta_{yu}+\xi_{yz}^4-\xi_{yy}^3=0\\&2\xi_y^3-\xi_x^2-\xi_t^1=0,\ \ 2\xi_z^4-\xi_x^2-\xi_t^1=0,\\&2\vartheta_{zu}-\xi_{zz}^4-\xi_{yy}^4=0,\\&3\xi_x^2u^2-3\xi_t^1u^2-6\vartheta u-6\vartheta_x+2\xi_t^2=0,\\&3\vartheta_xu^3+3\vartheta_{xx}u^2+2\vartheta_{zz}+2\vartheta_{yy}+2\vartheta_{tx}=0,\\&3\xi_t^1u^3+9\vartheta u^2-3\xi_{tx}^1u^2+6\vartheta_xu+2\vartheta_{tu}+\xi_{tz}^4+\xi_{ty}^3=0.\end{aligned}$$

Thus, solving the system of equations via computer package, we arrive at the solutions given in Eq. (2.6) and so ends the proof of Theorem 2.1. In addition, Eq. (2.6) furnishes eight classical symmetries of 3D-extKPLEq (1.5) [69]. Therefore, we give a corollary:

Corollary 2.1 The classical Lie point infinitesimal projectable symmetries of Eq. (1.5) are computed as

$$\left.\begin{aligned}&Q_1=\frac{\partial}{\partial t},\quad Q_2=\frac{\partial}{\partial x},\quad Q_3=\frac{\partial}{\partial y},\\&Q_4=\frac{\partial}{\partial z},Q_5=y\frac{\partial}{\partial z}-z\frac{\partial}{\partial y},\\&Q_6=2t\frac{\partial}{\partial z}-z\frac{\partial}{\partial x},Q_7=2t\frac{\partial}{\partial y}-y\frac{\partial}{\partial x},\\&Q_8=3t\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}+2y\frac{\partial}{\partial y}+2z\frac{\partial}{\partial z}-u\frac{\partial}{\partial u}.\end{aligned}\right\}\quad(2.11)$$

Hence, 3D-extKPLEq (1.5) admits an eight-dimensional vector space, which constitutes its Lie algebra G and whose basis is structured as $\{Q_1,Q_2,Q_3,\ldots,Q_8\}$.

2.2 Calculation of Lie group transformations of Eq. (1.5)

We contemplate the exponentiation of the vector fields Eq. (2.11) by computing the flow or one parameter group generated by Eq. (2.11) via the Lie equations [32, 33, 71]

$$\left.\begin{aligned}&\frac{d\bar t}{d\varepsilon}=\xi^1(\bar t,\bar x,\bar y,\bar z,\bar u),\quad \bar t|_{\eth=0}=t,\\&\frac{d\bar x}{d\varepsilon}=\xi^2(\bar t,\bar x,\bar y,\bar z,\bar u),\quad \bar x|_{\eth=0}=x,\\&\frac{d\bar y}{d\varepsilon}=\xi^3(\bar t,\bar x,\bar y,\bar z,\bar u),\quad \bar y|_{\eth=0}=y,\\&\frac{d\bar z}{d\varepsilon}=\xi^4(\bar t,\bar x,\bar y,\bar z,\bar u),\quad \bar z|_{\eth=0}=z,\\&\frac{d\bar u}{d\varepsilon}=(\bar t,\bar x,\bar y,\bar z,\bar u),\quad \bar u|_{\eth=0}=u.\end{aligned}\right\}$$

Let us take, for instance, the rotation generator Q6, given as

$$Q_5=y\frac{\partial}{\partial z}-z\frac{\partial}{\partial y}.\quad(2.12)$$

The associated Lie equations are:

$$\frac{d\bar z}{d\varepsilon}=\bar y,\quad(2.13)$$

$$\frac{d\bar y}{d\varepsilon}=-\bar z,\quad(2.14)$$

with the initial criteria$(t\dot{Z}, x\dot{Z}, y\dot{Z}, z\dot{Z}, u\dot{Z})|_{\varepsilon=0} \to (t, x, y, z, u)$.

We first notice that

$$\frac{d^2\bar{z}}{d\varepsilon^2} = \frac{d\bar{y}}{d\varepsilon} = -\bar{z}. \tag{2.15}$$

Thus, we have a system of ordinary differential equations (ODEs). Now, from Eq. (2.15), we consider

$$\frac{d^2\bar{z}}{d\varepsilon^2} = -\bar{z}. \tag{2.16}$$

Solving the second-order ODE in Eq. (2.15) gives the trigonometric relation

$$\bar{z} = A\cos(\varepsilon) + B\sin(\varepsilon). \tag{2.17}$$

Applying the initial condition to Eq. (2.17) gives $z = A$. So, Eq. (2.17) becomes

$$\bar{z} = z\cos(\varepsilon) + B\sin(\varepsilon). \tag{2.18}$$

Now, solving the first-order ODE Eq. (2.15) in the same vein, we achieve

$$\bar{y} = Bcos(\varepsilon) - Asin(\varepsilon) + C_0, \tag{2.19}$$

Application of the initial condition to Eq. (2.19) and taking integration constant C0 = 0 yields y = B. Therefore, Eqs (2.18) and (2.19) become, respectively,

$$\bar{z} = z\cos(\varepsilon) + y\sin(\varepsilon), \bar{y} = y\cos(\varepsilon) - z\sin(\varepsilon). \tag{2.20}$$

Hence, following the same procedure for other generators in Eq. (2.11), we achieve one parameter transformation group of 3D-extKPLEq (1.5). Thus, we arrive at a theorem to that effect, viz.:

Theorem 2.2 Let $T^i(t,x,y,z,u), i = 1,2,3,\cdots 8$ be the transformations group of one parameterϵgenerated by vectors $Q_1, Q_2, Q_3\ldots, Q_8$ in Eq. (2.11); then, for each of the vectors, we have accordingly

$$\left.\begin{array}{l}
T^1_\varepsilon: (\tilde{t}, \tilde{x}, \tilde{y}, \tilde{z}, \tilde{u}) \to (t + \varepsilon_1, x, y, z, u),\\
T^2_\varepsilon: (\tilde{t}, \tilde{x}, \tilde{y}, \tilde{z}, \tilde{u}) \to (t, x + \varepsilon_2, y, z, u),\\
T^3_\varepsilon: (\tilde{t}, \tilde{x}, \tilde{y}, \tilde{z}, \tilde{u}) \to (t, x, y + \varepsilon_3, z, u),\\
T^4_\varepsilon: (\tilde{t}, \tilde{x}, \tilde{y}, \tilde{z}, \tilde{u}) \to (t, x, y, z + \varepsilon_4, u),\\
T^5_\varepsilon: (\tilde{t}, \tilde{x}, \tilde{y}, \tilde{z}, \tilde{u}) \to (t, x, y\cos(\varepsilon)\\
-z\sin(\grave{o}), z\cos(\varepsilon) + y\sin(\varepsilon), u),\\
T^6_\varepsilon: (\tilde{t}, \tilde{x}, \tilde{y}, \tilde{z}, \tilde{u}) \to (t, x - t\varepsilon_6^2 - z\varepsilon_6, y, z + 2\varepsilon_6 t, u),\\
T^7_\varepsilon: (\tilde{t}, \tilde{x}, \tilde{y}, \tilde{z}, \tilde{u}) \to (t, x - t\varepsilon_7^2 - y\varepsilon_7, y + 2\varepsilon_7 t, z, u),\\
T^8_\varepsilon: (\tilde{t}, \tilde{x}, \tilde{y}, \tilde{z}, \tilde{u}) \to (te^{3\varepsilon_7}, xe^{\grave{o}_7}, ye^{2\varepsilon_7}, ze^{2\varepsilon_7}, ue^{-\varepsilon_7}),
\end{array}\right\}$$

where $\varepsilon \in R$ is regarded as the group parameter.

Theorem 2.3 Hence, supposing that$u(t,x,y,z) = \Theta(t,x,y,z)$ satisfies 3D-extKPLEq (1.5), in the same vein, the functions given in the structure

$$\left.\begin{array}{l}
u^1(t,x,y,z) = \Theta(t - \varepsilon_1, x, y, z),\\
u^2(t,x,y,z) = \Theta(t, x - \varepsilon_2, y, z),\\
u^3(t,x,y,z) = \Theta(t, x, y - \varepsilon_3,\\
z, u), u^4(t,x,y,z) = \Theta(t, x, y, z - \varepsilon_4),\\
u^5(t,x,y,z) = \Theta(t, x, ycos(\varepsilon)\\
+zsin(\grave{o}), zcos(\varepsilon) - ysin(\varepsilon)),\\
u^6(t,x,y,z) = e^{\varepsilon_6}\Theta(t, x + t\varepsilon_6^2 + z\varepsilon_6, y, z - 2\varepsilon_6 t),\\
u^7(t,x,y,z) = \Theta(t, x + t\varepsilon_7^2 + y\varepsilon_7, y - 2\varepsilon_7 t, z),\\
u^8(t,x,y,z) = e^{\varepsilon_7}\Theta(te^{3\varepsilon_7}, xe^{\varepsilon_7}, ye^{2\varepsilon_7}, ze^{2\varepsilon_7}),
\end{array}\right\}$$

will do, where $u^i(t,x,y,z) = T_i^{\grave{o}} \cdot \Theta(t,x,y,z)$, $i = 1,2,3,\ldots,8$ with ϵ << 1 regarded as any positive real number.

3. OPTIMAL SYSTEM

Let us suppose that G is a Lie group with gl regarded as its Lie algebra. Then, for any element $H \in G$, we have an inner automorphism defined as Ha −⟶ HHaH−1 on the Lie group G. In addition, this automorphism of G influences an automorphism of Lie algebra gl. Therefore, the group of all these automorphisms constitutes a Lie group that is referred to as the adjoint group and we denote it as G^A. Now, for any arbitrary vectors $P, Q \in gl$, we can define $AdP(Q): Q \to [P,Q]$, which is a linear map and also an automorphism of gl referred to as the inner derivation of gl. Moreover, for all $P, Q \in gl$, the algebra of all inner derivations $AdP(Q)$ alongside Lie bracket $[AdP, AdQ] = Ad[P, Q]$ is regarded as a Lie algebra gl^A commonly referred to as the adjoint algebra of gl whereas gl^A is the Lie algebra of G^A. Therefore, two algebras in gl are said to be conjugate if there exists a transformation G^A that takes one subalgebra into the other. Hence, the collection of all pairwise non-conjugate q-dimensional subalgebras forms the optimal system of subalgebras of order q.

3.1 Optimal system of Lie subalgebra of 3D-extKPLEq (1.5)

In this subsection, the construction of optimal system one-dimensional subalgebras can be carried out by imploring a global matrix of the adjoint transformations as recommended by Ovsiannikov [32]. This task tends to help in determining a list (which we refer to as an optimal system) of conjugacy inequivalent subalgebras possessing the property that stipulates that any other subalgebra is equivalent to a unique list under some element of the adjoint representation. Consequently, we construct an optimal system of Lie subalgebra of 3D-extKPLEq (1.5). We begin with the adjoint action presented via the Lie series [33]

$$Ad(\exp(\varepsilon Q_i))Q_j = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!}(adQ_i)(Q_j). \tag{3.21}$$

The commutator relations of the infinitesimal generators for 3D-extKPLEq (1.5) in Eq. (2.11) with the $(i;j)th$ entry in designating the Lie bracket $[Q_i, Q_j] = Q_iQ_j - Q_jQ_i$. Thus, we observe that Tab. 1 has zero elements in the diagonal, and it is thus said to be skew-symmetric. Besides, the generators $Q_1, Q_2, Q_3, \ldots, Q_8$ are linearly independent.

Tab. 1. Commutator table of the Lie algebra of 3D-extKPLEq (1.5) $[Q_i, Q_j]$

Q_1	Q_2	Q_3	Q_4	Q_5	Q_6	Q_7	Q_8	
Q_1	0	0	0	0	0	0	0	Q_1
Q_2	0	0	0	0	Q_3	0	$-Q_1$	$2Q_2$
Q_3	0	0	0	0	$-Q_2$	$-Q_1$	0	$2Q_3$
Q_4	0	0	0	0	0	$2Q_3$	$2Q_2$	$3Q_4$
Q_5	0	$-Q_3$	Q_2	0	0	Q_7	$-Q_6$	0
Q_6	0	0	Q_1	$-2Q_3$	Q_7	0	0	$-Q_6$
Q_7	0	Q_1	0	$-2Q_2$	$-Q_6$	0	0	$-Q_7$
Q_8	$-Q_1$	$-2Q_2$	$-2Q_3$	$-3Q_4$	0	Q_6	Q_7	0

From the commutator relations existent between vectors Eq. (2.11) and given in Tab. 1, these infinitesimal generators in Eq. (2.11) can be purveyed as a linear combination of Qi as

$$\left.\begin{aligned}Q = \alpha_1 Q_1 + \alpha_2 Q_2 + \alpha_3 Q_3 + \alpha_4 Q_4 + \alpha_5 Q_5\\ +\alpha_6 Q_6 + \alpha_7 Q_7 + \alpha_8 Q_8.\end{aligned}\right\} \quad (3.22)$$

Next, we achieve the adjoint representation relations, as revealed in Tab. 2. Engaging the Olver approach [33], one can generate the adjoint representation of 3D-extKPLEq (1.5) via symbolic computation from the commutator relations with the aid of the secured vector fields.

3.1.1 Construction of general invariants

In order to achieve the optimal system of Lie algebra R^8, it is expedient to secure the invariants to aid the selection of the required representative elements. Using Tab. 1, one gains the needed matrix representations of Ad(Qi). In the foregoing, we will refer to Tab. 2 as follows, where

$\Delta_0 = Q_2\cos(\varepsilon_5) + Q_3\sin(\varepsilon_5), \Delta_1 = Q_3\cos(\varepsilon_5) - Q_2\sin(\varepsilon_5), \Delta_2 = Q_6\cos(\varepsilon_5) - Q_7\sin(\varepsilon_5)$, $\Delta_3 = Q_6\sin(\varepsilon_5) + Q_7\cos(\varepsilon_5)$, $\Delta_4 = Q_4 + 2\varepsilon_6 Q_3 - \varepsilon_6^2 Q_1$, $\Delta_5 = Q_4 + 2\varepsilon_7 Q_2 - \varepsilon_7^2 Q_1$, $\Delta_6 = Q_2 - \varepsilon_7 Q_1$, and

$\Delta_7 = Q_3 - \varepsilon_6 Q_1$.

$$\left.\begin{aligned}&Ad_{\exp(\grave{o}Q)}(Q) = e^{-\varepsilon Q} Q e^{\varepsilon Q}\\ &= Q - \varepsilon[Q,Q] + \tfrac{1}{2!}\varepsilon^2[Q,[Q,Q]] - \cdots\\ &= (a_1 Q_1 + \cdots + a_n Q_n) - \varepsilon[b_1 Q_1 + \cdots\\ &+ b_n Q_n, a_1 Q_1 + \cdots + a_n Q_n] + O(\varepsilon^2)\\ &= \quad (a_1 Q_1 + \cdots + a_n Q_n)\\ &-\grave{o}(\Theta_1 Q_1 + \cdots + \Theta_n Q_n) + O(\varepsilon^2),\end{aligned}\right\} \quad (3.23)$$

with function $\Theta_i \equiv \Theta_i(\alpha_1, \ldots, \alpha_8, \beta_1, \ldots, \beta_8)$. We observe that the values of $\Theta_i, i = 1, \ldots, 8$ can be achieved via the commutator table. On inserting the partial sums $Q = \sum_i^8 a_i Q_i$ as well as $Q = \sum_j^8 b_j Q_j$ in Eq. (3.23), we obtain Θ_i as

$$\left.\begin{aligned}&\Theta_1 = -\alpha_1\beta_8 + a_2\beta_7 + \alpha_3\beta_6 - a_6\beta_3 - \alpha_7\beta_2 + \alpha_8\beta_1,\\ &\Theta_2 = -2\alpha_2\beta_8 + \alpha_3\beta_5 - 2\alpha_4\beta_7 - \alpha_5\beta_3 + 2\alpha_7\beta_4 + 2\alpha_8\beta_2,\\ &\Theta_3 = -\alpha_2 b_5 - 2\alpha_3\beta_8 - 2\alpha_4\beta_6 + \alpha_5\beta_2 + 2\alpha_6\beta_4 + 2\alpha_8\beta_3,\\ &\Theta_4 = -3\alpha_4\beta_8 + 3\alpha_8\beta_4, \Theta_5 = 0,\ \Theta_6 = \alpha_5\beta_7 + \alpha_6\beta_8\\ &-\alpha_7\beta_5 - \alpha_8\beta_6,\\ &\Theta_7 = -\alpha_5\beta_6 + \alpha_6\beta_5 + \alpha_7\beta_8 - \alpha_8\beta_7,\ \Theta_8 = 0.\end{aligned}\right\} \quad (3.24)$$

Tab. 2. Adjoint representation table of 3D-extKPLEq (1.5)

Ad	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6	Q_7	Q_8
Q_1	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6	Q_7	$Q_8 - \epsilon_1 Q_1$
Q_2	Q_1	Q_2	Q_3	Q_4	$Q_5 - \epsilon_2 Q_3$	Q_6	$Q_7 + \epsilon_2 Q_1$	$Q_8 - 2\epsilon_2 Q_2$
Q_3	Q_1	Q_2	Q_3	Q_4	$Q_5 + \epsilon_3 Q_2$	$Q_6 + \epsilon_3 Q_1$	Q_7	$Q_8 - 2\epsilon_3 Q_3$
Q_4	Q_1	Q_2	Q_3	Q_4	Q_5	$Q_6 - 2\epsilon_4 Q_3$	$Q_7 - 2\epsilon_4 Q_2$	$Q_8 - 3\epsilon_4 Q_4$
Q_5	Q_1	Δ_0	Δ_1	Q_4	Q_5	Δ_2	Δ_3	Q_8
Q_6	Q_1	Q_2	Δ_7	Δ_4	$Q_5 + \epsilon_6 Q_7$	Q_6	Q_7	$Q_8 + \epsilon_6 Q_6$
Q_7	Q_1	Δ_6	Q_3	Δ_5	$Q_5 - \epsilon_7 Q_6$	Q_6	Q_7	$Q_8 + \epsilon_7 Q_7$
Q_8	$e^{\epsilon_8} Q_1$	$e^{2\epsilon_8} Q_2$	$e^{2\epsilon_8} Q_3$	$e^{3\epsilon_8} Q_4$	Q_5	$e^{-\epsilon_8} Q_6$	$e^{-\epsilon_8} Q_7$	Q_8

Now, for any β_j, where $1 \leq j \leq 8$, it is required to have

$$\left.\begin{aligned}0 = \Theta_1\frac{\partial\Phi}{\partial a_1} + \Theta_2\frac{\partial\Phi}{\partial a_2} + \Theta_3\frac{\partial\Phi}{\partial a_3} + \Theta_4\frac{\partial\Phi}{\partial a_4} + \Theta_5\frac{\partial\Phi}{\partial a_5}\\ +\Theta_6\frac{\partial\Phi}{\partial a_6} + \Theta_7\frac{\partial\Phi}{\partial a_7} + \Theta_8\frac{\partial\Phi}{\partial a_8}.\end{aligned}\right\} \quad (3.25)$$

Thus, by equating the coefficients of all same powers of βj in Eq. (3.25), one achieves the needed eight DEs with regard to real-valued function invariant

$$\left.\begin{aligned}&b_1: \quad a_8\frac{\partial\Phi}{\partial a_1} = 0,\\ &b_2: \quad -a_7\frac{\partial\Phi}{\partial a_1} + 2a_8\frac{\partial\Phi}{\partial a_2} + a_5\frac{\partial\Phi}{\partial a_3} = 0,\\ &b_3: \quad -a_6\frac{\partial\Phi}{\partial a_1} - a_5\frac{\partial\Phi}{\partial a_2} + 2a_8\frac{\partial\Phi}{\partial a_3} = 0,\\ &b_4: \quad 2a_7\frac{\partial\Phi}{\partial a_2} + 2a_6\frac{\partial\Phi}{\partial a_3} + 3a_8\frac{\partial\Phi}{\partial a_4} = 0,\\ &b_5: \quad a_3\frac{\partial\Phi}{\partial a_2} - a_2\frac{\partial\Phi}{\partial a_3} - a_7\frac{\partial\Phi}{\partial a_6} + a_6\frac{\partial\Phi}{\partial a_7} = 0,\\ &b_6: \quad a_3\frac{\partial\Phi}{\partial a_1} - 2a_4\frac{\partial\Phi}{\partial a_3} - a_8\frac{\partial\Phi}{\partial a_6} - a_5\frac{\partial\Phi}{\partial a_7} = 0,\\ &b_7: \quad a_2\frac{\partial\Phi}{\partial a_1} - 2a_4\frac{\partial\Phi}{\partial a_2} + a_5\frac{\partial\Phi}{\partial a_6} - a_8\frac{\partial\Phi}{\partial a_7} = 0,\\ &b_8: \quad \left.\begin{aligned}-a_1\frac{\partial\Phi}{\partial a_1} - 2a_2\frac{\partial\Phi}{\partial a_2} - 2a_3\frac{\partial\Phi}{\partial a_3} - 3a_4\frac{\partial\Phi}{\partial a_4}\\ +a_6\frac{\partial\Phi}{\partial a_6} + a_7\frac{\partial\Phi}{\partial a_7} = 0.\end{aligned}\right\}\end{aligned}\right\} \quad (3.26)$$

On solving the system of equations displayed in Eq. (3.26), one secures the value of invariant Φ(α1, α2, α3, α4, α5, α6, α7, α8) = G(α5, α8), which is referred to as the general invariant function of Lie algebra R8. Here, function G is an arbitrary function depending on α5 and α8. In consequence, 3D-extKPLEq (1.5) has only two basic invariants.

3.1.2 Calculation of the adjoint matrix for Eq. (1.5)

Consider a linear map $F^{\epsilon}: h \mapsto h$ defined by $Q \mapsto Ad(exp(\varepsilon_i Q_i).Q)$ where $i = 1,2,3,\ldots,8$. The presentation of the matrix B_i^{ε} of $F_i^{\varepsilon}, i = 1,2,3,\ldots,8$ with regard to basis $\{Q_1, Q_2, Q_3, \ldots, Q_8\}$ [72] is given as:

$$B_1^{\varepsilon_1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\ -\varepsilon_1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$B_2^{\varepsilon_2}=\begin{pmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&-\varepsilon_2&0&1&0&0&0\\0&0&0&0&0&1&0&0\\\varepsilon_2&0&0&0&0&0&1&0\\0&-2\varepsilon_2&0&0&0&0&0&1\end{pmatrix},$$

$$B_4^{\varepsilon_4}=\begin{pmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&-2\varepsilon_4&0&0&1&0&0\\0&-2\varepsilon_4&0&0&0&0&1&0\\0&0&0&-3\varepsilon_4&0&0&0&1\end{pmatrix},$$

$$B_3^{\varepsilon_3}=\begin{pmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&\varepsilon_3&0&0&1&0&0&0\\\varepsilon_3&0&0&0&0&1&0&0\\0&0&0&0&0&0&1&0\\0&0&-2\varepsilon_3&0&0&0&0&1\end{pmatrix},$$

$$B_5^{\varepsilon_5}=\begin{pmatrix}1&0&0&0&0&0&0&0\\0&\cos(\varepsilon_5)&\sin(\varepsilon_5)&0&0&0&0&0\\0&-\sin(\varepsilon_5)&\cos(\varepsilon_5)&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&\cos(\varepsilon_5)&-\sin(\varepsilon_5)&0\\0&0&0&0&0&\sin(\varepsilon_5)&\cos(\varepsilon_5)&0\\0&0&0&0&0&0&0&1\end{pmatrix},$$

$$B_6^{\varepsilon_6}=\begin{pmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\-\varepsilon_6&0&1&0&0&0&0&0\\-\grave{o}_6^2&0&2\varepsilon_6&1&0&0&0&0\\0&0&0&0&1&0&\varepsilon_6&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&\varepsilon_6&0&1\end{pmatrix},$$

$$B_7^{\grave{o}_7}=\begin{pmatrix}1&0&0&0&0&0&0&0\\-\varepsilon_7&1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\-\varepsilon_7^2&2\varepsilon_7&0&1&0&0&0&0\\0&0&0&0&1&-\varepsilon_7&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&\varepsilon_7&1\end{pmatrix},$$

$$B_8^{\varepsilon_8}=\begin{pmatrix}e^{\varepsilon_8}&0&0&0&0&0&0&0\\0&e^{2v_8}&0&0&0&0&0&0\\0&0&e^{2v_8}&0&0&0&0&0\\0&0&0&e^{3\varepsilon_8}&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&e^{-\varepsilon_8}&0&0\\0&0&0&0&0&0&e^{-\varepsilon_8}&0\\0&0&0&0&0&0&0&1\end{pmatrix}.$$

Consequently, we obtained the global matrix associated with $B_i^{\grave{o}}$ of $F_i^{\varepsilon}, i=1,2,3,\ldots,8$

$$B^{\varepsilon}=\begin{pmatrix}e^{\varepsilon_8}&0&0&0&0&0&0&0\\B_{21}^{\varepsilon}&e^{2\varepsilon_8}\cos(\varepsilon_5)&e^{2\varepsilon_8}\sin(\varepsilon_5)&0&0&0&0&0\\B_{31}^{\varepsilon}&-e^{2\varepsilon_8}\sin(\varepsilon_5)&e^{2\varepsilon_8}\cos(\varepsilon_5)&0&0&0&0&0\\B_{41}^{\varepsilon}&2e^{2\varepsilon_8}\varepsilon_7&2e^{2\varepsilon_8}\varepsilon_6&e^{3\varepsilon_8}&0&0&0&0\\B_{51}^{\varepsilon}&B_{52}^{\varepsilon}&B_{53}^{\varepsilon}&0&1&-e^{-\varepsilon_8}\varepsilon_7&e^{-\varepsilon_8}\varepsilon_6&0\\B_{61}^{\varepsilon}&B_{62}^{\varepsilon}&B_{63}^{\varepsilon}&0&0&e^{-\varepsilon_8}\cos(\varepsilon_5)&-e^{-\varepsilon_8}\sin(\varepsilon_5)&0\\B_{71}^{\varepsilon}&B_{72}^{\varepsilon}&B_{73}^{\varepsilon}&0&0&e^{-\varepsilon_8}\sin(\grave{o}_5)&e^{-\varepsilon_8}\cos(\varepsilon_5)&0\\B_{81}^{\varepsilon}&B_{82}^{\varepsilon}&B_{83}^{\varepsilon}&-3e^{3\varepsilon_8}\varepsilon_4&0&e^{-\varepsilon_8}\varepsilon_6&e^{-\varepsilon_8}\varepsilon_7&1\end{pmatrix}$$

where

$B_{21}^{\varepsilon}=e^{\varepsilon_8}(-\sin(\varepsilon_5)\varepsilon_6-\cos(\varepsilon_5)\varepsilon_7),$
$B_{31}^{\varepsilon}=e^{\varepsilon_8}(\sin(\varepsilon_5)v_7-\cos(\varepsilon_5)\varepsilon_6)$
$B_{51}^{\varepsilon}=e^{\varepsilon_8}\begin{pmatrix}(\cos(\varepsilon_5)\varepsilon_2-\sin(\varepsilon_5)\varepsilon_3)\varepsilon_6\\-(\sin(\varepsilon_5)\varepsilon_2+\cos(\varepsilon_5)\varepsilon_3)\varepsilon_7\end{pmatrix}$
$B_{61}^{\varepsilon}=e^{\varepsilon_8}(\varepsilon_3+2\cos(\varepsilon_5)\varepsilon_4\varepsilon_6-2\sin(\varepsilon_5)\varepsilon_4\varepsilon_7),$
$B_{71}^{\varepsilon}=e^{\varepsilon_8}(\varepsilon_2+2\sin(\varepsilon_5)\varepsilon_4\varepsilon_6+2\cos(\varepsilon_5)\varepsilon_4\varepsilon_7)$
$B_{81}^{\varepsilon}=e^{\varepsilon_8}\begin{pmatrix}3\varepsilon_4\varepsilon_6^2-\begin{pmatrix}-2\sin(\varepsilon_5)\varepsilon_2\\-2\cos(\varepsilon_5)\varepsilon_3\end{pmatrix}\varepsilon_6+3\varepsilon_4\varepsilon_7^2\\-\grave{o}_1-\begin{pmatrix}2\sin(\varepsilon_5)\varepsilon_3\\-2\cos(\varepsilon_5)\varepsilon_2\end{pmatrix}\varepsilon_7\end{pmatrix},$

$B_{52}^{\varepsilon}=e^{2\varepsilon_8}(\sin(\varepsilon_5)\varepsilon_2+\cos(\varepsilon_5)\varepsilon_3),$
$B_{82}^{\varepsilon}=e^{2\varepsilon_8}(-2\cos(\varepsilon_5)\varepsilon_2+2\sin(\varepsilon_5)\varepsilon_3-6\varepsilon_4\varepsilon_7),$
$B_{53}^{\varepsilon}=e^{2\varepsilon_8}(\sin(\varepsilon_5)\varepsilon_3-\cos(\varepsilon_5)\varepsilon_2),$
$B_{83}^{\varepsilon}=e^{2\varepsilon_8}(-2\sin(\varepsilon_5)\varepsilon_2-2\cos(\varepsilon_5)\varepsilon_3-6\varepsilon_4\varepsilon_6),$
$B_{62}^{\varepsilon}=2e^{2\varepsilon_8}\sin(\varepsilon_5)\varepsilon_4,$
$B_{72}^{\varepsilon}=-2e^{2\varepsilon_8}\cos(\varepsilon_5)\varepsilon_4, B_{63}^{\varepsilon}=-2e^{2\varepsilon_8}\cos(\varepsilon_5)\varepsilon_4,$
$B_{73}^{\varepsilon}=-2e^{2\varepsilon_8}\sin(\varepsilon_5)\varepsilon_4, B_{41}^{\varepsilon}=-e^{\varepsilon_8}(\varepsilon_6^2+\varepsilon_7^2).$

3.1.3 Adjoint transformation equation

Here, we compute the adjoint transformation equation associated to 3D-extKPLEq (1.5). We present the adjoint transformation via the relation

$$(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots \tilde{\alpha}_7, \tilde{\alpha}_8) = (\alpha_1, \alpha_2, \cdots, \alpha_7, \alpha_8)B^{\varepsilon}, \quad (3.27)$$

where
$Q = \sum_i^8 a_i Q_i$ and $Q = \sum_i^8 a_i Q_i$ are equivalent under the adjoint action and B^{ε}represents the universal adjoint matrix. Thus we have

$$\tilde{a}_1 = a_1 e^{\varepsilon_8} - a_4 e^{\varepsilon_8}(\varepsilon_6^2 + \varepsilon_7^2) - a_2 e^{\varepsilon_8}\begin{pmatrix}\varepsilon_6(\sin(\varepsilon_5)) \\ +\varepsilon_7\cos(\varepsilon_5)\end{pmatrix}$$
$$+a_3 e^{\varepsilon_8}(\varepsilon_7\sin(\varepsilon_5) - \varepsilon_6\cos(\varepsilon_5))$$
$$+a_5 e^{\grave{o}_8}\begin{pmatrix}\varepsilon_6(\varepsilon_2\cos(\varepsilon_5) - \varepsilon_3\sin(\varepsilon_5)) \\ -\varepsilon_7(\varepsilon_2\sin(\varepsilon_5) + \varepsilon_3\cos(\varepsilon_5))\end{pmatrix}$$
$$+a_7 e^{\varepsilon_8}(\varepsilon_2 + 2\varepsilon_4\varepsilon_6\sin(\varepsilon_5) + 2\varepsilon_4\varepsilon_7\cos(\varepsilon_5))$$
$$+a_6 e^{\varepsilon_8}(\varepsilon_3 - 2\varepsilon_4\varepsilon_7\sin(\varepsilon_5) + 2\varepsilon_4\varepsilon_6\cos(\varepsilon_5))$$
$$+a_8 e^{\varepsilon_8}(3\varepsilon_4\varepsilon_6^2 + 3\varepsilon_4\varepsilon_7^2 - \varepsilon_1 + \varepsilon_6 A - \varepsilon_7(2\varepsilon_3\sin(\varepsilon_5)$$
$$-2\varepsilon_2\cos(\varepsilon_5)))\Big),$$
$$A = (2\varepsilon_2\sin(\varepsilon_5) + 2\varepsilon_3\cos(\varepsilon_5))$$

$$\tilde{a}_2 = 2a_4 e^{2\varepsilon_8}\varepsilon_7 - a_3 e^{2\varepsilon_8}\sin(\varepsilon_5) + 2a_6 e^{2\varepsilon_8}\varepsilon_4\sin(\varepsilon_5)$$
$$+a_2 e^{2\varepsilon_8}\cos(\varepsilon_5) - 2a_7 e^{2\varepsilon_8}\grave{o}_4\cos(\varepsilon_5)$$
$$+a_5 e^{2\varepsilon_8}(\varepsilon_2\sin(\varepsilon_5) + \varepsilon_3\cos(\varepsilon_5))$$
$$+a_8 e^{2\varepsilon_8}(-6\varepsilon_4\varepsilon_7 + 2\varepsilon_3\sin(\varepsilon_5) - 2\varepsilon_2\cos(\varepsilon_5)),$$

$$\tilde{a}_3 = 2a_4 e^{2\varepsilon_8}\varepsilon_6 + a_2 e^{2\varepsilon_8}\sin(\varepsilon_5) - 2a_7 e^{2\varepsilon_8}\varepsilon_4\sin(\varepsilon_5)$$
$$+a_3 e^{2\varepsilon_8}\cos(\varepsilon_5) - 2a_6 e^{2\varepsilon_8}\varepsilon_4\cos(\varepsilon_5)$$
$$+a_5 e^{2\varepsilon_8}(\varepsilon_3\sin(\varepsilon_5) - \varepsilon_2\cos(\varepsilon_5))$$
$$+a_8 e^{2\varepsilon_8}(-6\varepsilon_4\varepsilon_6 - 2\varepsilon_2\sin(\varepsilon_5) - 2\varepsilon_3\cos(\varepsilon_5)),$$

$$\left.\begin{aligned}&\tilde{a}_4 = a_4 e^{3\varepsilon_8} - 3a_8 e^{3\varepsilon_8}\grave{o}_4, \tilde{a}_5 = 0,\\ &\tilde{a}_6 = a_8 e^{-\varepsilon_8}\grave{o}_6 - a_5 e^{-\varepsilon_8}\grave{o}_7\\ &+a_7 e^{-\varepsilon_8}\sin(\varepsilon_5) + a_6 e^{-\varepsilon_8}\cos(\varepsilon_5),\\ &\tilde{a}_7 = a_5 e^{-\varepsilon_8}\varepsilon_6 + a_8 e^{-\varepsilon_8}\varepsilon_7\\ &-a_6(e^{-\varepsilon_8})\sin(\varepsilon_5) + a_7 e^{-\varepsilon_8}\cos(\varepsilon_5),\\ &\tilde{a}_8 = 0.\end{aligned}\right\} \quad (3.28)$$

Next, we engage the adjoint system in Eq. (3.28) in computing the one-dimensional subalgebra optimal system of 3D-extKPLEq (1.5).

3.1.4 Computation of 1-D subalgebra optimal system

Now, having secured the general invariant of 3D-extKPLEq (1.5) as $G(\alpha_5, \alpha_8)$ in a systematic way as earlier demonstrated, we calculate the optimal system of the equation using the invariant. We adopt the technique introduced in Hu et al.'s study [72] and so contemplate cases $\alpha_5 = 1$, $\alpha_8 = 1$ and $\alpha_5\alpha_8 = 0$ fundamentally on the bases of thesign of the invariants.

Case 1.

$\alpha_5 = 1, \alpha_8 = 1$

We choose based on this case, the representative element $Q = Q_5 + Q_8$.

On inserting the parameters $\tilde{\alpha}_i = 0, i = 1,2,3,4,6,7$ alongside $\tilde{\alpha}_i = 1, i = 5,8$ into the adjoint system in Eq. (3.28), we achieve the solution

$$\varepsilon_1 = a_1 - \tfrac{4}{15}a_4 a_6^2 - \tfrac{4}{15}a_4 a_7^2 - \tfrac{1}{5}a_2 a_6$$
$$+\tfrac{2}{5}a_2 a_7 + \tfrac{2}{5}a_3 a_6 + \tfrac{1}{5}a_3 a_7,$$
$$\varepsilon_2 = \tfrac{1}{5}a_3 - \tfrac{2}{15}a_6 a_4 - \tfrac{4}{15}a_7 a_4 + \tfrac{2}{5}a_2,$$
$$\varepsilon_3 = \tfrac{2}{15}a_7 a_4 - \tfrac{1}{5}a_2 - \tfrac{4}{15}a_6 a_4 + \tfrac{2}{5}a_3,$$
$$\varepsilon_4 = \tfrac{1}{3}a_4,$$
$$\varepsilon_6 = \tfrac{1}{2}a_6\sin(\varepsilon_5) - \tfrac{1}{2}a_7\cos(\varepsilon_5)$$
$$-\tfrac{1}{2}a_7\sin(\varepsilon_5) - \tfrac{1}{2}a_6\cos(\varepsilon_5),$$
$$\varepsilon_7 = \tfrac{1}{2}a_7\sin(\varepsilon_5) + \tfrac{1}{2}a_6\cos(\varepsilon_5)$$
$$+\tfrac{1}{2}a_6\sin(\varepsilon_5) - \tfrac{1}{2}a_7\cos(\varepsilon_5).$$

Case 2.

$\alpha_5\alpha_8 = 0$

In this case, we contemplate three different situations. These situations are treated in detail in the subsequent part of the research work.

Case 2.1.

$\alpha_5 = 0, \alpha_8 = 1$

We choose the representative element $Q = Q_8$. On substituting parametric values $\tilde{\alpha}_i = 0$ and $i = 1, 2, 3, \ldots, 7$ as well as $\tilde{\alpha}_8 = 1$ into the adjoint system in Eq. (3.28), we obtain the outcome

$$\varepsilon_1 = a_1 - \tfrac{1}{3}a_4 a_6^2 - \tfrac{1}{3}a_4 a_7^2 + \tfrac{1}{2}a_2 a_7 + \tfrac{1}{2}a_3 a_6,$$
$$\varepsilon_2 = \tfrac{1}{2}a_2 - \tfrac{1}{3}a_7 a_4, \varepsilon_3 = -\tfrac{1}{3}a_6 a_4 + \tfrac{1}{2}a_3,$$
$$\varepsilon_4 = \tfrac{1}{3}a_4, \varepsilon_6 = -a_7\sin(\varepsilon_5) - a_6\cos(\varepsilon_5),$$
$$\varepsilon_7 = a_6\sin(\varepsilon_5) - a_7\cos(\varepsilon_5).$$

Case 2.2.

$\alpha_5 = 1, \alpha_8 = 0$

In this subcase, we select the representative element $Q = Q_5$. On invoking parameters $\hat{\alpha}_i = 0$, $i = 1,2,3,4,6\ldots,8$ together with $\tilde{\alpha}_5 = 1$ into the adjoint system in Eq. (3.28), one secures

$$\varepsilon_2 = a_3 - 2a_6\varepsilon_4, \varepsilon_3 = -\tfrac{1}{a_6}(a_1 + a_7\varepsilon_2),$$
$$\varepsilon_4 = -\tfrac{1}{2a_6 a_4}(a_1 + a_7\varepsilon_2 - a_2 a_6), \varepsilon_4 = \tfrac{1}{3}a_4,$$
$$\varepsilon_6 = a_6\sin(\varepsilon_5) - a_7\cos(\varepsilon_5),$$
$$\varepsilon_7 = a_7\sin(\varepsilon_5) + a_6\cos(\varepsilon_5).$$

Case 2.3.

$\alpha_5 = 0, \alpha_8 = 0$

Here, we insert $\alpha_5 = 0$ and $\alpha_8 = 0$ into the system of partial differential equations elucidated in Eq. (3.26) and solve the resultant equations. So, we gain a new invariant function

$$\Phi(a_1, a_2, a_3, a_4, a_6, a_7) = H\left\{\sqrt[3]{a_4^2}(a_6^2 + a_7^2)\right\}.$$

Thus, we engage the new invariants to find more subalgebras of Eq. (1.5) in detail.

Case 2.3.1

$a_4 \neq 0, a_6^2 = 1 - a_7^2$

We select the representative element Q = $Q_4 + Q_6 - Q_7$. If one substitutes parameters $\tilde{\alpha}_i = 0$, i = 1, 2, 3, 6, 8 together with $\tilde{\alpha}_i = 1$, i = 4, 5, 7 into the adjoint system in Eq. (3.28), one can achieve the solution

$$\varepsilon_3 = -\frac{1}{16a_4}\begin{pmatrix}4\sqrt{2}a_4a_6^4\varepsilon_4^2 + \sqrt{2}a_4a_2^2a_6^2\\ +\sqrt{2}a_3^2a_4a_6^2 - 16a_3\varepsilon_4\end{pmatrix},$$
$$\varepsilon_5 = -\frac{1}{4}\pi, \varepsilon_6 = -\frac{1}{8}{a_6}^3(a_2 + a_3 - 2a_6\varepsilon_4),$$
$$\varepsilon_7 = -\frac{1}{8}a_6^3(a_2 - a_3 + 2a_6\varepsilon_4), \varepsilon_8 = ln\left(\frac{\sqrt{2}}{2}a_6\right).$$

Case 2.3.2.

$a_4 = 0, a_6^2 = 1 - a_7^2$

In this occurrence, we choose the representative element Q = $Q_6 - Q_7$. On substituting parametric values $\tilde{\alpha}_i = 0$, i = 1, 2, 3, 4, 5, 8 with $\tilde{\alpha}_6 = 1$, $\tilde{\alpha}_7 = -1$ into the adjoint system in Eq. (3.28), one then obtains a result that can be presented in the following manner:

$$\varepsilon_3 = -\frac{a_1}{a_6}, \varepsilon_4 = \frac{a_3}{2a_6},$$
$$\varepsilon_5 = \frac{1}{4}\pi, \varepsilon_8 = \ln\left(\frac{\sqrt{2}}{2}a_6\right).$$

Remark 3.1

We observe that other possible representatives from Case 2.3.2 have been obtained earlier, thereby contributing no additional subalgebra to the optimal system.

Next, we contemplate other possible cases (under Case 3) that can be secured by further scaling down on Q in Eq. (3.22) using the adjoint system in Eq. (3.28) in order to gain more subalgebras of 3D-extKPLEq (1.5).

Case 3.

$\boldsymbol{a_1 \neq 0, a_2 \neq 0, a_3 \neq 0}$

On scaling further vector Eq. (3.22), we select the representative element $Q = Q_1 + Q_2 + Q_3$.

Therefore, by inserting the parameters $\tilde{\alpha}_\mathrm{i} = 0, \mathrm{i} = 4,5, \ldots ,8$ and $\tilde{\alpha}_\mathrm{i} = 1, \mathrm{i} = 1,2,3$ into the adjoint system in Eq. (3.28), we obtain

$$\varepsilon_5 = \arctan\left(\sqrt{\frac{(a_2+a_3)^2}{a_2^2+a_3^2}}\right),$$
$$\varepsilon_6 = \sqrt{\frac{2a_1^2}{a_2^2+a_3^2}} - \left(\sqrt{\frac{2}{a_2^2+a_3^2}}\right)^{1/2} - \varepsilon_7,$$
$$\varepsilon_8 = \frac{1}{2}\ln\left(\sqrt{\frac{2}{a_2^2+a_3^2}}\right).$$

Case 3.1.

$a_1 = 0, a_2 \neq 0, a_3 \neq 0$

Here, we examine two possibilities. When $\alpha_1 = 0, \alpha_2 > 0$, $\alpha_3 > 0$, we select optimal representative element $Q = Q_2 + Q_3$. So, on substituting parameters $\tilde{\alpha}_i = 0, i = 1,4,5,\ldots,8$ with $\tilde{\boldsymbol{\alpha}}_i = 1, i = 1,2,3$ into the adjoint system in Eq. (3.28), we gain the solution

$$\varepsilon_5 = \frac{1}{4}\pi, \varepsilon_6 = -\varepsilon_7, \varepsilon_8 = \frac{1}{2}\ln\left(\frac{\sqrt{2}}{a_2}\right).$$

Next, we consider when $\alpha_1 = 0, \alpha_2 < 0, \alpha_3 < 0$, and so we select the optimal representative element Q = $-Q_2 - Q_3$ then by inserting the appropriate parametric values of $\tilde{\alpha}_i$ in Eq. (3.28) as earlier demonstrated, which yields the result

$$\varepsilon_5 = \frac{1}{4}\pi, \varepsilon_6 = -\varepsilon_7, \varepsilon_8 = \frac{1}{4}\ln\left(\frac{2}{a_2^2}\right).$$

Case 3.2.

$a_1 \neq 0,$

Further, we investigate two possible cases. So with $\alpha_1 > 0, \alpha_2 > 0, \alpha_3 = 0$, we select the optimal representative element $Q = Q_1 + Q_2$. Thus, on invoking parameters $\hat{\alpha}_i = 0, i = 3,4,5,\ldots,8$ with $\tilde{\alpha}_i = 1, i = 1,2$ into the adjoint system in Eq. (3.28), we obtain the outcome

$$\varepsilon_5 = 0, \varepsilon_7 = \frac{1}{\sqrt[3]{a_2^2}}(a_1\sqrt{a_2} - a_2),$$
$$\varepsilon_8 = -\frac{1}{2}\ln(a_2).$$

Now, when $\alpha_1 < 0, \alpha_2 < 0, \alpha_3 = 0$, we choose the optimal representative element $Q = -Q_1 - Q_2$. Then, on engaging the appropriate parametric values of $\tilde{\alpha}$ in Eq. (3.28), we secure the solution

$$\varepsilon_5 = 0, \varepsilon_7 = \frac{a_1}{a_2} - \sqrt{\left|-\frac{1}{a_2}\right|}, \varepsilon_8 = \frac{1}{2}\ln\left(\left|-\frac{1}{a_2}\right|\right).$$

Case 3.3.

$a_1 \neq 0, a_2 = 0, a_3 \neq 0$

We explore here two possible situations. Therefore, for $\alpha_1 > 0, \alpha_2 = 0, \alpha_3 > 0$, we choose the optimal representative element $Q = Q_1 + Q_3$. Hence, on inserting parametric values $\hat{\alpha}_i = 0, i = 2,4,5,\ldots,8$ with $\hat{\alpha}_i = 1, i = 1,3$ into the adjoint system in Eq. (3.28), we achieve the result

$$\varepsilon_5 = \pi, \varepsilon_6 = -\frac{1}{a_3}\left(a_1 + a_3\left|-\frac{1}{a_3}\right|\right),$$
$$\varepsilon_8 = \frac{1}{2}\ln\left(\left|-\frac{1}{a_3}\right|\right).$$

Next, we take on $\alpha_1 < 0, \alpha_2 = 0, \alpha_3 < 0$ and select the optimal representative element $Q = -Q_1 - Q_3$. The substitution of the relevant parametric values of $\tilde{\alpha}$ in Eq. (3.28) purveyed the outcome

$$\varepsilon_5 = 0, \varepsilon_6 = \frac{a_1}{a_3} - a_3\sqrt{\left|-\frac{1}{a_3}\right|},$$
$$\varepsilon_8 = \frac{1}{2}\ln\left(\left|-\frac{1}{a_3}\right|\right).$$

Moreover, we further scale down and gain a single vector as part of the subalgebras of Eq. (1.5). On selecting representative element Q = Q_1 and solving Eq. (3.28) with adequate values of $\tilde{\alpha}$, we obtain $\grave{o}_8 = -ln(\alpha_1)$. Engaging representative element Q = Q_2, and following the same process, we gain

$$\varepsilon_5 = \pi, \varepsilon_7 = 0, \varepsilon_8 = \frac{1}{2}\ln\left(-\frac{1}{a_2}\right),$$

Finally, by selecting the optimal representative element $Q = Q_3$ and solving the system in Eq. (3.28) for relevant parametric values of α̃, one obtains the solution $\varepsilon_5 = \varepsilon_6 = 0$, and $\varepsilon_8 = -\frac{1}{2}\ln(a_3)$. In consequence, if we admit the discrete symmetry $(t, x, y, z, u) \mapsto (-t, -x, -y, -z, u)$, albeit not in the connected component of the identity of the full symmetry group that, for instance, maps $-Q_2 - Q_3$ to $Q_2 + Q_3$, we reduce the number of inequivalent subalgebras [33]. Thus, we arrive at the following theorem:

Theorem 3.1

The optimal system of one-dimensional Lie subalgebra for 3D-extKPLEq (1.5) comprises members listed as:

$$Q_1;\ Q_2;\ Q_3;\ Q_5 + Q_8;\ Q_2 + Q_3;$$
$$Q_6 - Q_7;\ Q_5; Q_8; Q_1 + Q_2 + Q_3;\cdot$$
$$Q_4 + Q_6 - Q_7; Q_1 + Q_2; Q_1 + Q_3$$

Remark 3.2

It is to be noted that the usual tradition is to invoke the subalgebras to obtain invariant solutions to the model under consideration but various invariant solutions associated with the subalgebras presented in Theorem 3.1 have been copiously explored, as can be seen in the study of Adeyemo and Khalique [70]. Thus, we do not consider using the subalgebras to find invariants anymore but to compute the conserved vectors related to the vectors in the subsequent part of the research paper. This is an interesting aspect of this study.

4. CONSERVATION LAWS OF 3D-extKPLEq (1.5)

In this section, we perform the computation of conserved vectors of Eq. (1.5) for the presented Lie subalgebras in Theorem 3.1 by exploiting Ibragimov's theorem [73, 74] for conserved quantities.

4.1 Formal Lagrangian and adjoint equation

Ibragimov [74] gave a new theorem on the conserved vectors of a DE. The theorem validates any system of DE whereby the number of the equation equals the number of dependent variables. In addition, the underlying theorem requires no existence of classical Lagrangian. Ibragimov's approach basically suggests that every infinitesimal generator is related to a conserved vector. Moreover, it is conceptualised based on adjoint equations for nonlinear DEs. We give a detailed outline of the theorem as follows:

Theorem 4.1

[74] We contemplate an NLPDE system whose adjoint is the system

$$\left.\begin{aligned}\Omega_\alpha^*(x, \Psi, \Theta, \Psi_{(1)}, \Theta_{(1)}, \cdots, \Psi_{(s)}, \Theta_{(s)}) \equiv \frac{\delta(\Theta^\beta \Omega_\beta)}{\delta \Psi^\alpha}\\ = 0\end{aligned}\right\} \tag{4.29}$$

with α = 1, . . . , p and β = 1, . . . , p for a system of p equations defined as

$$\Omega_\alpha(x, \Psi, \Psi_{(1)}, \Psi_{(2)}, \Psi_{(3)}, \cdots, \Psi_{(s)}) = 0, \tag{4.30}$$

where Θ = Θ(x) with q independent as well as p dependent variables given, respectively, as $x = (x^1, x^2, \cdots, x^q)$ and $\Psi = (\Psi^1, \Psi^2, \cdots, \Psi^p)$, alongside the variational derivative popularly called the Euler-Lagrange operator, for each α.

Theorem 4.2

Let us suppose that a system of p of Eq. (4.30) is contemplated; then, the system of adjoint presented as Eq. (4.29) inherits the Lie point symmetries of Eq. (4.30), which implies that, if the system in Eq. (4.30) admits a group of point transformation with a generator given as $X = \xi^i \partial/\partial t + \eta\, \partial/\partial \Psi^\alpha$ such that the adjoint system in Eq. (4.29) admits operator X with an extension to the variables Θ^α via the relation

$$Y = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial \Psi^\alpha} + \eta_*^\alpha \frac{\partial}{\partial \Theta^\alpha} \tag{4.31}$$

with suitably selected $\eta_*^\alpha = \eta_*^\alpha(x, \Psi, \Theta)$.

The functions ξ^i as well as η^α signify the infinitesimal generator coefficients that are depending on x alongside Ψ. We are aware that the coefficient η^α is given in Ibragimov's study [74] as

$$\eta^\alpha = -\left[\lambda^\alpha \Theta^\beta + \Theta^\alpha D_i(\xi^i)\right] \tag{4.32}$$

where the computation of λ_β^α can be carried out by invoking the relation

$$Q(\Omega_\alpha) = \lambda_\beta^\alpha \Omega_\beta. \tag{4.33}$$

In consequence, a conserved quantity for the system comprising both the equation and its adjoint is produced, with T^i possessing the conserved vectors' components $T = (T^1, \cdots, T^n)$, which we can decided by the relation

$$T^i = \xi^i \mathcal{L} + W^\alpha \frac{\delta \mathcal{L}}{\delta \Psi_i^\alpha} + \sum_{s \geq 1} D_{i_1} \dots D_{i_s}(W^\alpha) \frac{\delta e \mathcal{L}}{\delta e \Psi_{i i_1 i_2 \dots i_s}^\alpha}, \tag{4.34}$$
$$i = 1, \dots, n$$

with formal Lagrangian L of system Ω and Ω* together with Lie characteristicfunction W^α stated, respectively, as

$$\mathcal{L} = \Theta^\alpha \Omega_\alpha(x, \Psi, \Psi_{(1)}, \cdots, \Psi_{(s)}) \tag{4.35}$$

and

$$W^\alpha = \eta^\alpha - \xi^j \Psi_j^\alpha, \alpha = 1, \cdots, p, j = 1, \cdots, q. \tag{4.36}$$

Remark 4.1

We remark that a system of Eq. (4.30) is self-adjoint if the replacement Θ = Ψ in the system of adjoint Eq. (4.29) produces the same system as that of Eq. (4.30). For a more detailed understanding of the proof and more information on the results presented here, the reader is directed to refer to the studies of Ibragimov [73, 74].

4.1.1 Application of Ibragimov's theorem to derive conserved quantities

The interesting fact about the concept of engaging Ibragimov's theorem of conservation law is that, to every Lie point symmetry of a given DE, it is believed that there exists a unique conserved quantity. In consequence, we devote this subsection to secure conservation laws related to the underlying equation via Ibragimov's conserved theorem [2, 73–75]. Using the salient information provided in the given references, we have the theorem:

Theorem 4.3

The adjoint equation of 3D-extKPLEq (1.5) is expressed as

$$\left.\begin{aligned}G^* &\equiv v_{tx}-\tfrac{3}{2}v_xu^3+\tfrac{3}{2}v_{xx}u^2+3u_{xx}v_x\\&+3u_xv_{xx}+v_{yy}+v_{zz}=0\end{aligned}\right\} \tag{4.37}$$

and the formal Lagrangian given as

$$\mathcal{L}=vG\equiv v\begin{pmatrix}u_{tx}+\frac{3}{2}u_xu^3+\frac{3}{2}u_{xx}u^2+3u_x^2u\\+3u_xu_{xx}+u_{yy}+u_{zz}\end{pmatrix}, \tag{4.38}$$

where

$$\left.\begin{aligned}G&=u_{tx}+\tfrac{3}{2}u_xu^3+\tfrac{3}{2}u_{xx}u^2+3u_x^2u\\&+3u_xu_{xx}+u_{yy}+u_{zz}.\end{aligned}\right\}. \tag{4.39}$$

Proof

Suppose that we define the adjoint of Eq. (1.5) as [74]

$$\left.\begin{aligned}G^*&=\frac{\delta}{\delta u}\left\{v\begin{pmatrix}u_{tx}+\frac{3}{2}u_xu^3+\frac{3}{2}u_{xx}u^2+3u_x^2u\\+3u_xu_{xx}+u_{yy}+u_{zz}\end{pmatrix}\right\}\\&=\begin{bmatrix}\frac{\partial}{\partial u}-D_x\frac{\partial}{\partial u_x}+D_x^2\frac{\partial}{\partial u_{xx}}+D_xD_t\frac{\partial}{\partial u_{xt}}\\+D_y^2\frac{\partial}{\partial u_{yy}}+D_z^2\frac{\partial}{\partial u_{zz}}\end{bmatrix}\\&\times\left\{v\begin{pmatrix}\frac{3}{2}u_xu^3+\frac{3}{2}u_{xx}u^2+3u_x^2u+3u_xu_{xx}\\+u_{tx}+u_{yy}+u_{zz}\end{pmatrix}\right\}\end{aligned}\right\}$$

$$=v_{tx}-\tfrac{3}{2}v_xu^3+\tfrac{3}{2}v_{xx}u^2+3u_{xx}v_x+3u_xv_{xx}+v_{yy}+v_{zz}=0. \tag{4.40}$$

We are aware that we have introduced a new variable $v=v(t,x,y,z)$. In consonance with the approach adopted in Ibragimov's study [74], Eq. (1.5) alongside its adjoint Eq. (4.40) possesses a Lagrangian L presented in the structure L = *vG* as given in Theorem 4.3. We notice specifically that ∂L/δ*u* = G* whereas ∂L/δ*v* = *G*. Now, all the symmetries $Q_1,\dots,Q_8$ as well as the Lie subalgebras computed in Section 3 are admitted by Eq. (1.5). Thus, the subalgebras of Eq. (1.5) are extended to the new variable *v*(*t, x, y, z*), which implies that they become

$$Y=\xi^1\frac{\partial}{\partial t}+\xi^2\frac{\partial}{\partial x}+\xi^3\frac{\partial}{\partial y}+\xi^4\frac{\partial}{\partial z}+\eta\frac{\partial}{\partial u}+\eta^*\frac{\partial}{\partial v}, \tag{4.41}$$

where

$$\begin{aligned}\eta^*&\equiv\eta^*(t,x,y,z,u,v)\\&=-\left\{\begin{aligned}&\lambda+D_t(\xi^1)+D_x(\xi^2)\\&+D_y(\xi^3)+D_z(\xi^4)\end{aligned}\right\}v.\end{aligned} \tag{4.42}$$

We decide parameter *λ* via the relation

$$Q[2](G)=\lambda G \tag{4.43}$$

with $Q^{[2]}$ denoting the generators that Eq. (2.11) prolonged to all of the various derivatives in Eq. (1.5), that is

$$\begin{aligned}Q^{[2]}&=Q+\zeta^x\frac{\partial}{\partial u_x}+\zeta^{tx}\frac{\partial}{\partial u_{tx}}+\zeta^{xx}\frac{\partial}{\partial u_{xx}}\\&+\zeta^{yy}\frac{\partial}{\partial u_{yy}}+\zeta^{zz}\frac{\partial}{\partial u_{zz}}.\end{aligned} \tag{4.44}$$

In this case, the vector field

$$\left.\begin{aligned}Q&=\xi^1\partial/\partial t+\xi^2\partial/\partial x+\xi^3\partial/\partial y\\&+\xi^4\partial/\partial z+\eta\,\partial/\partial u\end{aligned}\right\}$$

and the functions $\xi^1,\xi^2,\cdots,\xi^4$ alongside *η* are depending on (t,x,y,z,u).

Moreover, coefficient functions $\zeta^x,\zeta^{tx},\zeta^{xx},\zeta^{yy}$ and ζ^{zz} are explicated through the formulae given as

$$\left.\begin{aligned}\zeta^x&=D_x(\eta)-u_tD_x(\xi^1)-u_xD_x(\xi^2)\\&-u_yD_x(\xi^3)-u_zD_x(\xi^4),\\\zeta^{tx}&=D_t(\zeta^x)-u_{tx}D_t(\xi^1)-u_{xx}D_t(\xi^2)\\&-u_{xy}D_t(\xi^3)-u_{xz}D_t(\xi^4),\\\zeta^{xx}&=D_x(\zeta^x)-u_{tx}D_x(\xi^1)-u_{xx}D_x(\xi^2)\\&-u_{xy}D_x(\xi^3)-u_{xz}D_x(\xi^4),\\\zeta^{yy}&=D_y(\zeta^y)-u_{ty}D_y(\xi^1)-u_{xy}D_y(\xi^2)\\&-u_{yy}D_y(\xi^3)-u_{yz}D_y(\xi^4),\\\zeta^{zz}&=D_z(\zeta^z)-u_{tz}D_z(\xi^1)-u_{xz}D_z(\xi^2)\\&-u_{yz}D_z(\xi^3)-u_{zz}D_z(\xi^4),\end{aligned}\right\} \tag{4.45}$$

where the total derivatives appearing in Eq. (4.45) are given as

$$\left.\begin{aligned}D_t&=\partial_t+u_t\,\partial_u+u_{tt}\,\partial_ut+u_{xt}\,\partial_ux+\cdots,\\D_x&=\partial_x+u_x\,\partial_u+u_{xt}\,\partial_ut+u_{xx}\,\partial_ux+\cdots,\\D_y&=\partial_y+u_y\,\partial_u+u_{ty}\,\partial_ut+u_{xy}\,\partial_ux+\cdots,\\D_z&=\partial_z+u_z\,\partial_u+u_{tz}\,\partial_ut+u_{xz}\,\partial_ux+\cdots.\end{aligned}\right\} \tag{4.46}$$

Readers are directed to the given references for acquiring an understanding of the detailed steps involved in the calculation of conserved vectors of NLPDEs via the considered technique. For instance, in the case of Lie subalgebra $Q_1,\cdots,Q_3$ and $Q_4\equiv Q_5+Q_8$ one can compute as follows:

Case $Q_1,\dots,Q_3$

For the time translation symmetry
$Q_1=\partial/\partial t$,
we have functions

$$\xi^1=1,\xi^2=\xi^3=\xi^4=\eta=0.$$

Therefore, one can easily observe that $\zeta^x=\zeta^{xx}=\zeta^{tx}=\zeta^{yy}=\zeta^{zz}=0$.

Thus, we have

$Q^{[2]}(G)=0G$, that is *λ* = 0.

Furthermore, from Eq. (4.42) we obtain $\eta^*=0$, which makes the new subalgebra Eq. (2.11) to retain the structure of Q_1, that is $Y_1=\partial/\partial t$. Resultant to the fact that the coefficients of the generators are all constants, the translation symmetries conserve their structures, and this leads us to the conclusion that $Y_2=\partial/\partial x, Y_3=\partial/\partial y$ and $Y_4=\partial/\partial z$.

Case $Q_4=Q_5+Q_8$

$$\left.\begin{aligned}Q_4&\equiv Q_5+Q_8=3t\,\partial/\partial t+x\,\partial/\partial x\\&+(2y-z)\,\partial/\partial y+(y+2z)\,\partial/\partial z-u\,\partial/\partial u\end{aligned}\right\},$$

where $\xi^1=3t,\xi^2=x,\ \xi^3=2y-z,\ \xi^4=y+2z$ and $\eta=-u$. Thus, on engaging Eq. (4.45), the result of our calculation reveals

$$\begin{aligned}&\zeta^x=-2u_x,\ \zeta^{tx}=-5u_{tx},\ \zeta^{xx}=-3u_{xx},\\&\zeta^{yy}=-2u_{yz}-5u_{yy},\ \zeta^{zz}=2u_{yz}-5u_{zz}\end{aligned}.$$

From Eqs (4.43)–(4.46), we have

$$Q_4^{[2]}(G)=\begin{pmatrix}-5u_{tx}-\frac{15}{2}u^2u_{xx}-15uu_x^2-\frac{15}{2}u^3u_x\\ -15u_xu_{xx}-5u_{yy}-5u_{zz}\end{pmatrix}$$
$$=-5\begin{pmatrix}u_{tx}+\frac{3}{2}u_xu^3+\frac{3}{2}u_{xx}u^2+3u_x^2u\\ +3u_xu_{xx}+u_{yy}+u_{zz}\end{pmatrix}$$
$$=-5G.$$

We can see then that $\lambda=-5$. In consequence

$$\left.\begin{aligned}&\eta^*=-\{-5+D_t(3t)+D_x(x)+D_y(2y-z)\\&+D_z(y+2z)\}v\\&=-3v.\end{aligned}\right\}$$

Therefore, we arrive at the extended vector field

$$\left.\begin{aligned}&Y_4=3t\,\partial/\partial t+x\,\partial/\partial x+(2y-z)\,\partial/\partial y\\&+(y+2z)\,\partial/\partial z-u\,\partial/\partial u-5v\,\partial/\partial v\end{aligned}\right\}$$

and the related characteristic function

$$W=-3tu_t-xu_x-(y+2z)u_z-(2y-z)u_y-u.$$

On taking the same steps outlined above with the relation Eq. (4.34), one constructs the conserved vectors [2, 75] corresponding to the secured optimal system of 12 Lie subalgebras as subsequently presented:

$$\left.\begin{aligned}&T_1^t=\frac{3}{2}vu_xu^3+\frac{3}{2}vu_{xx}u^2+3vuu_x^2+vu_{zz}\\&+vu_{yy}+3vu_xu_{xx}+\frac{1}{2}vu_{tx}+\frac{1}{2}u_tv_x,\end{aligned}\right\}$$

$$\left.\begin{aligned}&T_1^x=\frac{3}{2}u_tv_xu^2-\frac{3}{2}vu_tu^3-\frac{3}{2}vu_{tx}u^2\\&-3u_tu_xuv-3u_xu_{tx}v-\frac{1}{2}u_{tt}v+3u_tu_xv_x+\frac{1}{2}u_tv_t,\end{aligned}\right\}$$
$$T_1^y=u_tv_y-u_{ty}v,T_1^z=u_tv_z-u_{tz}v;$$

$$\left.\begin{aligned}&T_2^t=\frac{1}{2}u_xv_x-\frac{1}{2}u_{xx}v,T_2^x=\frac{3}{2}u_xv_xu^2\\&+u_{zz}v+u_{yy}v+\frac{1}{2}u_{tx}v+\frac{1}{2}v_tu_x+3u_x^2v_x,\end{aligned}\right\}$$
$$T_2^y=u_xv_y-u_{xy}v,T_2^z=u_xv_z-u_{xz}v;$$

$$\left.\begin{aligned}&T_3^t=\frac{1}{2}u_yv_x-\frac{1}{2}u_{xy}v,\\&T_3^x=-\frac{3}{2}u_yu^3v+\frac{3}{2}u_yv_xu^2-\frac{3}{2}u_{xy}u^2v\\&-3u_xu_yuv-3u_xu_{xy}v-\frac{1}{2}u_{ty}v+\frac{1}{2}v_tu_y\\&+3u_xu_yv_x,\end{aligned}\right\}$$

$$\left.\begin{aligned}&T_3^y=\frac{3}{2}u_xu^3v+\frac{3}{2}u_{xx}u^2v+3u_x^2uv\\&+u_{zz}v+3u_xu_{xx}v+u_{tx}v+u_yv_y,\end{aligned}\right\}$$
$$T_3^z=u_yv_z-u_{yz}v;$$

$$\left.\begin{aligned}&T_8^y=3yu_xu^3v+3yu_{xx}u^2v+6yu_x^2uv+v_yu\\&+2yu_{zz}v-3u_yv-2zu_{yz}v-xu_{xy}v\\&+6yu_xu_{xx}v-3tu_{ty}v+2yu_{tx}v\\&+3tu_tv_y+xu_xv_y+2zu_zv_y+2yu_yv_y,\end{aligned}\right\}$$

$$\left.\begin{aligned}&T_8^z=3zu_xu^3v+3zu_{xx}u^2v+6zu_x^2uv+v_zu\\&-3u_zv-2yu_{yz}v+2zu_{yy}v-xu_{xz}v\\&+6zu_xu_{xx}v-3tu_{tz}v+2zu_{tx}v\\&+3tu_tv_z+xu_xv_z+2yu_yv_z+2zu_zv_z;\end{aligned}\right\}$$

$$\left.\begin{aligned}&T_9^t=\frac{3}{2}vu_xu^3+\frac{3}{2}vu_{xx}u^2+3vu_x^2u+vu_{zz}\\&+vu_{yy}+\frac{1}{2}u_yv_x+\frac{1}{2}u_xv_x-\frac{1}{2}vu_{xy}\\&-\frac{1}{2}vu_{xx}+3vu_xu_{xx}+\frac{1}{2}v_xu_t+\frac{1}{2}vu_{tx},\end{aligned}\right\}$$

$$\left.\begin{aligned}&T_9^y=\frac{3}{2}vu_xu^3+\frac{3}{2}vu_{xx}u^2+3vu_x^2u+vu_{zz}\\&+u_yv_y+v_yu_x-vu_{xy}+3vu_xu_{xx}+v_yu_t\\&-vu_{ty}+vu_{tx},T_9^z=v_zu_y-vu_{yz}\\&+v_zu_x-vu_{xz}+v_zu_t-vu_{tz};\end{aligned}\right\}$$

$$\begin{aligned}&T_4^t=\frac{9}{2}tvu_xu^3+\frac{9}{2}tvu_{xx}u^2+9tvu_x^2u\\&+\frac{1}{2}v_xu+3tvu_{zz}+3tvu_{yy}-vu_x\\&+\frac{1}{2}yu_zv_x+zu_zv_x+yu_yv_x-\frac{1}{2}zu_yv_x\\&+\frac{1}{2}xu_xv_x-\frac{1}{2}yvu_{xz}-zvu_{xz}\\&-yvu_{xy}+\frac{1}{2}zvu_{xy}-\frac{1}{2}xvu_{xx}\\&+9tvu_xu_{xx}+\frac{3}{2}tv_xu_t+\frac{3}{2}tvu_{tx},\end{aligned}$$

$$\left.\begin{aligned}&T_4^x=\frac{3}{2}zvu_yu^3-\frac{3}{2}vu^4-\frac{3}{2}yvu_zu^3-3zvu_zu^3\\&-3yvu_yu^3+\frac{3}{2}v_xu^3-\frac{9}{2}tvu_tu^3-6vu_xu^2\\&+\frac{3}{2}yu_zv_xu^2+3zu_zv_xu^2+3yu_yv_xu^2\\&-\frac{3}{2}zu_yv_xu^2+\frac{3}{2}xu_xv_xu^2-\frac{3}{2}yvu_{xz}u^2\\&-3zvu_{xz}u^2-3yvu_{xy}u^2+\frac{3}{2}zvu_{xy}u^2\\&+\frac{9}{2}tv_xu_tu^2-\frac{9}{2}tvu_{tx}u^2-3yvu_zu_xu-6zvu_zu_xu\\&-6yvu_yu_xu+3zvu_yu_xu+3u_xv_xu-9tvu_xu_tu\\&+\frac{1}{2}v_tu-6vu_x^2+xvu_{zz}+xvu_{yy}+3xu_x^2v_x\\&+3yu_zu_xv_x+6zu_zu_xv_x+6yu_yu_xv_x\\&-3zu_yu_xv_x-3yvu_xu_{xz}-6zvu_xu_{xz}-6yvu_xu_{xy}\\&+3zvu_xu_{xy}-2vu_t+9tu_xv_xu_t+\frac{1}{2}yu_zv_t+zu_zv_t\\&+yu_yv_t-\frac{1}{2}zu_yv_t+\frac{1}{2}xu_xv_t+\frac{3}{2}tu_tv_t-\frac{1}{2}yvu_{tz}\\&-zvu_{tz}-yvu_{ty}+\frac{1}{2}zvu_{ty}+\frac{1}{2}xvu_{tx}\\&-9tvu_xu_{tx}-\frac{3}{2}tvu_{tt},\end{aligned}\right\}$$

$$\left.\begin{aligned}&T_4^y=3yvu_xu^3-\frac{3}{2}zvu_xu^3+3yvu_{xx}u^2-\frac{3}{2}zvu_{xx}u^2\\&+6yvu_x^2u-3zvu_x^2u+v_yu-vu_z+2yvu_{zz}-zvu_{zz}\\&-3vu_y+yu_zv_y+2zu_zv_y+2yu_yv_y-zu_yv_y-yvu_{yz}\\&-2zvu_{yz}+xv_yu_x-xvu_{xy}+6yvu_xu_{xx}-3zvu_xu_{xx}\\&+3tv_yu_t-3tvu_{ty}+2yvu_{tx}-zvu_{tx},\end{aligned}\right\}$$

$$\left.\begin{aligned}&T_4^z=\frac{3}{2}yvu_xu^3+3zvu_xu^3+\frac{3}{2}yvu_{xx}u^2+3zvu_{xx}u^2\\&+3yvu_x^2u+6zvu_x^2u+v_zu-3vu_z+yu_zv_z+2zu_zv_z\\&+vu_y+2yv_zu_y-zv_zu_y-2yvu_{yz}+zvu_{yz}+yvu_{yy}\\&+2zvu_{yy}+xv_zu_x-xvu_{xz}+3yvu_xu_{xx}+6zvu_xu_{xx}\\&+3tv_zu_t-3tvu_{tz}+yvu_{tx}+2zvu_{tx};\end{aligned}\right\}$$

$$\left.\begin{aligned}&T_5^t=\frac{1}{2}u_xv_x-\frac{1}{2}u_{xy}v-\frac{1}{2}u_{xx}v+\frac{1}{2}u_yv_x,\\&T_5^x=\frac{3}{2}u_yv_xu^2-\frac{3}{2}u_yu^3v+\frac{3}{2}u_xv_xu^2-\frac{3}{2}u_{xy}u^2v\\&-3u_xu_yuv+u_{zz}v+u_{yy}vv-3u_xu_{xy}-\frac{1}{2}u_{ty}v\\&+\frac{1}{2}u_{tx}v+\frac{1}{2}v_tu_x+\frac{1}{2}v_tu_y+3u_xu_yv_x+3u_x^2v_x,\end{aligned}\right\}$$

$$\left.\begin{aligned}&T_5^y=\frac{3}{2}u_xu^3v+\frac{3}{2}u_{xx}u^2v+3u_x^2uv+u_{zz}v\\&-u_{xy}v+3u_xu_{xx}v+u_{tx}v+u_xv_y+u_yv_y,\end{aligned}\right\}$$

$$T_5^z = u_x v_z - u_{yz} v - u_{xz} v + u_y v_z;$$

$$\left.\begin{array}{l} T_6^t = t u_z v_x - t u_y v_x + \frac{1}{2} y u_x v_x - \frac{1}{2} z u_x v_x \\ -t v u_{xz} + t v u_{xy} - \frac{1}{2} y v u_{xx} + \frac{1}{2} z v u_{xx}, \end{array}\right\}$$

$$\left.\begin{array}{l} T_6^x = 3tvu_y u^3 - 3tvu_z u^3 + 3tu_z v_x u^2 - 3tu_y v_x u^2 \\ +\frac{3}{2} y u_x v_x u^2 + t u_z v_t - \frac{3}{2} z u_x v_x u^2 - 3tvu_{xz} u^2 \\ +3tvu_{xy} u^2 - 6tvu_z u_x u + 6tvu_y u_x u - vu_z + yvu_{zz} \\ -zvu_{zz} + vu_y + yvu_{yy} - zvu_{yy} + 3yu_x^2 v_x - 3zu_x^2 v_x \\ +6tu_z u_x v_x - 6tu_y u_x v_x - 6tvu_x u_{xz} + 6tvu_x u_{xy} - tu_y v_t \\ +\frac{1}{2} y u_x v_t - \frac{1}{2} z u_x v_t - tvu_{tz} + tvu_{ty} + \frac{1}{2} yvu_{tx} - \frac{1}{2} zvu_{tx}, \end{array}\right\}$$

$$\left.\begin{array}{l} T_6^y = yv_y u_x - 3tvu_x u^3 - 3tvu_{xx} u^2 - 6tvu_x^2 u \\ -2tvu_{zz} + 2tu_z v_y - 2tu_y v_y - 2tvu_{yz} - vu_x \\ -zv_y u_x - yvu_{xy} + zvu_{xy} - 6tvu_x u_{xx} - 2tvu_{tx}, \end{array}\right\}$$

$$\left.\begin{array}{l} T_6^z = 3tvu_x u^3 + 3tvu_{xx} u^2 + 6tvu_x^2 u + 2tu_z v_z \\ -2tv_z u_y + 2tvu_{yz} + 2tvu_{yy} + vu_x + yv_z u_x \\ -zv_z u_x - yvu_{xz} + zvu_{xz} + 6tvu_x u_{xx} + 2tvu_{tx}; \end{array}\right\}$$

$$T_7^t = -\frac{1}{2} y u_{xz} v + \frac{1}{2} z u_{xy} v + \frac{1}{2} y u_z v_x - \frac{1}{2} z u_y v_x,$$

$$\left.\begin{array}{l} T_7^x = -\frac{3}{2} y u_z u^3 v + \frac{3}{2} z u_y u^3 v + \frac{3}{2} y u_z v_x u^2 - \frac{3}{2} z u_y v_x u^2 \\ -\frac{3}{2} y u_{xz} u^2 v + \frac{3}{2} z u_{xy} u^2 v - 3yu_x u_z uv + 3zu_x u_y uv \\ -3yu_x u_{xz} v + 3zu_x u_{xy} v - \frac{1}{2} y u_{tz} v + \frac{1}{2} z u_{ty} v \\ +\frac{1}{2} y v_t u_z - \frac{1}{2} z v_t u_y + 3yu_x u_z v_x - 3zu_x u_y v_x, \end{array}\right\}$$

$$\left.\begin{array}{l} T_7^y = -\frac{3}{2} z u_x u^3 v - \frac{3}{2} z u_{xx} u^2 v - 3zu_x^2 uv - u_z v - zu_{zz} v \\ -yu_{yz} v - 3zu_x u_{xx} v - zu_{tx} v + yu_z v_y - zu_y v_y, \end{array}\right\}$$

$$\left.\begin{array}{l} T_7^z = \frac{3}{2} y u_x u^3 v + \frac{3}{2} y u_{xx} u^2 v + 3yu_x^2 uv \\ +u_y v + zu_{yz} v + yu_{yy} v + 3yu_x u_{xx} v \\ +yu_{tx} v + yu_z v_z - zu_y v_z; \end{array}\right\}$$

$$\left.\begin{array}{l} T_8^t = \frac{9}{2} t u_x u^3 v + \frac{9}{2} t u_{xx} u^2 v + 9tu_x^2 uv + \frac{1}{2} v_x u \\ +3tu_{zz} v + 3tu_{yy} v - u_x v - zu_{xz} v - yu_{xy} v \\ -\frac{1}{2} x u_{xx} v + 9tu_x u_{xx} v + \frac{3}{2} t u_{tx} v + \frac{3}{2} t u_t v_x \\ +yu_y v_x + zu_z v_x + \frac{1}{2} x u_x v_x, \end{array}\right\}$$

$$\left.\begin{array}{l} T_8^x = -\frac{3}{2} v u^4 - 3zvu_z u^3 - 3yvu_y u^3 + \frac{3}{2} v_x u^3 \\ -\frac{9}{2} tvu_t u^3 - 6vu_x u^2 + 3zu_z v_x u^2 + 3yu_y v_x u^2 \\ +\frac{3}{2} x u_x v_x u^2 - 3zvu_{xz} u^2 - 3yvu_{xy} u^2 \\ +\frac{9}{2} tv_x u_t u^2 - \frac{9}{2} tvu_{tx} u^2 - 6zvu_z u_x u \\ -6yvu_y u_x u + 3u_x v_x u - 9tvu_x u_t u + \frac{1}{2} v_t u \\ -6vu_x^2 + xvu_{zz} + xvu_{yy} + 3xu_x^2 v_x + 6zu_z u_x v_x \\ +6yu_y u_x v_x - 6zvu_x u_{xz} - 6yvu_x u_{xy} \\ -2vu_t + 9tu_x v_x u_t + zu_z v_t + yu_y v_t \\ +\frac{1}{2} x u_x v_t + \frac{3}{2} t u_t v_t - zvu_{tz} - yvu_{ty} \\ +\frac{1}{2} xvu_{tx} - 9tvu_x u_{tx} - \frac{3}{2} tvu_{tt}, \end{array}\right\}$$

$$\left.\begin{array}{l} T_9^x = \frac{3}{2} u_y v_x u^2 - \frac{3}{2} vu_y u^3 - \frac{3}{2} vu_t u^3 + \frac{3}{2} u_x v_x u^2 \\ -\frac{3}{2} vu_{xy} u^2 + \frac{3}{2} v_x u_t u^2 - \frac{3}{2} vu_{tx} u^2 - 3vu_y u_x u \\ -3vu_x u_t u + vu_{zz} + vu_{yy} + 3u_x^2 v_x + 3u_y u_x v_x \\ -3vu_x u_{xy} + 3u_x v_x u_t + \frac{1}{2} u_y v_t + \frac{1}{2} u_x v_t \\ +\frac{1}{2} u_t v_t - \frac{1}{2} vu_{ty} + \frac{1}{2} vu_{tx} - 3vu_x u_{tx} - \frac{1}{2} vu_{tt}, \end{array}\right\}$$

$$\left.\begin{array}{l} T_9^x = \frac{3}{2} u_y v_x u^2 - \frac{3}{2} vu_y u^3 - \frac{3}{2} vu_t u^3 + \frac{3}{2} u_x v_x u^2 \\ -\frac{3}{2} vu_{xy} u^2 + \frac{3}{2} v_x u_t u^2 - \frac{3}{2} vu_{tx} u^2 - 3vu_y u_x u \\ -3vu_x u_t u + vu_{zz} + vu_{yy} + 3u_x^2 v_x + 3u_y u_x v_x \\ -3vu_x u_{xy} + 3u_x v_x u_t + \frac{1}{2} u_y v_t + \frac{1}{2} u_x v_t \\ +\frac{1}{2} u_t v_t - \frac{1}{2} vu_{ty} + \frac{1}{2} vu_{tx} - 3vu_x u_{tx} - \frac{1}{2} vu_{tt}, \end{array}\right\}$$

$$\left.\begin{array}{l} T_{10}^t = tu_z v_x + \frac{1}{2} u_z v_x - tu_y v_x + \frac{1}{2} y u_x v_x \\ -\frac{1}{2} z u_x v_x - tvu_{xz} - \frac{1}{2} vu_{xz} + tvu_{xy} \\ -\frac{1}{2} yvu_{xx} + \frac{1}{2} zvu_{xx}, T_{10}^x = 3tvu_y u^3 \\ -3tvu_z u^3 - \frac{3}{2} vu_z u^3 + 3tu_z v_x u^2 \\ +\frac{3}{2} u_z v_x u^2 - 3tu_y v_x u^2 + \frac{3}{2} y u_x v_x u^2 \\ -\frac{3}{2} z u_x v_x u^2 - 3tvu_{xz} u^2 - \frac{3}{2} vu_{xz} u^2 \\ +3tvu_{xy} u^2 - 6tvu_z u_x u - 3vu_z u_x u \\ +6tvu_y u_x u - vu_z + yvu_{zz} - zvu_{zz} \\ +vu_y + yvu_{yy} - zvu_{yy} + 3yu_x^2 v_x \\ -3zu_x^2 v_x + 6tu_z u_x v_x + 3u_z u_x v_x \\ -6tu_y u_x v_x - 6tvu_x u_{xz} - 3vu_x u_{xz} \\ +6tvu_x u_{xy} + tu_z v_t + \frac{1}{2} u_z v_t - tu_y v_t \\ +\frac{1}{2} y u_x v_t - \frac{1}{2} z u_x v_t - tvu_{tz} \\ -\frac{1}{2} vu_{tz} + tvu_{ty} + \frac{1}{2} yvu_{tx} - \frac{1}{2} zvu_{tx}, \end{array}\right\}$$

$$\left.\begin{array}{l} T_{10}^x = 2tu_z v_y - 3tvu_x u^3 - 3tvu_{xx} u^2 \\ -6tvu_x^2 u - 2tvu_{zz} + u_z v_y - 2tu_y v_y \\ -2tvu_{yz} - vu_{yz} - vu_x + yv_y u_x - zv_y u_x \\ -yvu_{xy} + zvu_{xy} - 6tvu_x u_{xx} - 2tvu_{tx}, \end{array}\right\}$$

$$\left.\begin{array}{l} T_{10}^z = 3tvu_x u^3 + \frac{3}{2} vu_x u^3 + 3tvu_{xx} u^2 + \frac{3}{2} vu_{xx} u^2 \\ +6tvu_x^2 u + 3vu_x^2 u + 2tu_z v_z + u_z v_z - 2tv_z u_y \\ +2tvu_{yz} + 2tvu_{yy} + vu_{yy} + vu_x + yv_z u_x \\ -zv_z u_x - yvu_{xz} + zvu_{xz} + 6tvu_x u_{xx} \\ +3vu_x u_{xx} + 2tvu_{tx} + vu_{tx}; \end{array}\right\}$$

$$\left.\begin{array}{l} T_{11}^t = \frac{3}{2} u_x u^3 v + \frac{3}{2} u_{xx} u^2 v + 3u_x^2 uv + u_{zz} v + u_{yy} v \\ -\frac{1}{2} u_{xx} v + 3u_x u_{xx} v + \frac{1}{2} u_{tx} v + \frac{1}{2} u_t v_x + \frac{1}{2} u_x v_x, \end{array}\right\}$$

$$\left.\begin{array}{l} T_{11}^x = \frac{3}{2} u_x v_x u^2 - \frac{3}{2} u_t u^3 v + \frac{3}{2} u_t v_x u^2 - \frac{3}{2} u_{tx} u^2 v \\ -3u_t u_x uv + u_{zz} v + u_{yy} v + \frac{1}{2} u_{tx} v - 3u_x u_{tx} v \\ -\frac{1}{2} u_{tt} v + 3u_t u_x v_x + \frac{1}{2} v_t u_x + \frac{1}{2} u_t v_t + 3u_x^2 v_x, \end{array}\right\}$$

$$\left.\begin{array}{l} T_{11}^y = u_x v_y - u_{xy} v - u_{ty} v + u_t v_y, \\ T_{11}^z = u_x v_z - u_{xz} v - u_{tz} v + u_t v_z; \end{array}\right\}$$

$$\left.\begin{aligned}T_{12}^{t} &= \tfrac{3}{2}vu_xu^3+\tfrac{3}{2}vu_{xx}u^2+3vu_x^2u+vu_{zz}+vu_{yy}\\&+\tfrac{1}{2}u_yv_x-\tfrac{1}{2}vu_{xy}+3vu_xu_{xx}+\tfrac{1}{2}v_xu_t+\tfrac{1}{2}vu_{tx},\end{aligned}\right\}$$

$$\left.\begin{aligned}T_{12}^{x} &= \tfrac{3}{2}v_xu_tu^2-\tfrac{3}{2}vu_yu^3-\tfrac{3}{2}vu_tu^3+\tfrac{3}{2}u_yv_xu^2\\&-\tfrac{3}{2}vu_{xy}u^2-\tfrac{3}{2}vu_{tx}u^2-3vu_yu_xu-3vu_xu_tu\\&+3u_yu_xv_x-3vu_xu_{xy}+3u_xv_xu_t+\tfrac{1}{2}u_yv_t\\&+\tfrac{1}{2}u_tv_t-\tfrac{1}{2}vu_{ty}-3vu_xu_{tx}-\tfrac{1}{2}vu_{tt},\end{aligned}\right\}$$

$$\left.\begin{aligned}T_{12}^{y} &= \tfrac{3}{2}vu_xu^3+\tfrac{3}{2}vu_{xx}u^2+3vu_x^2u+vu_{zz}\\&+u_yv_y+3vu_xu_{xx}+v_yu_t-vu_{ty}+vu_{tx},\end{aligned}\right\}$$

$$T_{12}^{z}=v_zu_y-vu_{yz}+v_zu_t-vu_{tz}.$$

Remark 4.2

We observe that there is the possibility of obtaining solutions to the model 3D-extKPLEq (1.5) by exploring the double reduction method via the obtained conservation laws but our focus lies in generating the optimal system and using the same to calculate conserved vectors having applications in the fields of physical science.

4.2 Application of conserved quantities in physical sciences

Conservation laws [2, 75, 76], on the other hand, are a subject of pertinent engrossment in physics, inclusive of theoretical as well as quantum mechanics. This study investigates the conserved quantities of Eq. (1.5) where the results reveal that the model possesses conservation of momentum and energy. In isolated systems, physical quantities such as charge, mass, energy, and angular momentum along linear momentum are conserved. Conserved quantities can be invoked in carrying out integrability checks on DEs. In addition, we have among others the fact that conserved quantities play a key part in the establishment of existence as well as the uniqueness of solutions and linearisation mappings. They are also utilised in the global behaviour of solutions and stability analysis.

In addition, conserved quantities play a leading role in the evolution of numerical techniques. They also furnish a crucial starting point in securing non-locally related systems and potential variables. In particular, a conserved quantity is fundamental in the investigation of a given DE, which implies that it holds for any posed data, whether initial conditions, boundary conditions or both. Furthermore, the conservation laws' structure is such that it does not depend on coordinates, since it involves a contact transformation mapping one to the other.

Conservation law, which is otherwise called the law of conservation, in physics, refers to a principle that stipulates that a certain physical property (also referred to as a measurable quantity) within an isolated physical system remains unchanged over time. In classical physics, these types of laws are involved in governing energy, momentum, mass, angular momentum and electric charge. Moreover, in particle physics, various other conservation laws are applicable in relation to properties associated with subatomic particles that are interestingly invariant during the occurrence of interactions. One of the important functions attributable to conservation laws is the fact that they make it possible to make predictions concerning the macroscopic behaviour of a system without necessarily having to contemplate the microscopic details of the course of a chemical reaction or physical process [77].

Based on the understanding that energy and momentum are among the constituents of the physical quantities conserved by the obtained vectors, we provide some applications of these quantities.

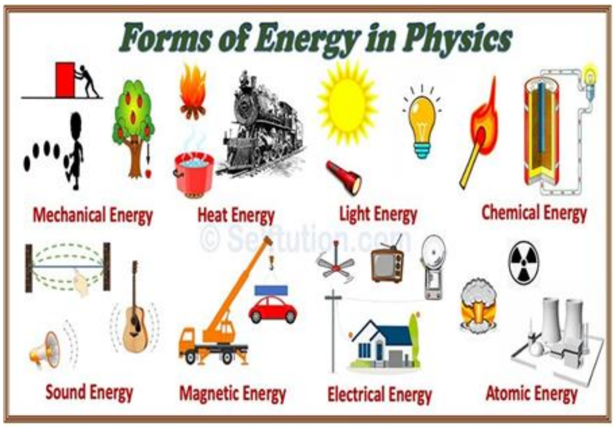

Fig. 1. Diagrammatic representation of various forms of energy in physics [78]

The law of conservation of energy reveals that energy can be neither created nor destroyed. However, its diverse forms (mechanical, kinetic, chemical, etc.) can be changed into others. It is to be noted that the sum of all forms of energy in an isolated system, therefore, remains unaltered. For instance, for a falling body, there is an existing constant amount of energy, although the form of the energy changes from one state to another, which is from potential to kinetic. Following the theory of relativity, mass and energy are revealed to be equivalent. Therefore, the rest of the mass of a body may be considered a form of potential energy, and part of this can later be converted into other forms of energy [78].

If one takes into cognizance all forms of energy (see Fig. 1), the total energy possessed by an isolated system always remains unchanged (constant). It is to be noted that all the forms of energy comply with the law of conservation of energy. Therefore, in summary, one can infer that the conservation of energy law explicates that in a closed system, that is, a system that is isolated from its surrounding environments, the total energy of a system is conserved. Therefore, in an isolated system such as the universe, if there exists a loss of energy in some part of the system, there must be a compensating gain of an equal amount of energy in some other part of the universe [79].

Furthermore, the application of the conservation law of momentum is significant concerning the solution of collision problems. We may take, for example, the operation of rockets, which showcases and exhibits the conservation of momentum: the increased forward momentum of the rocket is scientifically proven to be equal but opposite in sign to the involved momentum of the discharged exhaust gases.

Thus, one notices that the rocket fuel burns to push the exhaust gases downward, which consequently causes the rocket to be pushed upward. A rocket is a spacecraft or an aircraft vehicle that typically makes use of thrust generated by the rocket engine for its flight. A rocket is generally set in motion into outer space and this occasions various scientific researches as well as development applications. Rockets are kept running via some forms of fuel that is typically comprised of liquid hydrogen, hydrogen peroxide, liquid oxygen, hydrazine, and so on. The chemical conformation of the rocket fuels generates an enormous amount of energy that can assist the massive body of the vehicle to shoot straight up and also get launched into space.

In addition, the working mechanism of motorboats adopts the same principle; in this case, the water is pushed backwards, as a result of which the vehicle also gets pushed forward as part of the reaction so as to ensure momentum conservation [80].

In the case of real-life applications, we consider the example of a balloon, where it is observed that the particles of gas move rapidly, thus occasioning collisions with each other as well as the walls of the balloon. Although the particles themselves move slower and faster when they gain or lose momentum as they collide, the total momentum of the system is unaltered. The general law of physics is the conservation of momentum, and this stipulates that the value of momentum remains fixed in an isolated collection of objects. Alternatively, the conservation of momentum law can be delineated as a physical quantity that stays unaltered both before and after the collision of two or more objects that are present in a system. In more direct terms, this implies that, in the case of a collision between two or more bodies, the value of momentum before the collision and that which emerges after the collision are equal in magnitude [81].

More so, the conservation law of energy can be observed in the examples of energy transference that occur on a daily basis. An example is the use of flowing water for the generation of electricity. When water falls from an upper level, this provides an opportunity to productively tap into the conversion that can be facilitated of potential energy into another form of energy, namely kinetic energy. In turn, this energy is then utilised to rotate the turbine of a generator as the principle of function underlying a hydroelectric power plant's operation, resulting in the consequent generation of electricity.

5. CONCLUDING REMARKS

In this article, we investigated and discussed group invariance properties associated with a nonlinear wave Eq. (1.5) in three dimensions by invoking the Lie group theoretic technique, which gives eight Lie point symmetries. In addition, we presented infinitesimal generators, and Lie point symmetries along the corresponding optimal system of one-dimensional Lie subalgebras. Later, the conserved vectors associated to the Lie subalgebras obtained earlier were achieved via Ibragimov's conserved vectors theorem using the formal Lagrangian of the equation under study. It is undeniable that conservation laws are key ingredients in the deduction of the physical aspects of the model under study. As a result, some conservation laws that are well-known in physics, such as those pertaining to the conservation of energy and momentum, are derived in this study. In addition, applications of these quantities with regard to physics and mathematics are outlined. We declare here that, in addition to the fact that the presented formal calculations and analysis of results make the present study a more trustworthy source of referral, investigations carried out in this research work are more robust and comprehensive than those in Lu et al.'s study [67]. Having comprehensively explained the significance of the results achieved in this study, we declare confidently that they are going to be very useful in classical physics, other physical sciences and various engineering fields.

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