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GROWTH AND OSCILLATION OF SOME POLYNOMIALS GENERATED BY SOLUTIONS OF COMPLEX DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we continue the study of some properties on the growth and oscillation of solutions of linear differential equations with entire coefficients of the type

$$f'' + A(z)f' + B(z)f = 0$$

and

$$f^{(k)} + A_{k-2}(z)f^{(k-2)} + \ldots + A_0(z)f = 0.$$

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1. INTRODUCTION AND MAIN RESULTS

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of Nevanlinna's value distribution theory (see [11, 13, 18]). In addition, we will use $\lambda(f)$ and $\overline{\lambda}(f)$ to denote respectively the exponents of convergence of the zero-sequence and distinct zeros of a meromorphic function f, $\rho(f)$ to denote the order of growth of f. A meromorphic function $\varphi(z)$ is called a small function with respect to $f(z)$ if $T(r,\varphi) = o(T(r,f))$ as $r \to +\infty$ except possibly a set of r of finite linear measure, where $T(r,f)$ is the Nevanlinna characteristic function of f.

Definition 1.1 ([9,18])**.** Let f be a meromorphic function. Then the hyper-order of $f(z)$ is defined as

$$\rho_2(f) = \limsup_{r\to+\infty} \frac{\log\log T(r,f)}{\log r}.$$

Definition 1.2 ([11, 16])**.** The type of a meromorphic function f of order ρ $(0<\rho<\infty)$ is defined as

$$\tau(f) = \limsup_{r\to+\infty} \frac{T(r,f)}{r^{\rho}}.$$

If f is an entire function, then the type of f of order ρ $(0<\rho<\infty)$ is defined as

$$\tau_M(f) = \limsup_{r\to+\infty} \frac{\log M(r,f)}{r^{\rho}},$$

where $M(r,f) = \max_{|z|=r} |f(z)|$.

Definition 1.3 ([9,18])**.** Let f be a meromorphic function. Then the hyper-exponent of convergence of the zeros sequence of $f(z)$ is defined as

$$\lambda_2(f) = \limsup_{r\to+\infty} \frac{\log\log N\left(r,\frac{1}{f}\right)}{\log r},$$

where $N\left(r,\frac{1}{f}\right)$ is the counting function of zeros of $f(z)$ in $\{z : |z| \le r\}$. Similarly, the hyper-exponent of convergence of the sequence of distinct zeros of $f(z)$ is defined as

$$\overline{\lambda}_2(f) = \limsup_{r\to+\infty} \frac{\log\log \overline{N}\left(r,\frac{1}{f}\right)}{\log r},$$

where $\overline{N}\left(r,\frac{1}{f}\right)$ is the counting function of distinct zeros of $f(z)$ in $\{z : |z| \le r\}$.

It is well-known that the study of the properties of solutions of complex differential equations is an interesting topic. The growth and oscillation theory for complex differential equations in the plane were firstly investigated by Bank and Laine in 1982–1983 (see [1, 2]). In [7], Z.X. Chen began to consider the fixed points of solutions of the linear differential equation

$$f'' + A(z)f = 0, \tag{1.1}$$

where A is a polynomial and transcendental entire function with finite order. In [15], the authors have investigated the relations between the solutions of (1.1) and small functions. They showed that $w = d_1 f_1 + d_2 f_2$ keeps the same properties of the growth and oscillation of f_j $(j = 1, 2)$, where f_1 and f_2 are two linearly independent solutions of (1.1), $d_j(z)$ $(j = 1, 2)$ are entire functions of finite order and obtained the following results:

Theorem 1.4 ([15]). *Let $A(z)$ be a transcendental entire function of finite order. Let $d_j(z)$ $(j = 1, 2)$ be finite order entire functions that are not all vanishing identically such that $\max\{\rho(d_1), \rho(d_2)\} < \rho(A)$. If f_1 and f_2 are two linearly independent solutions of* (1.1)*, then the polynomial of solutions $w = d_1 f_1 + d_2 f_2$ satisfies*

$$\rho(w) = \rho(f_j) = \infty \quad (j = 1, 2)$$

and

$$\rho_2(w) = \rho_2(f_j) = \rho(A) \quad (j = 1, 2).$$

Theorem 1.5 ([15]). *Under the hypotheses of Theorem 1.4, let $\varphi(z) \not\equiv 0$ be an entire function with finite order such that $\psi(z) \not\equiv 0$. If f_1 and f_2 are two linearly independent solutions of* (1.1)*, then the polynomial of solutions w satisfies*

$$\overline{\lambda}(w - \varphi) = \lambda(w - \varphi) = \rho(f_j) = \infty \quad (j = 1, 2)$$

and

$$\overline{\lambda}_2(w - \varphi) = \lambda_2(w - \varphi) = \rho(A),$$

where

$$\begin{aligned}
\psi(z) &= \frac{2\left(d_1 d_2 d_2' - d_2^2 d_1'\right)}{h}\varphi^{(3)} + \phi_2\varphi'' + \phi_1\varphi' + \phi_0\varphi,\\
\phi_2 &= \frac{3d_2^2 d_1'' - 3d_1 d_2 d_2''}{h},\\
\phi_1 &= \frac{2d_1 d_2 d_2' A + 6d_2 d_1' d_2'' - 6d_2 d_2' d_1'' - 2d_2^2 d_1' A}{h},\\
\phi_0 &= \frac{2d_2 d_1' d_2''' - 2d_1 d_2' d_2''' - 3d_1 d_2 d_2'' A - 3d_2 d_1'' d_2'' + 2d_1 d_2 d_2' A'}{h}\\
&\quad - \frac{4d_2 d_1' d_2' A - 6d_1' d_2' d_2'' + 3d_1\left(d_2''\right)^2 + 4d_1\left(d_2'\right)^2 A + 3d_2^2 d_1'' A}{h}\\
&\quad + \frac{6\left(d_2'\right)^2 d_1'' - 2d_2^2 d_1' A'}{h},\\
h &= \begin{vmatrix}
d_1 & 0 & d_2 & 0\\
d_1' & d_1 & d_2' & d_1\\
d_1'' - d_1 A & 2d_1' & d_2'' - d_2 A & 2d_2'\\
d_1''' - 3d_1' A - d_1 A' & d_1'' - d_1 A + 2d_1'' & d_1''' - 3d_2' A - d_2 A' & d_1'' - d_2 A - 2d_2''
\end{vmatrix}.
\end{aligned}$$

It is a natural to ask: *Can we obtain the same results as in Theorem 1.4 and Theorem 1.5 for higher order linear differential equations?* In this paper, we give a partial answer to this question. We consider first the complex differential equation

$$f'' + A(z)f' + B(z)f = 0, \tag{1.2}$$

where $A(z)$ and $B(z)$ are entire functions of finite order. Before we state our results we define $h(z)$ and $\psi(z)$ by

$$h = \begin{vmatrix} H_1 & H_2 & H_3 & H_4 \\ H_5 & H_6 & H_7 & H_8 \\ H_9 & H_{10} & H_{11} & H_{12} \\ H_{13} & H_{14} & H_{15} & H_{16} \end{vmatrix},$$

where

$$H_1 = d_1, \ \ H_2 = 0, \ \ H_3 = d_2, \ \ H_4 = 0, \ \ H_5 = d_1', \ \ H_6 = d_1, \ \ H_7 = d_2', \ \ H_8 = d_2,$$
$$H_9 = d_1'' - d_1 B, \ \ H_{10} = 2d_1' - d_1 A, \ \ H_{11} = d_2'' - d_2 B, \ \ H_{12} = 2d_2' - d_2 A,$$
$$H_{13} = d_1^{(3)} - 3d_1' B + d_1 AB - d_1 B', \ \ H_{14} = 3d_1'' - 2d_1' A - d_1 B + d_1 A^2 - d_1 A',$$
$$H_{15} = d_2^{(3)} - 3d_2' B + d_2 AB - d_2 B', \ \ H_{16} = 3d_2'' - 2d_2' A - d_2 B + d_2 A^2 - d_2 A',$$

$$\psi(z) = 2\frac{\left(d_1 d_2 d_2' - d_2^2 d_1'\right)}{h}\varphi^{(3)} + \phi_2 \varphi'' + \phi_1 \varphi' + \phi_0 \varphi, \tag{1.3}$$

where $\varphi \not\equiv 0$, d_j $(j = 1, 2)$ are entire functions of finite order and

$$\phi_2 = \frac{2\left(d_1 d_2 d_2' - d_2^2 d_1'\right) A - 3d_1 d_2 d_2'' + 3d_2^2 d_1''}{h}, \tag{1.4}$$

$$\begin{aligned} \phi_1 = \frac{1}{h} & \left[6d_2 \left(d_1' d_2'' - d_2' d_1''\right) + 2d_2 \left(d_1 d_2' - d_2 d_1'\right) B \right. \\ & \left. + 2d_2 \left(d_1 d_2' - d_2 d_1'\right) A' + 3d_2 \left(d_2 d_1'' - d_1 d_2''\right) A\right], \end{aligned} \tag{1.5}$$

$$\begin{aligned} \phi_0 = \frac{1}{h} & [(d_1 d_2' d_2'' - 3d_2 d_2' d_1'' + 2d_2 d_1' d_2'') A \\ & + \left(4d_1 (d_2')^2 + 3d_2^2 d_1'' - 3d_1 d_2 d_2'' - 4d_2 d_1' d_2'\right) B + 2\left(d_2 d_1' d_2' - d_1 (d_2')^2\right) A' \\ & + 2\left(d_1 d_2 d_2' - d_2^2 d_1'\right) B' + 6(d_2')^2 d_1'' - 2d_1 d_2' d_2''' \\ & + 2d_2 d_1' d_2''' - 3d_2 d_1'' d_2'' - 6d_1' d_2' d_2'' + 3d_1 (d_2'')^2]. \end{aligned} \tag{1.6}$$

Theorem 1.6. *Let $A(z)$ and $B(z)$ be entire functions of finite order such that $\rho(A) < \rho(B)$ and $\tau(A) < \tau(B) < +\infty$ if $\rho(B) = \rho(A) > 0$. Let $d_j(z)$ $(j = 1, 2)$ be entire functions that are not all vanishing identically such that* $\max\{\rho(d_1), \rho(d_2)\} < \rho(B)$. *If f_1 and f_2 are two nontrivial linearly independent solutions of* (1.2)*, then the polynomial of solutions $w = d_1 f_1 + d_2 f_2$ satisfies*

$$\rho(w) = \rho(f_1) = \rho(f_2) = \infty$$

and

$$\rho_2(w) = \rho(B).$$

Theorem 1.7. *Under the hypotheses of Theorem 1.6, let $\varphi(z) \not\equiv 0$ be an entire function with finite order such that $\psi(z) \not\equiv 0$. If f_1 and f_2 are two nontrivial linearly independent solutions of* (1.2)*, then the polynomial of solutions $w = d_1 f_1 + d_2 f_2$ satisfies*

$$\overline{\lambda}(w - \varphi) = \lambda(w - \varphi) = \infty \tag{1.7}$$

and

$$\overline{\lambda}_2(w - \varphi) = \lambda_2(w - \varphi) = \rho(B). \tag{1.8}$$

Theorem 1.8. *Let $A(z)$ and $B(z)$ be entire functions of finite order such that $\rho(A) < \rho(B)$. Let $d_j(z), b_j(z)$ $(j = 1, 2)$ be finite order entire functions such that $d_1(z)b_2(z) - d_2(z)b_1(z) \not\equiv 0$. If f_1 and f_2 are two nontrivial linearly independent solutions of* (1.2), *then*

$$\rho\left(\frac{d_1 f_1 + d_2 f_2}{b_1 f_1 + b_2 f_2}\right) = \infty$$

and

$$\rho_2\left(\frac{d_1 f_1 + d_2 f_2}{b_1 f_1 + b_2 f_2}\right) = \rho(B).$$

We consider now the complex differential equation

$$f^{(k)} + A_{k-2}(z)f^{(k-2)} + \ldots + A_1(z)f' + A_0(z)f = 0, \tag{1.9}$$

where $A_0(z), \ldots, A_{k-2}(z)$ are entire functions. It is clear that the technique of the proof which is used in Theorem 1.6 is not efficient for higher order linear differential equations. Then, the study of growth and oscillation of the polynomial of solutions

$$P_k(f) = \sum_{i=1}^{k} d_i f_i, \tag{1.10}$$

where $d_i(z)$ $(i = 1, \ldots, k)$ are entire functions of finite order that are not all vanishing identically is more difficult for k^{th}-order linear differential equations. We will give here some sufficient conditions which ensure that $P_k(f)$ has infinite order.

Theorem 1.9. *Let $k \geq 3$, and let $f_1(z), \ldots, f_k(z)$ be linearly independent solutions of* (1.9) *such that $\lambda(f_i) < \infty$ $(i = 1, \ldots, k)$, and $d_i(z)$ $(i = 1, \ldots, k)$ are entire functions of finite order not all vanishing identically, let A_0 be a transcendental entire function and $A_1, \ldots, A_{k-2}$ are entire functions of order less than $\rho(A_0)$ if $\rho(A_0) > 0$, and are polynomials if $\rho(A_0) = 0$. Then $P_k(f)$ is of infinite order.*

From Theorem 1.9, we can obtain the following result.

Theorem 1.10. *Let $k \geq 3$, and let $f_1(z), \ldots, f_k(z)$ $(k \geq m)$ be linearly independent solutions of* (1.9) *such that $\lambda(f_i) < \infty$ $(i = 1, \ldots, k)$, and let $d_i(z)$ $(i = 1, \ldots, k)$, $b_j(z)$ $(j = 1, \ldots, m)$ be entire functions of finite order such that*

$$Q_{k,m}(f) = \frac{\sum\limits_{i=1}^{k} d_i f_i}{\sum\limits_{j=1}^{m} b_j f_j}$$

is a irreducible rational function in f_i $(i = 1, \ldots, k)$. Under the hypotheses of Theorem 1.9, $Q_{k,m}(f)$ has infinite order.

2. AUXILIARY LEMMAS

Lemma 2.1 ([4,6])**.** *Let $A_0, A_1, \ldots, A_{k-1}, F \not\equiv 0$ be finite order meromorphic functions. If f is a meromorphic solution of the equation*

$$f^{(k)} + A_{k-1} f^{(k-1)} + \ldots + A_1 f' + A_0 f = F$$

with $\rho(f) = +\infty$ and $\rho_2(f) = \rho$, then f satisfies

$$\overline{\lambda}(f) = \lambda(f) = \rho(f) = +\infty,$$
$$\overline{\lambda}_2(f) = \lambda_2(f) = \rho_2(f) = \rho.$$

Lemma 2.2 ([10])**.** *Let $A_0(z), \ldots, A_{k-1}(z)$ be entire functions of finite order such that*

$$\max\{\rho(A_j) : j = 1, \ldots, k-1\} < \rho(A_0).$$

Then every solution $f \not\equiv 0$ of the equation

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \ldots + A_1(z) f' + A_0(z) f = 0 \tag{2.1}$$

satisfies $\rho(f) = +\infty$ and $\rho_2(f) = \rho(A_0)$.

Lemma 2.3. *Let $A(z)$ and $B(z)$ be entire functions of finite order such that $\rho(A) < \rho(B)$. If f_1 and f_2 are two nontrivial linearly independent solutions of* (1.2)*. Then $\frac{f_1}{f_2}$ is of infinite order and*

$$\rho_2\left(\frac{f_1}{f_2}\right) = \rho(B).$$

Proof. Suppose that f_1 and f_2 are two nontrivial linearly independent solutions of (1.2). Since $\rho(B) > \rho(A)$, then by Lemma 2.2 we have

$$\rho(f_1) = \rho(f_2) = +\infty, \quad \rho_2(f_1) = \rho_2(f_2) = \rho(B). \tag{2.2}$$

On the other hand,

$$\left(\frac{f_1}{f_2}\right)' = -\frac{W(f_1, f_2)}{f_2^2}, \tag{2.3}$$

where $W(f_1, f_2) = f_1 f_2' - f_2 f_1'$ is the Wronskian of f_1 and f_2. By using (1.2), we obtain that

$$W'(f_1, f_2) = -A(z) W(f_1, f_2), \tag{2.4}$$

which implies that

$$W(f_1, f_2) = K \exp\left(-\int A(z)dz\right), \tag{2.5}$$

where $\int A(z)dz$ is the primitive of $A(z)$ and $K \in \mathbb{C}\setminus\{0\}$. By (2.3) and (2.5), we have

$$\left(\frac{f_1}{f_2}\right)' = -K \frac{\exp(-\int A(z)dz)}{f_2^2}. \tag{2.6}$$

Since $\rho(f_2) = +\infty$ and $\rho_2(f_2) = \rho(B) > \rho(A)$, then from (2.6) we obtain

$$\rho\left(\frac{f_1}{f_2}\right) = +\infty, \quad \rho_2\left(\frac{f_1}{f_2}\right) = \rho(B).$$

□

Lemma 2.4 ([14])**.** *Let f and g be meromorphic functions such that $0 < \rho(f), \rho(g) < \infty$ and $\tau(f), \tau(g) < \infty$. Then we have:*

(i) *If $\rho(f) > \rho(g)$, then we obtain*

$$\tau(f+g) = \tau(fg) = \tau(f).$$

(ii) *If $\rho(f) = \rho(g)$ and $\tau(f) \neq \tau(g)$, then we get*

$$\rho(f+g) = \rho(fg) = \rho(f) = \rho(g).$$

Lemma 2.5 ([3])**.** *Let $k \geq 3$, and let f, g be linearly independent solutions of (1.9), where A_0 is a transcendental entire function and $A_1, \ldots, A_{k-2}$ are entire functions of order less than $\rho(A_0)$ if $\rho(A_0) > 0$, and are polynomials if $\rho(A_0) = 0$. Then $u = \frac{f}{g}$ has infinite order.*

Lemma 2.6 ([18])**.** *Suppose $f_j(z)$ $(j = 1, \ldots, n+1)$ and $g_k(z)$ $(k = 1, \ldots, n)(n \geq 1)$ are entire functions satisfying the following conditions:*

(i) $\sum\limits_{j=1}^{n} f_j(z)e^{g_j(z)} = f_{n+1}(z)$.

(ii) *The order of $f_j(z)$ is less than the order of $e^{g_k(z)}$ for $1 \leq j \leq n+1$, $1 \leq k \leq n$. Furthermore, the order of $f_j(z)$ is less than the order of $e^{g_h(z)-g_k(z)}$ for $n \geq 2$ and $1 \leq j \leq n+1$, $1 \leq h < k \leq n$.*

Then $f_j(z) \equiv 0$ $(j = 1, 2, \ldots, n+1)$.

Lemma 2.7 ([12])**.** *Let f be a solution of equation (2.1) where the coefficients $A_j(z)$ $(j = 0, \ldots, k-1)$ are analytic functions in the disc $\Delta_R = \{z \in \mathbb{C} : |z| < R\}$, $0 < R \leq \infty$. Let $n_c \in \{1, \ldots, k\}$ be the number of nonzero coefficients $A_j(z)$ $(j = 0, \ldots, k-1)$, and let $\theta \in [0, 2\pi)$ and $\varepsilon > 0$. If $z_\theta = \nu e^{i\theta} \in \Delta_R$ is such that $A_j(z_\theta) \neq 0$ for some $j = 0, \ldots, k-1$, then for all $\nu < r < R$,*

$$\left|f\left(re^{i\theta}\right)\right| \leq C \exp\left(n_c \int_{\nu}^{r} \max_{j=0,\ldots,k-1} \left|A_j\left(te^{i\theta}\right)\right|^{\frac{1}{k-j}} dt\right), \tag{2.7}$$

where $C > 0$ is a constant satisfying

$$C \leq (1+\varepsilon) \max_{j=0,\ldots,k-1} \left(\frac{\left|f^{(j)}(z_\theta)\right|}{(n_c)^j \max\limits_{n=0,\ldots,k-1} |A_n(z_\theta)|^{\frac{j}{k-n}}}\right).$$

The following lemma is a special case of the result due to T.B. Cao, J.F. Xu and Z.X. Chen in [5].

Lemma 2.8 ([5]). *Let f be a meromorphic function with finite order $0 < \rho(f) < \infty$ and type $0 < \tau(f) < \infty$. Then for any given $\beta < \tau(f)$ there exists a subset I of $[1, +\infty)$ that has infinite logarithmic measure such that $T(r, f) > \beta r^{\rho(f)}$ holds for all $r \in I$.*

Lemma 2.9. *Let $A(z)$ and $B(z)$ be entire functions such that $\rho(B) = \rho$ $(0 < \rho < \infty)$, $\tau(B) = \tau$ $(0 < \tau < \infty)$, and let $\rho(A) < \rho(B)$ and $\tau(A) < \tau(B)$ if $\rho(A) = \rho(B)$. If $f \not\equiv 0$ is a solution of the differential equation*

$$f'' + A(z)f' + B(z)f = 0,$$

then $\rho(f) = +\infty$ and $\rho_2(f) = \rho(B)$.

Proof. If $\rho(A) < \rho(B)$ then by Lemma 2.2, we obtain the result. We prove only the case when $\rho(A) = \rho(B) = \rho$ and $\tau(A) < \tau(B)$. Since $f \not\equiv 0$, then

$$B = -\left(\frac{f''}{f} + A\frac{f'}{f}\right). \tag{2.8}$$

Suppose that f is of finite order, then by (2.8) and the lemma of the logarithmic derivative ([11])

$$T(r, B) \le T(r, A) + O(\log r)$$

which implies the contradiction

$$\tau(B) \le \tau(A).$$

Hence $\rho(f) = \infty$. By using inequality (2.7) for $R = \infty$, we have

$$\rho_2(f) \le \max\{\rho(A), \rho(B)\} = \rho(B). \tag{2.9}$$

On the other hand, since $\rho(f) = \infty$, then by (2.8) and the lemma of the logarithmic derivative

$$T(r, B) \le T(r, A) + S(r, f), \tag{2.10}$$

where $S(r, f) = O(\log T(r, f)) + O(\log r)$, possibly outside a set $E_0 \subset (0, +\infty)$ with a finite linear measure. By $\tau(A) < \tau(B)$, we choose α_0, α_1 satisfying $\tau(A) < \alpha_1 < \alpha_0 < \tau(B)$ such that for $r \to +\infty$, we have

$$T(r, A) \le \alpha_1 r^{\rho}. \tag{2.11}$$

By Lemma 2.8, there exists a subset $E_1 \subset [1, +\infty)$ of infinite logarithmic measure such that

$$T(r, B) > \alpha_0 r^{\rho}. \tag{2.12}$$

By (2.10)–(2.12), we obtain for all $r \in E_1 \backslash E_0$

$$(\alpha_0 - \alpha_1) r^{\rho} \le O(\log T(r, f)) + O(\log r)$$

which implies

$$\rho(B) = \rho \leq \rho_2(f). \tag{2.13}$$

By using (2.9) and (2.13), we obtain $\rho_2(f) = \rho(B)$. □

Remark 2.10. Lemma 2.9 was obtained by J. Tu and C.F. Yi in [17] for higher order linear differential equations by using the type τ_M.

Lemma 2.11 ([8])**.** *Let $A_0, A_1, \ldots, A_{k-1}$ be entire functions satisfying:*

(i) $\rho(A_j) < \rho(A_0) < \infty$ $(j = 1, \ldots, k-1)$ *or*

(ii) A_0 *being transcendental function with* $\rho(A_0) = 0$, *and* $A_1, \ldots, A_{k-1}$ *being polynomials.*

Then every solution $f \not\equiv 0$ of equation (2.1) satisfies $\rho(f) = \infty$.

3. PROOF OF THE THEOREMS

Proof of Theorem 1.6. Suppose that f_1 and f_2 are two nontrivial linearly independent solutions of (1.2) and that

$$w = d_1 f_1 + d_2 f_2. \tag{3.1}$$

Then, by Lemma 2.9, we have

$$\rho(f_1) = \rho(f_2) = \infty$$

and

$$\rho_2(f_1) = \rho_2(f_2) = \rho(B).$$

Suppose that $d_1 = cd_2$, where c is a complex number. Then, by (3.1), we obtain

$$w = cd_2 f_1 + d_2 f_2 = (cf_1 + f_2)\, d_2.$$

Since $f = cf_1 + f_2$ is a solution of (1.2) and $\rho(d_2) < \rho(B)$, then we have

$$\rho(w) = \rho(cf_1 + f_2) = \infty$$

and

$$\rho_2(w) = \rho_2(cf_1 + f_2) = \rho(B).$$

Suppose now that $d_1 \not\equiv cd_2$ where c is a complex number. Differentiating both sides of (3.1), we obtain

$$w' = d_1' f_1 + d_1 f_1' + d_2' f_2 + d_2 f_2'. \tag{3.2}$$

Differentiating both sides of (3.2), we have

$$w'' = d_1'' f_1 + 2d_1' f_1' + d_1 f_1'' + d_2'' f_2 + 2d_2' f_2' + d_2 f_2''. \tag{3.3}$$

Substituting $f_j'' = -A(z)f_j' - B(z)f_j$ $(j = 1, 2)$ into equation (3.3), we obtain

$$w'' = (d_1'' - d_1 B)\, f_1 + (2d_1' - d_1 A)\, f_1' + (d_2'' - d_2 B)\, f_2 + (2d_2' - d_2 A)\, f_2'. \tag{3.4}$$

Differentiating both sides of (3.4) and substituting $f_j'' = -A(z)f_j' - B(z)f_j \ (j = 1, 2)$, we get

$$\begin{aligned} w''' &= \left(d_1^{(3)} - 3d_1'B + d_1\left(AB - B'\right)\right) f_1 \\ &+ \left(3d_1'' - 2d_1'A + d_1\left(A^2 - A' - B\right)\right) f_1' \\ &+ \left(d_2^{(3)} - 3d_2'B + d_2\left(AB - B'\right)\right) f_2 \\ &+ \left(3d_2'' - 2d_2'A + d_2\left(A^2 - A' - B\right)\right) f_2'. \end{aligned} \tag{3.5}$$

By (3.1)–(3.5), we have

$$\begin{cases} w = d_1 f_1 + d_2 f_2, \\ w' = d_1' f_1 + d_1 f_1' + d_2' f_2 + d_2 f_2', \\ w'' = \left(d_1'' - d_1 B\right) f_1 + \left(2d_1' - d_1 A\right) f_1' + \left(d_2'' - d_2 B\right) f_2 + \left(2d_2' - d_2 A\right) f_2', \\ w''' = \left(d_1^{(3)} - 3d_1'B + d_1\left(AB - B'\right)\right) f_1 + \left(3d_1'' - 2d_1'A + d_1\left(A^2 - A' - B\right)\right) f_1' \\ + \left(d_2^{(3)} - 3d_2'B + d_2\left(AB - B'\right)\right) f_2 + \left(3d_2'' - 2d_2'A + d_2\left(A^2 - A' - B\right)\right) f_2'. \end{cases} \tag{3.6}$$

To solve this system of equations, we need first to prove that $h \not\equiv 0$. By simple calculations, we obtain

$$\begin{aligned} h &= \begin{vmatrix} H_1 & H_2 & H_3 & H_4 \\ H_5 & H_6 & H_7 & H_8 \\ H_9 & H_{10} & H_{11} & H_{12} \\ H_{13} & H_{14} & H_{15} & H_{16} \end{vmatrix} \\ &= 2\left(d_1 d_2' - d_2 d_1'\right)^2 B + \left(d_2^2 d_1' d_1'' + d_1^2 d_2' d_2'' - d_1 d_2 d_1' d_2'' - d_1 d_2 d_2' d_1''\right) A \\ &\quad - 2\left(d_1 d_2' - d_2 d_1'\right)^2 A' + 2d_1 d_2 d_1' d_2''' + 2d_1 d_2 d_2' d_1''' - 6d_1 d_2 d_1'' d_2'' \\ &\quad - 6d_1 d_1' d_2' d_2'' - 6d_2 d_1' d_2' d_1'' \\ &\quad + 6d_1 (d_2')^2 d_1'' + 6d_2 (d_1')^2 d_2'' - 2d_2^2 d_1' d_1''' - 2d_1^2 d_2' d_2''' + 3d_1^2 (d_2'')^2 + 3d_2^2 (d_1'')^2. \end{aligned} \tag{3.7}$$

It is clear that $\left(d_1 d_2' - d_2 d_1'\right)^2 \not\equiv 0$ because $d_1 \neq c d_2$. Since $\max\left\{\rho\left(d_1\right), \rho\left(d_2\right)\right\} < \rho(B)$ and $\left(d_1 d_2' - d_2 d_1'\right)^2 \not\equiv 0$, then by using Lemma 2.4 we can deduce

$$\rho\left(h\right) = \rho(B) > 0. \tag{3.8}$$

Hence $h \not\equiv 0$. By Cramer's method, we have

$$\begin{aligned} f_1 &= \frac{\begin{vmatrix} w & H_2 & H_3 & H_4 \\ w' & H_6 & H_7 & H_8 \\ w'' & H_{10} & H_{10} & H_{12} \\ w^{(3)} & H_{14} & H_{15} & H_{16} \end{vmatrix}}{h} \\ &= 2\frac{\left(d_1 d_2 d_2' - d_2^2 d_1'\right)}{h} w^{(3)} + \phi_2 w'' + \phi_1 w' + \phi_0 w, \end{aligned} \tag{3.9}$$

where ϕ_j $(j = 0, 1, 2)$ are meromorphic functions of finite order which are defined in (1.4)–(1.6). Suppose now $\rho(w) < \infty$, then by (3.9) we obtain $\rho(f_1) < \infty$ which is a contradiction. Hence $\rho(w) = \infty$. By (3.1), we have $\rho_2(w) \leq \rho(B)$. Suppose that $\rho_2(w) < \rho(B)$, then by (3.9) we obtain $\rho_2(f_1) < \rho(B)$, which is a contradiction. Hence $\rho_2(w) = \rho(B)$. □

Proof of Theorem 1.7. By Theorem 1.6, we have $\rho(w) = \infty$ and $\rho_2(w) = \rho(B)$. Set $g(z) = d_1 f_1 + d_2 f_2 - \varphi$. Since $\rho(\varphi) < \infty$, then we have $\rho(g) = \rho(w) = \infty$ and $\rho_2(g) = \rho_2(w) = \rho(B)$. In order to prove $\overline{\lambda}(w-\varphi) = \lambda(w-\varphi) = \infty$ and $\overline{\lambda}_2(w-\varphi) = \lambda_2(w-\varphi) = \rho(B)$, we need to prove only $\overline{\lambda}(g) = \lambda(g) = \infty$ and $\overline{\lambda}_2(g) = \lambda_2(g) = \rho(B)$. By $w = g + \varphi$, we get from (3.9)

$$f_1 = 2\frac{\left(d_1 d_2 d_2' - d_2^2 d_1'\right)}{h} g^{(3)} + \phi_2 g'' + \phi_1 g' + \phi_0 g + \psi, \tag{3.10}$$

where

$$\psi = 2\frac{\left(d_1 d_2 d_2' - d_2^2 d_1'\right)}{h} \varphi^{(3)} + \phi_2 \varphi'' + \phi_1 \varphi' + \phi_0 \varphi.$$

Substituting (3.10) into equation (1.2), we obtain

$$\frac{2\left(d_1 d_2 d_2' - d_2^2 d_1'\right)}{h} g^{(5)} + \sum_{j=0}^{4} \beta_j g^{(j)} = -\left(\psi'' + A(z)\psi' + B(z)\psi\right) = F(z).$$

where β_j $(j = 0, \dots, 4)$ are meromorphic functions of finite order. Since $\psi \not\equiv 0$ and $\rho(\psi) < \infty$, it follows that ψ is not a solution of (1.2), which implies that $F(z) \not\equiv 0$. Then, by applying Lemma 2.1, we obtain (1.7) and (1.8). □

Proof of Theorem 1.8. Suppose that f_1 and f_2 are two nontrivial linearly independent solutions of (1.2). Then by Lemma 2.3, we have

$$\rho\left(\frac{f_1}{f_2}\right) = +\infty, \quad \rho_2\left(\frac{f_1}{f_2}\right) = \rho(B).$$

Set $g = \frac{f_1}{f_2}$. Then

$$w(z) = \frac{d_1(z) f_1(z) + d_2(z) f_2(z)}{b_1(z) f_1(z) + b_1(z) f_2(z)} = \frac{d_1(z) g(z) + d_2(z)}{b_1(z) g(z) + b_2(z)}. \tag{3.11}$$

It follows that

$$\begin{aligned} &\rho(w) \leq \rho(g) = +\infty, \\ &\rho_2(w) \leq \max\left\{\rho_2(d_j), \rho_2(b_j)\,(j = 1, 2), \rho_2(g)\right\} = \rho_2(g). \end{aligned} \tag{3.12}$$

On the other hand, we have

$$g(z) = -\frac{b_2(z) w(z) - d_2(z)}{b_1(z) w(z) - d_1(z)},$$

which implies that

$$\begin{aligned} +\infty &= \rho(g) \leq \rho(w), \\ \rho_2(g) &\leq \max\left\{\rho_2(d_j), \rho_2\left(b_j\right)(j=1,2), \rho_2(w)\right\} = \rho_2(w). \end{aligned} \tag{3.13}$$

By using (3.12) and (3.13), we obtain

$$\rho(w) = \rho(g) = +\infty, \quad \rho_2(w) = \rho_2(g) = \rho(B).$$

□

Proof of Theorem 1.9. Under the conditions of Theorem 1.9 and Lemma 2.11 we have

$$\rho\left(f_j\right) = \infty \quad (j = 1, \ldots, k).$$

By Hadamard factorization,

$$f_j(z) = \Pi_j e^{h_j(z)} \quad (j = 1, \ldots, k), \tag{3.14}$$

where Π_j is the canonical product of zeros of $f_j(z)$ such that

$$\lambda\left(f_j\right) = \rho\left(\Pi_j\right) < \infty$$

and $h_j(z)$ $(j = 1, \ldots, k)$ are transcendental entire functions. Suppose that $P_k(f)$ is of finite order, then

$$P_k(f) = \Pi_{k+1} e^{h_{k+1}(z)}, \tag{3.15}$$

where $h_{k+1}(z)$ is a polynomial and Π_{k+1} is the canonical product of zeros of $P_k(f)$. By (3.14) and (3.15), we have

$$\sum_{j=1}^{k} d_j \Pi_j e^{h_j(z)} = \Pi_{k+1} e^{h_{k+1}(z)}. \tag{3.16}$$

Since $h_j(z)$ $(j = 1, \ldots, k)$ are transcendental entire functions, then by (3.16), we obtain

$$\max\left\{\rho\left(d_j \Pi_j\right)(j = 1, \ldots, k), \rho\left(\Pi_{k+1} e^{h_{k+1}}\right)\right\} < \rho\left(e^{h_j}\right) = \infty. \tag{3.17}$$

By using Lemma 2.5, for any two linearly independent solutions f_p and f_q, where $1 \leq p < q \leq k$, the quotient $\frac{f_p}{f_q}$ has infinite order. Then

$$\max\left\{\rho\left(d_j \Pi_j\right)(j = 1, \ldots, k), \rho\left(\Pi_{k+1} e^{h_{k+1}}\right)\right\} < \rho\left(e^{h_p - h_q}\right) = \infty. \tag{3.18}$$

By (3.17), (3.18) and Lemma 2.6, we have $d_j \equiv 0$ $(j = 1, \ldots, k)$ which is a contradiction. Hence $P_k(f)$ is of infinite order. □

Proof of Theorem 1.10. Suppose that $Q_{k,m}(f)$ is of finite order. Then

$$\frac{\sum_{i=1}^{k} d_i f_i}{\sum_{j=1}^{m} b_j f_j} = \Pi_{k+1} e^{h_{k+1}(z)}, \tag{3.19}$$

where $h_{k+1}(z)$ is a polynomial and Π_{k+1} is the canonical product of zeros of $Q_{k,m}(f)$. The equality (3.19) is equivalent to

$$\sum_{i=1}^{m}\left(d_i - b_i\Pi_{k+1}e^{h_{k+1}(z)}\right) f_i + d_{m+1}f_{m+1} + \ldots + d_k f_k = 0. \tag{3.20}$$

By Hadamard factorization,

$$f_i(z) = \Pi_i e^{h_i(z)}\,(i = 1, \ldots, k)\,.$$

By the same reasoning as Theorem 1.9, and by using Lemma 2.6 we obtain $d_i \equiv 0$ $(i = 1, \ldots, k)$ which is a contradiction. Hence $Q_{k,m}(f)$ is of infinite order. □

4. OPEN QUESTION

It is interesting to study the hyper-order and oscillation of the combination

$$P_k(f) = \sum_{i=1}^{k} d_i f_i,$$

where f_i $(i = 1, \ldots, k)$ are linearly independent solutions of (1.9) and d_i $(i = 1, \ldots, k)$ are entire functions of finite order not all vanishing identically.

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