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GROUP DECISION MAKING USING INTERVAL-VALUED INTUITIONISTIC FUZZY SOFT MATRIX AND CONFIDENT WEIGHT OF EXPERTS

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Abstract

This article proposes an algorithmic approach for multiple attribute group decision making (MAGDM) problems using interval-valued intuitionistic fuzzy soft matrix (IVIFSM) and confident weight of experts. We propose a novel concept for assigning confident weights to the experts based on cardinals of interval-valued intuitionistic fuzzy soft sets (IVIFSSs). The confident weight is assigned to each of the experts based on their preferred attributes and opinions, which reduces the chances of biasness. Instead of using medical knowledgebase, the proposed algorithm mainly relies on the set of attributes preferred by the group of experts. To make the set of preferred attributes more important, we use combined choice matrix, which is combined with the individual IVIFSM to produce the corresponding product IVIFSM. This article uses IVIFSMs for representing the experts' opinions. IVIFSM is the matrix representation of IVIFSS and IVIFSS is a natural combination of interval-valued intuitionistic fuzzy set (IVIFS) and soft set. Finally, the performance of the proposed algorithm is validated using a case study from real life.

1 Introduction

Molodtsov [1] introduced soft set theory as a generic mathematical tool for dealing with uncertain problems, which cannot be handled using traditional mathematical tools such as theory of probability [2], theory of fuzzy sets [3], rough set theory [4], and the interval mathematics. All these theories have their own inherent difficulties due to the inadequacy of the parameterization. Molodtsov's soft set [1] is free from such kind of difficulties, which can be used for approximate description of objects without any restriction. Due to this absence of restriction on the approximate description, soft set theory has been emerging as a convenient and easily applicable tool in practice.

Since its introduction, soft set theory has been successfully applied in many different fields such as decision making [5-12], data analysis [13], forecasting [14], simulation [15], optimization [16], texture classification [17], etc. Maji et al. [18] presented some operations for soft sets and also discussed their properties. They presented the concept of fuzzy soft set (FSS) [19] which is based on a combination of the fuzzy set and soft set models. Yang et al. [20] introduced the concept of the interval-valued fuzzy soft set (IVFSS) by combining the interval-valued fuzzy set and soft set and then explained a decision making algorithm based on IVFSS. Decision making problems were solved first by Maji and Roy [11] using soft sets. They [12] studied a soft set theoretic approach to deal with decision making and introduced the concept of choice value. Combining soft set [1] with intuitionistic fuzzy set [25, 33], Maji et al. [21–23] introduced the concept of intuitionistic fuzzy soft sets (IFSS). The suitability of IFSS for the decision making applications is found in [24]. Das and Kar [31] proposed an algorithmic approach for group decision making based on IFSS. The authors [31] have used cardinals of IFSS as a novel concept for assigning confident weight to the set of experts. Cagman and Enginoglu [6-7] pioneered the concept of soft matrix to represent a soft set. Mao et al. [30] presented the concept of intuitionistic fuzzy soft matrix (IFSM) and applied it in group decision making problems. Jiang et al. [26] introduced IVIFSS by combining interval-valued intuitionistic fuzzy set and soft set and discussed their properties. Jiang et al. [9] presented an adjustable approach to IFSS based decision making by using level soft sets of IFSSs. Feng et al. [27] extended the level soft sets method to interval-valued fuzzy soft sets. Qin et al. [28] generalized the approaches introduced by Feng et al. [27] and Jiang et al. [9]. They [28] defined the notion of reduct IFSSs and presented an adjustable approach for decision making based on IVIFSS. Zhang et al. [29] investigated the decision making problems based on IVIFSS. They [29] developed an adjustable approach to IVIFSSs based decision making using level soft sets. Recently, Das et al. in [32] have introduced IVIFSM and applied it to multiple attribute decision making problems. Das and Kar [34-35] have also introduced the concept of intuitionistic multi fuzzy soft set (IMFSS) and hesitant fuzzy soft set (HFSS) and applied them in decision making problem.

Preceding discussion narrates that a few decision making approaches [26, 28, 29, 32] have been developed by the researchers in the context of IVIFSS. In [26], Jiang et al. introduced IVIFSSs and discussed their various operations and properties. In [28], authors presented an adjustable approach to IVIFSS based decision making using reduct IFSS and level soft sets of IFSS. Zhang et al. in [29] developed an adjustable approach to IVIFSS based decision making. They [29] also defined the concept of the weighted IVIFSS, where they emphasized the importance of the parameters in IVIFSS. Then they presented a decision making approach using weighted IVIFSS. Authors introduced IVIFSM in [32] and studied a novel algorithmic approach focussing on the choice parameters of different experts. Since decision makers/experts express their opinions based on the available information and domain of expertise of different experts are different, so a prioritising mechanism of various experts are necessary to introduce a quality decision making paradigm. Due to lack of information or limited domain knowledge, experts often prefer to express their opinions only for a subset of attributes instead of the entire set of attributes. As a result, some decision makers may express their opinions to more numbers of attributes, while others show their interest only for a few attributes. Normally, more importance can be assigned to those experts who encompass more attributes. In another case, an expert who is more confident about some set of attributes can be assigned more importance. For having better decision making outcome in MAGDM problems, importance or weight of various decision makers imparts a huge significance in the entire process. But as per our knowledge, this mechanism can be found only in [31], where authors have used a novel concept in terms of confident weight assigning mechanism of experts in the framework of IFSS. None have implemented the idea presented in [31] for group decision making using IVIFSS. Motivated by this idea, we have introduced a MAGDM algorithm using confident weight assigning of experts in the context of IVIFSS. The proposed algorithm mainly focuses on the choice parameters/attributes of various experts and computes the confident weight of an expert based on her prescribed opinions. We have used cardinals of IVIFSS for measuring the weight. The proposed confident weight also reduces the chance of biasing. Once the weights are assigned to individual experts' opinions, this article present a consensus reaching approach based on IVIFSM, combined choice matrix, product IVIFSM, score, and accuracy values. We have used a case study related to heart disease diagnosis, where two cases are shown. Case I shows the final outcome without assigning any weight, whereas case II shows the result by assigning weight. As per the experimental result, these two cases show different ordering of diseases.

Rest of this article is organized as follows. In section 2, we briefly review some basic notions and background of soft sets, FSSs, IVFSSs, IVIFSSs, and IVIFSMs. Section 3 presents IVIFSM and a few operations on it. The cardinal of IVIFSS is introduced in section 4 followed by the proposed algorithmic approach in section 5. A case study has been illustrated in section 6 to verify the practicability and effectiveness of the proposed method. Then a brief discussion on the results is given in section 7. Finally, key conclusions are drawn in section 8.

2 Preliminaries

This section briefly reviews some basic concepts related with this article.

2.1 Soft Set

Let U be an initial universal set and E be a set of parameters. Let *P(U)* denotes the power set of U and $A \subseteq E$. A pair $(F_{\{A\}}, E)$ is called a soft set [1] over U, where $F_{\{A\}}$ is a mapping given by $F_{\{A\}} : E \to P(U)$such that $F_{\{A\}}(e) = \emptyset$ if $e \notin A$. In other words, a soft set over U is a mapping from parameters to *P(U)*, and it is not a set, but a parameterized family of subsets of U. For any parameter $e \in A, F_{\{A\}}(e)$ may be considered as the set of e-approximate elements of the soft set $(F_{\{A\}}, E)$. For illustration, let us consider the following example.

Example 1. Let U be the set of five diseases (Viral fever, Malaria, Typhoid, Gastric ulcer, Pneumonia) given by $U = \{d_1, d_2, d_3, d_4, d_5\}$ and E be the set of five symptoms given by E= {Temperature, Headache, Stomach pain, Cough, Chest pain} $=\{s_1, s_2, s_3, s_4, s_5\}$.

Let $A = \{s_1, s_2, s_3\} \subset E$. Now consider that $F_{\{A\}}$ is a mapping given by, $F_{\{A\}}(s_1) = \{d_1, d_2\}$, $F_{\{A\}}(s_2) = \{d_1, d_3\}$, $F_{\{A\}}(s_3) = \{d_2, d_4\}$.

Then the soft set $(F_{\{A\}}, E) = \{(s_1, d_1, d_2),$ $(s_2, d_1, d_3), (s_3, \{d_2, d_4\}), (s_4, \{\emptyset\}), (s_5, \{\emptyset\})$. A fuzzy soft set can also be represented in the form of a two dimensional table. Table 1, given below, represents the soft set $(F_{\{A\}}, E)$.

Table 1. Tabular representation of $(F_{\{A\}}, E)$

U/E	s_1	s_2	s_3	s_4	s_5
d_1	1	1	0	0	0
d_2	1	0	1	0	0
d_3	0	1	0	0	0
d_4	0	0	1	0	0
d_5	0	0	0	0	0

2.2 Fuzzy soft set

Let U be an initial universe and E be a set of parameters (which are fuzzy variables). Let *FS(U)* denotes the set of all fuzzy sets of U and $A \subset E$. A pair $\left(\widehat{F}_{\{A\}}, E\right)$ is called a fuzzy soft set (FSS) [19] over U, where $\widehat{F}_{\{A\}}$ is a mapping given by $\widehat{F}_{\{A\}} : E \to FS(U)$ such that $\widehat{F}_{\{A\}}(e) = \widehat{\emptyset}$if $e \notin A$ where $\widehat{\emptyset}$ is null fuzzy set.

A fuzzy soft set is a parameterized family of fuzzy subsets of U. Its universe is the set of all fuzzy sets of U, i.e., *FS(U)*. A fuzzy soft set can be considered a special case of a soft set because it is still a mapping from parameters to a universe. The difference between fuzzy soft set and soft set is that in a fuzzy soft set, the universe to be considered is the set of fuzzy subsets of U.

Example 2. Let U and E remain same as in Example 1.

Here $A = \{s_1, s_2, s_3, s_5\} \subset E$. Let

$$\widehat{F}_{\{A\}}(s_1) = \{d_1/0.2, d_2/0.4, d_3/0.9, d_4/0.7\},$$

$$\widehat{F}_{\{A\}}(s_2) = \{d_2/0.8, d_3/0.1, d_4/0.7\},$$

$$\widehat{F}_{\{A\}}(s_3) = \{d_1/0.6, d_2/0.2, d_3/0.8\},$$

and

$$\widehat{F}_{\{A\}}(s_5) = \{d_1/0.6, d_2/0.7, d_3/0.5, d_4/0.8\}.$$

Then the fuzzy soft set is given by

$$(\widehat{F}_{\{A\}}, E) = \left\{ \begin{array}{l} (s_1, \{d_1/0.2, d_2/0.4, d_3/0.9, d_4/0.7\}), \\ (s_2, \{d_2/0.8, d_3/0.1, d_4/0.7\}), \\ (s_3, \{d_1/0.6, d_2/0.2, d_3/0.8\}), \\ \left(s_4, \{\widehat{\emptyset}\}\right), \\ (s_5, \{d_1/0.6, d_2/0.7, d_3/0.5, d_4/0.8\}) \end{array} \right\}.$$

Tabular representation of the fuzzy soft set $(\widehat{F}_{\{A\}}, E)$ is shown in Table 2.

Table 2. Tabular representation of $(\widehat{F}_{\{A\}}, E)$

U/E	s_1	s_2	s_3	s_4	s_5
d_1	0.2	0	0.6	0	0.6
d_2	0.4	0.8	0.2	0	0.7
d_3	0.9	0.1	0.8	0	0.5
d_4	0.7	0.7	0	0	0.8
d_5	0	0	0	0	0

2.3 Intuitionistic fuzzy soft set (IFSS)

Let U be an initial universe and E be a set of parameters. Let *IFS (U)* denotes the set of all intuitionistic fuzzy sets of U and $A \subset E$. A pair $(\widehat{F}_{\{A\}}, E)$ is called an intuitionistic fuzzy soft set (IFSS) [22] over U, where $\widehat{F}_{\{A\}}$ is a mapping given by, $\widehat{F}_{\{A\}} : E \to IFS(U)$ so that $\widehat{F}_{\{A\}}(e) = \widehat{\emptyset}$ if $e \notin A$, where $\widehat{\emptyset}$ is null intuitionistic fuzzy set, i.e., the membership value of x, $\mu(x) = 0$; the non-membership value of x, $\nu(x) = 1$ and the indeterministic part of x, $\pi(x) = 0$, $\forall x \in \widehat{\emptyset}$.

Example 3. Let U, A and E are same as in Example 2. Let us take

$$\widehat{F}_{\{A\}}(s_1) = \{d_1/(0.2,0.5), d_2/(0.4,0.3), d_3/(0.9,0.1), d_4/(0.7,0.2)\},$$
$$\widehat{F}_{\{A\}}(s_2) = \{d_2/(0.8,0.2), d_3/(0.1,0.8), d_4/(0.7,0.2)\},$$
$$\widehat{F}_{\{A\}}(s_3) = \{d_1/(0.6,0.3), d_2/(0.2,0.6), d_3/(0.8,0.1)\}, and$$
$$\widehat{F}_{\{A\}}(s_5) = \{d_1/(0.6,0.1), d_2/(0.7,0.2), d_3/(0.5,0.2), d_4/(0.8,0.2)\}.$$

Then the IFSS is given below.

$$(\widehat{F}_{\{A\}}, E) = \left\{ \begin{array}{l} (s_1, \{d_1/(0.2,0.5), d_2/(0.4,0.3), d_3/(0.9,0.1), d_4/(0.7,0.2)\}), \\ (s_2, \{d_2/(0.8,0.2), d_3/(0.1,0.8), d_4/(0.7,0.2)\}), \\ (s_3, \{d_1/(0.6,0.3), d_2/(0.2,0.6), d_3/(0.8,0.1)\}), \\ \left(s_4, \widehat{\emptyset}\right), \\ (s_5, \{d_1/(0.6,0.1), d_2/(0.7,0.2), d_3/(0.5,0.2), d_4/(0.8,0.2)\}) \end{array} \right\}.$$

Tabular representation of the IFSS $(\widehat{F}_{\{A\}}, E)$ is shown in Table 3.

Table 3. Tabular representation of $(\widehat{F}_{\{A\}}, E)$

U/E	s_1	s_2	s_3	s_4	s_5
d_1	(0.2,0.5)	0	(0.6,0.3)	0	(0.6,0.1)
d_2	(0.4,0.3)	(0.8,0.2)	(0.2,0.6)	0	(0.7,0.2)
d_3	(0.9,0.1)	(0.1,0.8)	(0.8,0.1)	0	(0.5,0.2)
d_4	(0.7,0.2)	(0.7,0.2)	0	0	(0.8,0.2)
d_5	0	0	0	0	0

2.4 Interval-valued intuitionistic fuzzy soft set (IVIFSS)

Let U be an initial universe and E be a set of parameters. $IVIF(U)$ denotes the set of all interval-valued intuitionistic fuzzy sets of U. Let $A \subseteq E$. A pair $(\widehat{F}_{\{A\}}, E)$ is an IVIFSS [26] over U, where $\widehat{F}_{\{A\}}$ is a mapping, given by

$$\widehat{F}_{\{A\}} : E \to IVIF(U).$$

An IVIFSS is a parameterized family of interval-valued intuitionistic fuzzy subsets of U. Thus, its universe is the set of all interval-valued intuitionistic fuzzy sets of U, i.e., $IVIF(U)$.

$$\forall \varepsilon \in A,\ F(\varepsilon)$$

is an interval-valued intuitionistic fuzzy set of U. $F(\varepsilon)$ is expressed as:

$$F(\varepsilon) = \{< x, [\mu^l_{F(\varepsilon)}(x), \mu^r_{F(\varepsilon)}(x)], [\nu^l_{F(\varepsilon)}(x), \nu^r_{F(\varepsilon)}(x)] > | x \in X\}.$$

Example 4. Let U, A and E are same as in Example 3. Let us take

$$\widehat{F}_{\{A\}}(s_1) = \left\{ d_1/\begin{pmatrix} (0.2,0.4) \\ (0.5,0.6) \end{pmatrix}, d_2/\begin{pmatrix} (0.4,0.5) \\ (0.3,0.4) \end{pmatrix}, d_3/\begin{pmatrix} (0.7,0.8) \\ (0.1,0.2) \end{pmatrix} d_4/\begin{pmatrix} (0.6,0.7) \\ (0.2,0.3) \end{pmatrix} \right\},$$

$$\widehat{F}_{\{A\}}(s_2) = \left\{ d_2/\begin{pmatrix} (0.6,0.8) \\ (0.1,0.2) \end{pmatrix}, d_3/\begin{pmatrix} (0.1,0.2), \\ (0.7,0.8) \end{pmatrix}, d_4/\begin{pmatrix} (0.5,0.7) \\ (0.2,0.3) \end{pmatrix} \right\},$$

$$\widehat{F}_{\{A\}}(s_3) = \left\{ d_1/\begin{pmatrix} (0.6,0.7) \\ (0.2,0.3) \end{pmatrix}, d_2/\begin{pmatrix} (0.2,0.3) \\ (0.6,0.7) \end{pmatrix}, d_3/\begin{pmatrix} (0.6,0.8) \\ (0.1,0.2) \end{pmatrix} \right\},$$

and

$$\widehat{F}_{\{A\}}(s_5) = \left\{ d_1/\begin{pmatrix} (0.6,0.7) \\ (0.1,0.2) \end{pmatrix}, d_2/\begin{pmatrix} (0.5.0.7) \\ (0.2,0.3) \end{pmatrix}, d_3/\begin{pmatrix} (0.4,0.6) \\ (0.2,0.3) \end{pmatrix}, d_4/\begin{pmatrix} (0.7,0.8) \\ (0.1,0.2) \end{pmatrix} \right\}.$$

Then the IVIFSS is given by

$$(\widehat{F}_{\{A\}}, E) = \left\{ \begin{array}{l} \left(s_1, \left\{ d_1/\begin{pmatrix} (0.2,0.4) \\ (0.5,0.6) \end{pmatrix}, d_2/\begin{pmatrix} (0.4,0.5) \\ (0.3,0.4) \end{pmatrix}, d_3/\begin{pmatrix} (0.7,0.8) \\ (0.1,0.2) \end{pmatrix} d_4/\begin{pmatrix} (0.6,0.7) \\ (0.2,0.3) \end{pmatrix} \right\}\right), \\ \left(s_2, \left\{ d_2/\begin{pmatrix} (0.6,0.8) \\ (0.1,0.2) \end{pmatrix}, d_3/\begin{pmatrix} (0.1,0.2), \\ (0.7,0.8) \end{pmatrix}, d_4/\begin{pmatrix} (0.5,0.7) \\ (0.2,0.3) \end{pmatrix} \right\}\right), \\ \left(s_3, \left\{ d_1/\begin{pmatrix} (0.6,0.7) \\ (0.2,0.3) \end{pmatrix}, d_2/\begin{pmatrix} (0.2,0.3) \\ (0.6,0.7) \end{pmatrix}, d_3/\begin{pmatrix} (0.6,0.8) \\ (0.1,0.2) \end{pmatrix} \right\}\right), \\ \left(s_4, \widehat{\emptyset}\right), \\ \left(s_5, \left\{ d_1/\begin{pmatrix} (0.6,0.7) \\ (0.1,0.2) \end{pmatrix}, d_2/\begin{pmatrix} (0.5.0.7) \\ (0.2,0.3) \end{pmatrix}, d_3/\begin{pmatrix} (0.4,0.6) \\ (0.2,0.3) \end{pmatrix}, d_4/\begin{pmatrix} (0.7,0.8) \\ (0.1,0.2) \end{pmatrix} \right\}\right) \end{array} \right\}.$$

Tabular representation of the IVIFSS$(\widehat{F}_{\{A\}}, E)$is shown in Table 4.

Table 4. Tabular representation of $(\widehat{F}_{\{A\}}, E)$

U/E	s_1	s_2	s_3	s_4	s_5
d_1	$\begin{pmatrix}(0.2,0.4)\\(0.5,0.6)\end{pmatrix}$	0	$\begin{pmatrix}(0.6,0.7)\\(0.2,0.3)\end{pmatrix}$	0	$\begin{pmatrix}(0.6,0.7)\\(0.1,0.2)\end{pmatrix}$
d_2	$\begin{pmatrix}(0.4,0.5)\\(0.3,0.4)\end{pmatrix}$	$\begin{pmatrix}(0.6,0.8)\\(0.1,0.2)\end{pmatrix}$	$\begin{pmatrix}(0.2,0.3)\\(0.6,0.7)\end{pmatrix}$	0	$\begin{pmatrix}(0.5.0.7)\\(0.2,0.3)\end{pmatrix}$
d_3	$\begin{pmatrix}(0.7,0.8)\\(0.1,0.2)\end{pmatrix}$	$\begin{pmatrix}(0.1,0.2),\\(0.7,0.8)\end{pmatrix}$	$\begin{pmatrix}(0.6,0.8)\\(0.1,0.2)\end{pmatrix}$	0	$\begin{pmatrix}(0.4,0.6)\\(0.2,0.3)\end{pmatrix}$
d_4	$\begin{pmatrix}(0.6,0.7)\\(0.2,0.3)\end{pmatrix}$	$\begin{pmatrix}(0.5,0.7)\\(0.2,0.3)\end{pmatrix}$	0	0	$\begin{pmatrix}(0.7,0.8)\\(0.1,0.2)\end{pmatrix}$
d_5	0	0	0	0	0

3 Interval-valued Intuitionistic Fuzzy Soft Matrix

Let $(\widehat{F}_{\{A\}}, E)$ be an IVIFSS over the initial universe U. Let E be a set of parameters and$A \subseteq E$. Then a subset of $U \times E$ is uniquely defined by $R_A = \{(u,e) : e \in A, u \in \widehat{F}_{\{A\}}(e)\}$,which is called a relation of $(\widehat{F}_{\{A\}}, E)$. The membership function of R_A is written as $\mu_{R_A} : U \times E \to \mathrm{Int}([0,1])$ and defined by

$$\mu_{R_A}(u,e) = \begin{cases} [\{\mu^l_{\widehat{F}_{\{A\}}(e)}(u), \mu^r_{\widehat{F}_{\{A\}}(e)}(u)\}, \{\nu^l_{\widehat{F}_{\{A\}}(e)}(u), \nu^r_{\widehat{F}_{\{A\}}(e)}(u)\}]\ if\ e \in A \\ [0,0],\ if\ e \notin A, \end{cases}$$

where $\mathrm{Int}([0,1])$stands for the set of all closed subintervals of $[0,1]$. $[\mu^l_{\widehat{F}_A(e)}(u), \mu^r_{\widehat{F}_A(e)}(u)]$ and $[\nu^l_{\widehat{F}_{\{A\}}(e)}(u), \nu^r_{\widehat{F}_{\{A\}}(e)}(u)]$ are respectively the interval-valued intuitionistic fuzzy membership and non-membership degrees of the object u associated with the parameter e.

Let $U = \{x_1, x_2, ..., x_m\}$ and $E = \{e_1, e_2, ..., e_n\}$. For simplicity, if we take the $[ij]^{th}$ entry of the relation

$$\widehat{F}\ \text{as} \widehat{a}_{ij} = [\{\mu^l_{\widehat{F}_{\{A\}}(e_j)}(x_i), \mu^r_{\widehat{F}_{\{A\}}(e_j)}(x_i)\}, \{\nu^l_{\widehat{F}_{\{A\}}(e_j)}(x_i), \nu^r_{\widehat{F}_{\{A\}}(e_j)}(x_i)\}],$$

then the matrix is defined as

$$[\widehat{a}_{ij}]_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

The above matrix is called an IVIFSM [32] of order $m \times n$ corresponding to the IVIFSS $(\widehat{F}_{\{A\}}, E)$ over U.

Example 5. The IVIFSM corresponding to Example 4 is given below.

$$[\widehat{a}_{ij}] = \begin{pmatrix} \begin{pmatrix} (0.2,0.4) \\ (0.5,0.6) \end{pmatrix} & \begin{pmatrix} (0,0) \\ (0,0) \end{pmatrix} & \begin{pmatrix} (0.6,0.7) \\ (0.2,0.3) \end{pmatrix} & \begin{pmatrix} (0,0) \\ (0,0) \end{pmatrix} & \begin{pmatrix} (0.6,0.7) \\ (0.1,0.2) \end{pmatrix} \\ \begin{pmatrix} (0.4,0.5) \\ (0.3,0.4) \end{pmatrix} & \begin{pmatrix} (0.6,0.8) \\ (0.1,0.2) \end{pmatrix} & \begin{pmatrix} (0.2,0.3) \\ (0.6,0.7) \end{pmatrix} & \begin{pmatrix} (0,0) \\ (0,0) \end{pmatrix} & \begin{pmatrix} (0.5.0.7) \\ (0.2,0.3) \end{pmatrix} \\ \begin{pmatrix} (0.7,0.8) \\ (0.1,0.2) \end{pmatrix} & \begin{pmatrix} (0.1,0.2), \\ (0.7,0.8) \end{pmatrix} & \begin{pmatrix} (0.6,0.8) \\ (0.1,0.2) \end{pmatrix} & \begin{pmatrix} (0,0) \\ (0,0) \end{pmatrix} & \begin{pmatrix} (0.4,0.6) \\ (0.2,0.3) \end{pmatrix} \\ \begin{pmatrix} (0.6,0.7) \\ (0.2,0.3) \end{pmatrix} & \begin{pmatrix} (0.5,0.7) \\ (0.2,0.3) \end{pmatrix} & \begin{pmatrix} (0,0) \\ (0,0) \end{pmatrix} & \begin{pmatrix} (0,0) \\ (0,0) \end{pmatrix} & \begin{pmatrix} (0.7,0.8) \\ (0.1,0.2) \end{pmatrix} \\ \begin{pmatrix} (0,0) \\ (0,0) \end{pmatrix} & \begin{pmatrix} (0,0) \\ (0,0) \end{pmatrix} & \begin{pmatrix} (0,0) \\ (0,0) \end{pmatrix} & \begin{pmatrix} (0,0) \\ (0,0) \end{pmatrix} & \begin{pmatrix} (0,0) \\ (0,0) \end{pmatrix} \end{pmatrix}.$$

3.1 Choice Matrix and Combined Choice Matrix

Choice Matrix is a square matrix whose rows and columns both indicate parameters. If β is a choice matrix, then its element $\beta(i,j)$ is defined as follows:

$$\beta(i,j)^P = \begin{cases} (1,\ 1),(1,1) \text{ when i}^{\text{th}} \text{ and j}^{\text{th}} \text{ parameters are the choice parameters of both the decision makers} \\ (0,\ 0),(0,0) \text{ otherwise, i.e., when at least one of the i}^{\text{th}} \text{ and j}^{\text{th}} \text{ parameters is not the choice of the decision makers.} \end{cases}$$

In *combined choice matrix*, rows indicate choice parameters of single decision maker, where columns indicate combined choice parameters (obtained by the intersection of their parameters sets) of decision makers.

3.2 Product of IVIFSM and Combined Choice Matrix

Product of IVIFSM and combined choice matrix is possible if the number of columns of IVIFSM $\widehat{A}$ be equal to the number of rows of the combined choice matrixβ. Then $\widehat{A}$ and β are said to be conformable for the product $(\widehat{A}\otimes\beta)$ and their product $(\widehat{A}\otimes\beta)$ is called product IVIFSM. The product $(\widehat{A}\otimes\beta)$ is also denoted simply by $\widehat{A}\beta$.

$$\text{If } \widehat{A} = [\widehat{a}_{ij}]_{m\times n} \text{ and } \beta = [\widehat{\beta}_{jk}]_{n\times p}, \text{ then } \widehat{A}\beta = [\widehat{c}_{ik}]_{m\times p},$$

where

$$\widehat{c}_{ik} = \begin{Bmatrix} [\max_{j=1}^{n} \min\{\mu^l_{\widehat{a}_{ij}}, \mu^l_{\widehat{\beta}_{jk}}\}, \max_{j=1}^{n} \min\{\mu^r_{\widehat{a}_{ij}}, \mu^r_{\widehat{\beta}_{jk}}\}], \\ [\min_{j=1}^{n} \min\{\nu^l_{\widehat{a}_{ij}}, \nu^l_{\widehat{\beta}_{jk}}\}, \min_{j=1}^{n} \min\{\nu^r_{\widehat{a}_{ij}}, \nu^r_{\widehat{\beta}_{jk}}\}] \end{Bmatrix}.$$

Example 6. Example for the product of IVIFSM and combined choice matrix is shown in Step 5 of both the case studies, explained in Section 6.

3.3 Addition of IVIFSMs

Two IVIFSMs $\widehat{A}$ and $\widehat{B}$ are said to be conformable for addition, if they have the same order and after addition, the sum is also an $IVIFSM$ of the same order. Now if both $\widehat{A} = (\widehat{a}_{ij})$ and $\widehat{B} = (\widehat{b}_{ij})$ be the same order $m \times n$, then the addition of $\widehat{A}$ and $\widehat{B}$ is denoted by $\widehat{A} \oplus \widehat{B}$ and is defined by

$$[\widehat{c}_{ij}] = [\widehat{a}_{ij}] \oplus [\widehat{b}_{ij}],\ where\ \widehat{c}_{ij} = \left\{ \begin{array}{l} [\max \{\mu^l_{a_{ij}}, \mu^l_{b_{ij}}\},\ \max \{\mu^r_{a_{ij}}, \mu^r_{b_{ij}}\}], \\ [\min \{\nu^l_{a_{ij}}, \nu^l_{b_{ij}}\},\ \min \{\nu^r_{a_{ij}}, \nu^r_{b_{ij}}\}] \end{array} \right\} \forall i, j.$$

Example 7. Example for the addition operation is shown in Step 6 of of both the case studies, explained in Section 6.

3.4 Complement of IVIFSM

Complement of an IVIFSM $(\widehat{a}_{ij})_{m\times n}$ is denoted by $(\widehat{a}_{ij})^c_{m\times n}$,where $(\widehat{a}_{ij})_{m\times n}$is the matrix representation of the interval-valued intuitionistic fuzzy soft set $(\widehat{F}_{\{A\}}, E)$. $(\widehat{a}_{ij})^c_{m\times n}$ is the matrix representation of the interval-valued intuitionistic fuzzy soft set $(\widehat{F}^c_{\neg\{A\}}, E)$and is defined as

$$(\widehat{a}_{ij})^c_{m\times n} = \left[\left(\mu^l_{a_{ij}}\right)^c, \left(\mu^r_{a_{ij}}\right)^c\right], \left[\left(\nu^l_{a_{ij}}\right)^c, \left(\nu^r_{a_{ij}}\right)^c\right] = [1-\mu^r_{a_{ij}}, 1-\mu^l_{a_{ij}}], [1-\nu^r_{a_{ij}}, 1-\nu^l_{a_{ij}}].$$

4 Cardinal Set of IVIFSS and Cardinal Score

The cardinal set of IVIFSS $(\widehat{F}_{\{A\}}, E)$ is denoted by $(c\widehat{F}_{\{A\}}, E)$ and defined by

$$(c\widehat{F}_{\{A\}}, E) = \{[\mu^l_{c\widehat{F}_{\{A\}}}(x), \mu^r_{c\widehat{F}_{\{A\}}}(x)], [\nu^l_{c\widehat{F}_{\{A\}}}(x), \nu^r_{c\widehat{F}_{\{A\}}}(x)]\}x : x \in E\}.$$

It is an interval-valued intuitionistic fuzzy set over E. The membership function $[\mu^l_{c\widehat{F}_{\{A\}}}(x), \mu^r_{c\widehat{F}_{\{A\}}}(x)]$ and non-membership function $[\nu^l_{c\widehat{F}_{\{A\}}}(x), \nu^r_{c\widehat{F}_{\{A\}}}(x)]$ of $(c\widehat{F}_{\{A\}}, E)$ are respectively defined by

$$\left[\mu^l_{c\widehat{F}_{\{A\}}}(x) = \frac{\sum_{d\in U} \mu^l_{\widehat{F}_{\{A\}}}(d)}{|U|}, \mu^r_{c\widehat{F}_{\{A\}}}(x) = \frac{\sum_{d\in U} \mu^r_{\widehat{F}_{\{A\}}}(d)}{|U|}\right] \forall x \in E$$

and

$$\left[\nu^l_{c\widehat{F}_{\{A\}}}(x) = \frac{\sum_{d\in U} \nu^l_{\widehat{F}_{\{A\}}}(d)}{|U|},\ \nu^r_{c\widehat{F}_{\{A\}}}(x) = \frac{\sum_{d\in U} \nu^r_{\widehat{F}_{\{A\}}}(d)}{|U|}\right] \forall x \in E.$$

Table 5. Cardinal set of IVIFSS

E	x_1	x_2		x_n
$c\widehat{F}_{\{A\}}$	$\begin{pmatrix} \mu^l_{c\widehat{F}_{\{A\}}}(x_1), \mu^r_{c\widehat{F}_{\{A\}}}(x_1) \\ \nu^l_{c\widehat{F}_{\{A\}}}(x_1), \nu^r_{c\widehat{F}_{\{A\}}}(x_1), \end{pmatrix}$	$\begin{pmatrix} \mu^l_{c\widehat{F}_{\{A\}}}(x_2), \mu^r_{c\widehat{F}_{\{A\}}}(x_2) \\ \nu^l_{c\widehat{F}_{\{A\}}}(x_2), \nu^r_{c\widehat{F}_{\{A\}}}(x_2), \end{pmatrix}$	...	$\begin{pmatrix} \mu^l_{c\widehat{F}_{\{A\}}}(x_n), \mu^r_{c\widehat{F}_{\{A\}}}(x_n) \\ \nu^l_{c\widehat{F}_{\{A\}}}(x_n), \nu^r_{c\widehat{F}_{\{A\}}}(x_n), \end{pmatrix}$

Here $|U|$is the cardinality of the universe U.

The sets of all cardinal sets of IVIFSS $(\widehat{F}_{\{A\}}, E)$ over U is denoted by $cIVIFS(E)$. Let $\widehat{F}_{\{A\}} \in IVIFS(U)$, $\widehat{cF}_{\{A\}} \in cIVIFS(U), E = \{x_1, x_2, ..., x_n\}$,and $A \subseteq E$, then $\widehat{cF}_{\{A\}}$is presented in Table 5.

$If a_{1j} = \left(\{\mu^l_{c\widehat{F}_{\{A\}}}(x_j), \mu^r_{c\widehat{F}_{\{A\}}}(x_j)\}, \{\nu^l_{c\widehat{F}_{\{A\}}}(x_j), \nu^r_{c\widehat{F}_{\{A\}}}(x_j)\}\right)$ for $j = 1, 2, ..., n$, then the cardinal set is uniquely characterized by a matrix $[a_{ij}]_{1\times n} = [a_{11}\ a_{12}\ ...\ a_{1n}]$,which is called the cardinal matrix of the cardinal set $c\widehat{F}_{\{A\}}$over E.

Cardinal score [36] of a cardinal set $c\widehat{F}_{\{A\}}$ is denoted by $S(c\widehat{F}_{\{A\}})$ and defined as

$$S(c\widehat{F}_{\{A\}}) = \frac{1}{2}\left(\sum_{x\in E}\mu^l_{c\widehat{F}_{\{A\}}}(x) + \sum_{x\in E}\mu^r_{c\widehat{F}_{\{A\}}}(x) - \sum_{x\in E}\nu^l_{c\widehat{F}_{\{A\}}}(x) - \sum_{x\in E}\nu^r_{c\widehat{F}_{\{A\}}}(x)\right).$$

Example 8. Suppose the IVIFSM $[P_1(i,j)]_{5\times 5}$for IVIFSS $(\widehat{F}_{\{P_1\}}, E)$ is as given below.

$$\{\mathrm{P}_1(i,j)\} = \begin{pmatrix} \begin{pmatrix}(0.3.0.7),\\(0.1,0.2)\end{pmatrix} & \begin{pmatrix}(0.3,0.4),\\(0.4,0.5)\end{pmatrix} & \begin{pmatrix}(0.5,0.6),\\(0.2,0.4)\end{pmatrix} & \begin{pmatrix}(0.5,0.6),\\(0.2,0.4)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} \\ \begin{pmatrix}(0.4,0.5),\\(0.3,0.5)\end{pmatrix} & \begin{pmatrix}(0.3,0.7),\\(0.2,0.3)\end{pmatrix} & \begin{pmatrix}(0.3,0.6),\\(0.1,0.3)\end{pmatrix} & \begin{pmatrix}(0.3,0.7),\\(0.1,0.2)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} \\ \begin{pmatrix}(0.3,0.6),\\(0.3,0.4)\end{pmatrix} & \begin{pmatrix}(0.4,0.6),\\(0.2,0.3)\end{pmatrix} & \begin{pmatrix}(0.4,0.6),\\(0.2,0.4)\end{pmatrix} & \begin{pmatrix}(0.4,0.6),\\(0.2,0.4)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} \\ \begin{pmatrix}(0.4,0.8),\\(0.1,0.2)\end{pmatrix} & \begin{pmatrix}(0.2,0.5),\\(0.3,0.4)\end{pmatrix} & \begin{pmatrix}(0.2,0.5),\\(0.3,0.4)\end{pmatrix} & \begin{pmatrix}(0.2,0.5),\\(0.3,0.4)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} \\ \begin{pmatrix}(0.3,0.5),\\(0.2,0.4)\end{pmatrix} & \begin{pmatrix}(0.2,0.3),\\(0.4,0.7)\end{pmatrix} & \begin{pmatrix}(0.2,0.3),\\(0.5,0.6)\end{pmatrix} & \begin{pmatrix}(0.3,0.4),\\(0.4,0.5)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} \end{pmatrix},$$

then

$$\mu^l_{c\widehat{F}_{\{P_1\}}}(x_1) = (0.3 + 0.4 + 0.3 + 0.4 + 0.3)/5 = 0.34$$
$$\mu^r_{c\widehat{F}_{\{P_1\}}}(x_1) = (0.7 + 0.5 + 0.6 + 0.8 + 0.5)/5 = 0.62$$
$$\nu^l_{c\widehat{F}_{\{P_1\}}}(x_1) = (0.1 + 0.3 + 0.3 + 0.1 + 0.2)/5 = 0.2$$
$$\nu^r_{c\widehat{F}_{\{P_1\}}}(x_1) = (0.2 + 0.5 + 0.4 + 0.2 + 0.4)/5 = 0.34$$

Similarly,

$$\mu^l_{c\widehat{F}_{\{A\}}}(x_2)=0.28,\mu^r_{c\widehat{F}_{\{A\}}}(x_2)=0.5,\nu^l_{c\widehat{F}_{\{A\}}}(x_2)=0.3,\nu^r_{c\widehat{F}_{\{A\}}}(x_2)=0.44$$
$$\mu^l_{c\widehat{F}_{\{A\}}}(x_3)=0.32,\mu^r_{c\widehat{F}_{\{A\}}}(x_3)=0.52,\nu^l_{c\widehat{F}_{\{A\}}}(x_3)=0.26,\nu^r_{c\widehat{F}_{\{A\}}}(x_3)=0.42$$
$$\mu^l_{c\widehat{F}_{\{A\}}}(x_4)=0.3,\mu^r_{c\widehat{F}_{\{A\}}}(x_4)=0.56,\nu^l_{c\widehat{F}_{\{A\}}}(x_4)=0.24,\nu^r_{c\widehat{F}_{\{A\}}}(x_4)=0.38$$

Cardinal matrix of the IVIFSM $[p_1(i,j)]_{5\times5}$is $[p_1(i,j)]_{1\times5}=[a_{11}\ a_{12}\ ...\ a_{15}]$, where

$$[p_1(i,j)]_{1\times5}=\left[\begin{pmatrix}(0.34,0.62)\\(0.2,0.34)\end{pmatrix}\begin{pmatrix}(0.28,0.5)\\(0.3,0.44)\end{pmatrix}\begin{pmatrix}(0.32,0.52)\\(0.26,0.42)\end{pmatrix}\begin{pmatrix}(0.3,0.56)\\(0.24,0.38)\end{pmatrix}\begin{pmatrix}(0,0)\\(0,0)\end{pmatrix}\right]$$

Cardinal score $S(c\widehat{F}_{\{A\}})$ is

$$S(c\widehat{F}_{\{A\}})=\frac{1}{2}\left(\sum_{x\in E}\mu^l_{c\widehat{F}_{\{A\}}}(x)+\sum_{x\in E}\mu^r_{c\widehat{F}_{\{A\}}}(x)-\sum_{x\in E}\nu^l_{c\widehat{F}_{\{A\}}}(x)-\sum_{x\in E}\nu^r_{c\widehat{F}_{\{A\}}}(x)\right)$$
$$=\frac{1}{2}(3.44-2.58)=0.43$$

5 Algorithmic Approach

The steps for the proposed approach are given below.

Step 1: Opinions of a set of experts / decision makers $P=\{p_1,p_2,...,p_k\}$ for a given set of alternatives $D=\{d_1,d_2,...,d_m\}$ and a set of attributes $S=\{s_1,s_2,...,s_n\}$ are represented using interval-valued intuitionistic fuzzy soft matrices.

Step 2: Cardinal matrix of each IVIFSM is explored and then cardinal score is computed.

Step 3: Cardinal score is multiplied with the corresponding IVIFSM to produce the normalized IVIFSM. Let $\left[\widehat{a}_{(ij)}\right]_{m\times n}$ be an IVIFSM and cardinal score is h, then the normalized IVIFSM, denoted by N_{IVIFSM}, is defined by$N_{IVIFSM}[\widehat{a}_{(ij)}]=\left[h*\widehat{a}_{(ij)}\right]_{m\times n}$,for $i=1,2,...,m$ and $j=1,2,...,n$.

Step 4: Choice matrix $\beta(i,j)^P$and combined choice matrix $\beta(i,j)^{P_C}$ of each of the decision makers$P=\{p_1,p_2,...,p_k\}$ are computed in the context of interval-valued intuitionistic fuzzy set based on their choice parameters / attributes.

Step 5: Product $IVIFSM\,(P_{IVIFSM})$ for each decision maker is calculated by multiplying the normalized $IVIFSM$with its combined choice matrix.

Step 6: Summation of these product $IVIFSMs$ is the resultant $IVIFSM$ (R_{IVIFSM}).

Step 7: Weight $W(d_i)$ of each alternative $d_i\{i=1,2,...,m\}$ is estimated by adding the membership and non-membership values of the entries of the respective row (i^{th} row).

Step 8:$\forall d_i\in D$, compute the score $S(d_i)$ of d_i, such that,

$$S(d_i)=\{(\mu^l_i-\nu^l_i)+(\mu^r_i-\nu^r_i)\}/2,\ d_i\in D\ \forall i.$$

Step 9: If $S(d_i)>S(d_j)\,\forall d_j\in D$, then alternative$d_i$is selected. If $\exists j$, such that,$S(d_i)=S(d_j)$, where $i\neq j$ for highest score value, then decision is made according to their accuracy values as described in step 10.

Step 10: Accuracy value $H(d_i)$, $i=1,2,...,m$, is defined as

$$H(d_i)=\{(\mu^l_i+\nu^l_i)+(\mu^r_i+\nu^r_i)\}/2,\ d_i\in D\,\forall i.$$

If $H(d_i)>H(d_j)\,\forall j$ for which $S(d_i)=S(d_j)$, defined in Step 9, alternative d_i is selected. If $H(d_i)=H(d_j)$for any$j$, then $d_i$and d_j both are selected.

6 Case Study

Let $D = \{d_1, d_2, d_3, d_4, d_5\}$ be the set of five stages of heart disease (Stage 'I', Stage 'II', Stage 'III', Stage 'IV', and Stage 'V'). Patients belonging to Stage 'I' are assumed not to be affected by heart disease. Patients belonging to Stage 'II' are in initial stage, patients belonging to Stage 'III' are in more unsafe stage than in stage 'II' and so on. Patients belonging to Stage 'V' are in the last stage of heart disease which is unrecoverable. Let E be the set of five symptoms (Chest pain, Palpitations, Dizziness, Fainting, Fatigue) given by

$$E = \{s_1, s_2, s_3, s_4, s_5\}.$$

Suppose that a group of three experts

$$P = \{p_1, p_2, p_3\}$$

are monitoring the symptoms of a patient as per their knowledgebase to reach a consensus about which stage is more likely to appear for the patient, where expert p_1 is aware of symptoms (s_1, s_2, s_3, s_4), p_2 is aware of symptoms ($s_1 s_2, s_3, s_5$), and p_3 is aware of symptoms (s_1, s_2, s_4, s_5). According to the symptoms or parameters observed by the three experts, we assume to have the information in IV-IFSSs

$$(\widehat{F}_{\{p_1\}}, E), (\widehat{F}_{\{p_2\}}, E),$$

and

$$(\widehat{F}_{\{p_3\}}, E)$$

for experts p_1, p_2, and p_3 respectively.

Let the IVIFSMs of the IVIFSSs

$$(\widehat{F}_{\{p_1\}}, E), (\widehat{F}_{\{p_2\}}, E), (\widehat{F}_{\{p_3\}}, E)$$

are respectively,

$$\{p_1(i,j)\} = \begin{pmatrix} \begin{pmatrix}(0.3.0.7),\\(0.1,0.2)\end{pmatrix} & \begin{pmatrix}(0.3,0.4),\\(0.4,0.5)\end{pmatrix} & \begin{pmatrix}(0.5,0.6),\\(0.2,0.4)\end{pmatrix} & \begin{pmatrix}(0.5,0.6),\\(0.2,0.4)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} \\ \begin{pmatrix}(0.4,0.5),\\(0.3,0.5)\end{pmatrix} & \begin{pmatrix}(0.3,0.7),\\(0.2,0.3)\end{pmatrix} & \begin{pmatrix}(0.3,0.6),\\(0.1,0.3)\end{pmatrix} & \begin{pmatrix}(0.3,0.7),\\(0.1,0.2)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} \\ \begin{pmatrix}(0.3,0.6),\\(0.3,0.4)\end{pmatrix} & \begin{pmatrix}(0.4,0.6),\\(0.2,0.3)\end{pmatrix} & \begin{pmatrix}(0.4,0.6),\\(0.2,0.4)\end{pmatrix} & \begin{pmatrix}(0.4,0.6),\\(0.2,0.4)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} \\ \begin{pmatrix}(0.4,0.8),\\(0.1,0.2)\end{pmatrix} & \begin{pmatrix}(0.2,0.5),\\(0.3,0.4)\end{pmatrix} & \begin{pmatrix}(0.2,0.5),\\(0.3,0.4)\end{pmatrix} & \begin{pmatrix}(0.2,0.5),\\(0.3,0.4)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} \\ \begin{pmatrix}(0.3,0.5),\\(0.2,0.4)\end{pmatrix} & \begin{pmatrix}(0.2,0.3),\\(0.4,0.7)\end{pmatrix} & \begin{pmatrix}(0.2,0.3),\\(0.5,0.6)\end{pmatrix} & \begin{pmatrix}(0.3,0.4),\\(0.4,0.5)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} \end{pmatrix},$$

$$\{p_2(i,k)\} = \begin{pmatrix} \begin{pmatrix}(0.3,0.5),\\(0.3,0.4)\end{pmatrix} & \begin{pmatrix}(0.5,0.6),\\(0.3,0.4)\end{pmatrix} & \begin{pmatrix}(0.2,0.5),\\(0.3,0.4)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.5,0.7),\\(0.2,0.3)\end{pmatrix} \\ \begin{pmatrix}(0.4,0.5),\\(0.3,0.5)\end{pmatrix} & \begin{pmatrix}(0.4,0.7),\\(0.2,0.3)\end{pmatrix} & \begin{pmatrix}(0.3,0.5),\\(0.2,0.3)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.4,0.5),\\(0.3,0.4)\end{pmatrix} \\ \begin{pmatrix}(0.3,0.6),\\(0.3,0.4)\end{pmatrix} & \begin{pmatrix}(0.7,0.8),\\(0.1,0.2)\end{pmatrix} & \begin{pmatrix}(0.2,0.4),\\(0.3,0.5)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.4,0.6),\\(0.2,0.4)\end{pmatrix} \\ \begin{pmatrix}(0.2,0.6),\\(0.1,0.2)\end{pmatrix} & \begin{pmatrix}(0.5,0.7),\\(0.1,0.2)\end{pmatrix} & \begin{pmatrix}(0.3,0.6),\\(0.2,0.3)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.5,0.6),\\(0.3,0.4)\end{pmatrix} \\ \begin{pmatrix}(0.4,0.5),\\(0.2,0.4)\end{pmatrix} & \begin{pmatrix}(0.2,0.3),\\(0.4,0.5)\end{pmatrix} & \begin{pmatrix}(0.4,0.7),\\(0.1,0.3)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.4,0.5),\\(0.3,0.4)\end{pmatrix} \end{pmatrix},$$

$$\{p_3(i,l)\} = \begin{pmatrix} \begin{pmatrix}(0.3,0.5),\\(0.3,0.4)\end{pmatrix} & \begin{pmatrix}(0.2,0.7),\\(0.1,0.3)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.3,0.6),\\(0.3,0.4)\end{pmatrix} & \begin{pmatrix}(0.2,0.4),\\(0.5,0.6)\end{pmatrix} \\ \begin{pmatrix}(0.3,0.7),\\(0.2,0.3)\end{pmatrix} & \begin{pmatrix}(0.3,0.5),\\(0.3,0.5)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.4,0.8),\\(0.1,0.2)\end{pmatrix} & \begin{pmatrix}(0.7,0.8),\\(0.1,0.2)\end{pmatrix} \\ \begin{pmatrix}(0.5,0.7),\\(0.1,0.3)\end{pmatrix} & \begin{pmatrix}(0.5,0.6),\\(0.3,0.4)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.5,0.6),\\(0.3,0.4)\end{pmatrix} & \begin{pmatrix}(0.4,0.6),\\(0.3,0.4)\end{pmatrix} \\ \begin{pmatrix}(0.4,0.6),\\(0.2,0.3)\end{pmatrix} & \begin{pmatrix}(0.4,0.5),\\(0.3,0.4)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.3,0.7),\\(0.2,0.3)\end{pmatrix} & \begin{pmatrix}(0.3,0.4),\\(0.4,0.5)\end{pmatrix} \\ \begin{pmatrix}(0.1,0.2),\\(0.5,0.7)\end{pmatrix} & \begin{pmatrix}(0.4,0.6),\\(0.3,0.4)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.2,0.3),\\(0.5,0.6)\end{pmatrix} & \begin{pmatrix}(0.1,0.4),\\(0.4,0.5)\end{pmatrix} \end{pmatrix}.$$

Here i (1,2,...,5) is used to represent an alternative, i.e., a stage of heart disease and the indices j, k, l (1,2,...,5) are used for attributes, i.e., symptoms. These are the input information, given by the experts.

In this case study, we consider two cases. In the first case, we use non-normalized IVIFSM and normalized IVIFSM in the second case.

Case 1: It uses non-normalized IVIFSMs as input.

[Step 2 & Step 3]: These steps are not applicable in Case1.

[Step 4] The combined choice matrices for p_1, p_2, and p_3 are respectively

$$s_{\{p_1\}} \overset{s_{\{p_2 \wedge p_3\}}}{\begin{pmatrix} \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} \\ \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} \\ \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} \\ \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} \\ \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} \end{pmatrix}}$$

$$s_{\{p_2\}} \overset{s_{\{p_1 \wedge p_3\}}}{\begin{pmatrix} \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} \\ \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} \\ \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} \\ \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} \\ \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} \end{pmatrix}}$$

$$s_{\{p_3\}} \overset{s_{\{p_1 \wedge p_2\}}}{\begin{pmatrix} \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} \\ \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} \\ \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} \\ \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} \\ \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} \end{pmatrix}}$$

[Step 5] The product (as per the rule of multiplication of IVIFSMs) of IVIFSMs (non-normalized) and combined choice matrices are given below.

$$\{p_1(i,j)\}\begin{pmatrix}
\begin{pmatrix}(0.3.0.7),\\(0.1,0.2)\end{pmatrix} & \begin{pmatrix}(0.3,0.4),\\(0.4,0.5)\end{pmatrix} & \begin{pmatrix}(0.5,0.6),\\(0.2,0.4)\end{pmatrix} & \begin{pmatrix}(0.5,0.6),\\(0.2,0.4)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix}\\
\begin{pmatrix}(0.4,0.5),\\(0.3,0.5)\end{pmatrix} & \begin{pmatrix}(0.3,0.7),\\(0.2,0.3)\end{pmatrix} & \begin{pmatrix}(0.3,0.6),\\(0.1,0.3)\end{pmatrix} & \begin{pmatrix}(0.3,0.7),\\(0.1,0.2)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix}\\
\begin{pmatrix}(0.3,0.6),\\(0.3,0.4)\end{pmatrix} & \begin{pmatrix}(0.4,0.6),\\(0.2,0.3)\end{pmatrix} & \begin{pmatrix}(0.4,0.6),\\(0.2,0.4)\end{pmatrix} & \begin{pmatrix}(0.4,0.6),\\(0.2,0.4)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix}\\
\begin{pmatrix}(0.4,0.8),\\(0.1,0.2)\end{pmatrix} & \begin{pmatrix}(0.2,0.5),\\(0.3,0.4)\end{pmatrix} & \begin{pmatrix}(0.2,0.5),\\(0.3,0.4)\end{pmatrix} & \begin{pmatrix}(0.2,0.5),\\(0.3,0.4)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix}\\
\begin{pmatrix}(0.3,0.5),\\(0.2,0.4)\end{pmatrix} & \begin{pmatrix}(0.2,0.3),\\(0.4,0.7)\end{pmatrix} & \begin{pmatrix}(0.2,0.3),\\(0.5,0.6)\end{pmatrix} & \begin{pmatrix}(0.3,0.4),\\(0.4,0.5)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix}
\end{pmatrix}
\otimes s_{\{p_1\}}
\overset{s_{\{p_2\wedge p_3\}}}{\begin{pmatrix}
\begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix}\\
\begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix}\\
\begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix}\\
\begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix}\\
\begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix}
\end{pmatrix}}$$

$$=\begin{pmatrix}
\begin{pmatrix}(0.5.0.7),\\(0.1,0.2)\end{pmatrix} & \begin{pmatrix}(0.5.0.7),\\(0.1,0.2)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.5.0.7),\\(0.1,0.2)\end{pmatrix}\\
\begin{pmatrix}(0.4,0.7),\\(0.1,0.2)\end{pmatrix} & \begin{pmatrix}(0.4,0.7),\\(0.1,0.2)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.4,0.7),\\(0.1,0.2)\end{pmatrix}\\
\begin{pmatrix}(0.4,0.6),\\(0.2,0.3)\end{pmatrix} & \begin{pmatrix}(0.4,0.6),\\(0.2,0.3)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.4,0.6),\\(0.2,0.3)\end{pmatrix}\\
\begin{pmatrix}(0.4,0.8),\\(0.1,0.2)\end{pmatrix} & \begin{pmatrix}(0.4,0.8),\\(0.1,0.2)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.4,0.8),\\(0.1,0.2)\end{pmatrix}\\
\begin{pmatrix}(0.3,0.5),\\(0.2,0.4)\end{pmatrix} & \begin{pmatrix}(0.3,0.5),\\(0.2,0.4)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.3,0.5),\\(0.2,0.4)\end{pmatrix}
\end{pmatrix}$$

$$\{p_2(i,k)\}\begin{pmatrix}
\begin{pmatrix}(0.3,0.5),\\(0.3,0.4)\end{pmatrix} & \begin{pmatrix}(0.5,0.6),\\(0.3,0.4)\end{pmatrix} & \begin{pmatrix}(0.2,0.5),\\(0.3,0.4)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.5,0.7),\\(0.2,0.3)\end{pmatrix}\\
\begin{pmatrix}(0.4,0.5),\\(0.3,0.5)\end{pmatrix} & \begin{pmatrix}(0.4,0.7),\\(0.2,0.3)\end{pmatrix} & \begin{pmatrix}(0.3,0.5),\\(0.2,0.3)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.4,0.5),\\(0.3,0.4)\end{pmatrix}\\
\begin{pmatrix}(0.3,0.6),\\(0.3,0.4)\end{pmatrix} & \begin{pmatrix}(0.7,0.8),\\(0.1,0.2)\end{pmatrix} & \begin{pmatrix}(0.2,0.4),\\(0.3,0.5)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.4,0.6),\\(0.2,0.4)\end{pmatrix}\\
\begin{pmatrix}(0.2,0.6),\\(0.1,0.2)\end{pmatrix} & \begin{pmatrix}(0.5,0.7),\\(0.1,0.2)\end{pmatrix} & \begin{pmatrix}(0.3,0.6),\\(0.2,0.3)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.5,0.6),\\(0.3,0.4)\end{pmatrix}\\
\begin{pmatrix}(0.4,0.5),\\(0.2,0.4)\end{pmatrix} & \begin{pmatrix}(0.2,0.3),\\(0.4,0.5)\end{pmatrix} & \begin{pmatrix}(0.4,0.7),\\(0.1,0.3)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.4,0.5),\\(0.3,0.4)\end{pmatrix}
\end{pmatrix}
\otimes s_{\{p_2\}}
\overset{s_{\{p_1\wedge p_3\}}}{\begin{pmatrix}
\begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix}\\
\begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix}\\
\begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix}\\
\begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix}\\
\begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix}
\end{pmatrix}}$$

$$=\begin{pmatrix}
\begin{pmatrix}(0.5,0.7),\\(0.2,0.3)\end{pmatrix} & \begin{pmatrix}(0.5,0.7),\\(0.2,0.3)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.5,0.7),\\(0.2,0.3)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix}\\
\begin{pmatrix}(0.4,0.7),\\(0.2,0.3)\end{pmatrix} & \begin{pmatrix}(0.4,0.7),\\(0.2,0.3)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.4,0.7),\\(0.2,0.3)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix}\\
\begin{pmatrix}(0.7,0.8),\\(0.1,0.2)\end{pmatrix} & \begin{pmatrix}(0.7,0.8),\\(0.1,0.2)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.7,0.8),\\(0.1,0.2)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix}\\
\begin{pmatrix}(0.5,0.7),\\(0.1,0.2)\end{pmatrix} & \begin{pmatrix}(0.5,0.7),\\(0.1,0.2)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.5,0.7),\\(0.1,0.2)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix}\\
\begin{pmatrix}(0.4,0.7),\\(0.1,0.3)\end{pmatrix} & \begin{pmatrix}(0.4,0.7),\\(0.1,0.3)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.4,0.7),\\(0.1,0.3)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix}
\end{pmatrix}$$

$$\{p_3(i,l)\}\begin{pmatrix} \begin{pmatrix}(0.3,0.5),\\(0.3,0.4)\end{pmatrix} & \begin{pmatrix}(0.2,0.7),\\(0.1,0.3)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.3,0.6),\\(0.3,0.4)\end{pmatrix} & \begin{pmatrix}(0.2,0.4),\\(0.5,0.6)\end{pmatrix} \\ \begin{pmatrix}(0.3,0.7),\\(0.2,0.3)\end{pmatrix} & \begin{pmatrix}(0.3,0.5),\\(0.3,0.5)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.4,0.8),\\(0.1,0.2)\end{pmatrix} & \begin{pmatrix}(0.7,0.8),\\(0.1,0.2)\end{pmatrix} \\ \begin{pmatrix}(0.5,0.7),\\(0.1,0.3)\end{pmatrix} & \begin{pmatrix}(0.5,0.6),\\(0.3,0.4)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.5,0.6),\\(0.3,0.4)\end{pmatrix} & \begin{pmatrix}(0.4,0.6),\\(0.3,0.4)\end{pmatrix} \\ \begin{pmatrix}(0.4,0.6),\\(0.2,0.3)\end{pmatrix} & \begin{pmatrix}(0.4,0.5),\\(0.3,0.4)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.3,0.7),\\(0.2,0.3)\end{pmatrix} & \begin{pmatrix}(0.3,0.4),\\(0.4,0.5)\end{pmatrix} \\ \begin{pmatrix}(0.1,0.2),\\(0.5,0.7)\end{pmatrix} & \begin{pmatrix}(0.4,0.6),\\(0.3,0.4)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.2,0.3),\\(0.5,0.6)\end{pmatrix} & \begin{pmatrix}(0.1,0.4),\\(0.4,0.5)\end{pmatrix} \end{pmatrix} \otimes s_{\{p_3\}} \overset{s_{\{p_1 \wedge p_2\}}}{\begin{pmatrix} \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} \\ \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} \\ \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} \\ \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} \\ \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} \end{pmatrix}}$$

$$= \begin{pmatrix} \begin{pmatrix}(0.3,0.7),\\(0.1,0.3)\end{pmatrix} & \begin{pmatrix}(0.3,0.7),\\(0.1,0.3)\end{pmatrix} & \begin{pmatrix}(0.3,0.7),\\(0.1,0.3)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} \\ \begin{pmatrix}(0.7,0.8),\\(0.1,0.2)\end{pmatrix} & \begin{pmatrix}(0.7,0.8),\\(0.1,0.2)\end{pmatrix} & \begin{pmatrix}(0.7,0.8),\\(0.1,0.2)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} \\ \begin{pmatrix}(0.5,0.7),\\(0.1,0.3)\end{pmatrix} & \begin{pmatrix}(0.5,0.7),\\(0.1,0.3)\end{pmatrix} & \begin{pmatrix}(0.5,0.7),\\(0.1,0.3)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} \\ \begin{pmatrix}(0.4,0.7),\\(0.2,0.3)\end{pmatrix} & \begin{pmatrix}(0.4,0.7),\\(0.2,0.3)\end{pmatrix} & \begin{pmatrix}(0.4,0.7),\\(0.2,0.3)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} \\ \begin{pmatrix}(0.4,0.6),\\(0.3,0.4)\end{pmatrix} & \begin{pmatrix}(0.4,0.6),\\(0.3,0.4)\end{pmatrix} & \begin{pmatrix}(0.4,0.6),\\(0.3,0.4)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} \end{pmatrix}$$

[Step 6] The sum of the product IVIFSMs is

$$\begin{pmatrix} \begin{pmatrix}(0.5.0.7),\\(0.1,0.2)\end{pmatrix} & \begin{pmatrix}(0.5.0.7),\\(0.1,0.2)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.5.0.7),\\(0.1,0.2)\end{pmatrix} \\ \begin{pmatrix}(0.4,0.7),\\(0.1,0.2)\end{pmatrix} & \begin{pmatrix}(0.4,0.7),\\(0.1,0.2)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.4,0.7),\\(0.1,0.2)\end{pmatrix} \\ \begin{pmatrix}(0.4,0.6),\\(0.2,0.3)\end{pmatrix} & \begin{pmatrix}(0.4,0.6),\\(0.2,0.3)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.4,0.6),\\(0.2,0.3)\end{pmatrix} \\ \begin{pmatrix}(0.4,0.8),\\(0.1,0.2)\end{pmatrix} & \begin{pmatrix}(0.4,0.8),\\(0.1,0.2)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.4,0.8),\\(0.1,0.2)\end{pmatrix} \\ \begin{pmatrix}(0.3,0.5),\\(0.2,0.4)\end{pmatrix} & \begin{pmatrix}(0.3,0.5),\\(0.2,0.4)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.3,0.5),\\(0.2,0.4)\end{pmatrix} \end{pmatrix} \oplus \begin{pmatrix} \begin{pmatrix}(0.5,0.7),\\(0.2,0.3)\end{pmatrix} & \begin{pmatrix}(0.5,0.7),\\(0.2,0.3)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.5,0.7),\\(0.2,0.3)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} \\ \begin{pmatrix}(0.4,0.7),\\(0.2,0.3)\end{pmatrix} & \begin{pmatrix}(0.4,0.7),\\(0.2,0.3)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.4,0.7),\\(0.2,0.3)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} \\ \begin{pmatrix}(0.7,0.8),\\(0.1,0.2)\end{pmatrix} & \begin{pmatrix}(0.7,0.8),\\(0.1,0.2)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.7,0.8),\\(0.1,0.2)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} \\ \begin{pmatrix}(0.5,0.7),\\(0.1,0.2)\end{pmatrix} & \begin{pmatrix}(0.5,0.7),\\(0.1,0.2)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.5,0.7),\\(0.1,0.2)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} \\ \begin{pmatrix}(0.4,0.7),\\(0.1,0.3)\end{pmatrix} & \begin{pmatrix}(0.4,0.7),\\(0.1,0.3)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.4,0.7),\\(0.1,0.3)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} \end{pmatrix} \oplus$$

$$\begin{pmatrix} \begin{pmatrix}(0.3,0.7),\\(0.1,0.3)\end{pmatrix} & \begin{pmatrix}(0.3,0.7),\\(0.1,0.3)\end{pmatrix} & \begin{pmatrix}(0.3,0.7),\\(0.1,0.3)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} \\ \begin{pmatrix}(0.7,0.8),\\(0.1,0.2)\end{pmatrix} & \begin{pmatrix}(0.7,0.8),\\(0.1,0.2)\end{pmatrix} & \begin{pmatrix}(0.7,0.8),\\(0.1,0.2)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} \\ \begin{pmatrix}(0.5,0.7),\\(0.1,0.3)\end{pmatrix} & \begin{pmatrix}(0.5,0.7),\\(0.1,0.3)\end{pmatrix} & \begin{pmatrix}(0.5,0.7),\\(0.1,0.3)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} \\ \begin{pmatrix}(0.4,0.7),\\(0.2,0.3)\end{pmatrix} & \begin{pmatrix}(0.4,0.7),\\(0.2,0.3)\end{pmatrix} & \begin{pmatrix}(0.4,0.7),\\(0.2,0.3)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} \\ \begin{pmatrix}(0.4,0.6),\\(0.3,0.4)\end{pmatrix} & \begin{pmatrix}(0.4,0.6),\\(0.3,0.4)\end{pmatrix} & \begin{pmatrix}(0.4,0.6),\\(0.3,0.4)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix}(0.5,0.7),\\(0.1,0.2)\end{pmatrix} & \begin{pmatrix}(0.5,0.7),\\(0.1,0.2)\end{pmatrix} & \begin{pmatrix}(0.3,0.7),\\(0.1,0.3)\end{pmatrix} & \begin{pmatrix}(0.5,0.7),\\(0.2,0.3)\end{pmatrix} & \begin{pmatrix}(0.5,0.7),\\(0.1,0.2)\end{pmatrix} \\ \begin{pmatrix}(0.7,0.8),\\(0.1,0.2)\end{pmatrix} & \begin{pmatrix}(0.7,0.8),\\(0.1,0.2)\end{pmatrix} & \begin{pmatrix}(0.7,0.8),\\(0.1,0.2)\end{pmatrix} & \begin{pmatrix}(0.4,0.7),\\(0.2,0.3)\end{pmatrix} & \begin{pmatrix}(0.4,0.7),\\(0.1,0.2)\end{pmatrix} \\ \begin{pmatrix}(0.7,0.8),\\(0.1,0.2)\end{pmatrix} & \begin{pmatrix}(0.7,0.8),\\(0.1,0.2)\end{pmatrix} & \begin{pmatrix}(0.5,0.7),\\(0.1,0.3)\end{pmatrix} & \begin{pmatrix}(0.7,0.8),\\(0.1,0.2)\end{pmatrix} & \begin{pmatrix}(0.4,0.6),\\(0.2,0.3)\end{pmatrix} \\ \begin{pmatrix}(0.5,0.8),\\(0.1,0.2)\end{pmatrix} & \begin{pmatrix}(0.5,0.8),\\(0.1,0.2)\end{pmatrix} & \begin{pmatrix}(0.4,0.7),\\(0.2,0.3)\end{pmatrix} & \begin{pmatrix}(0.5,0.7),\\(0.1,0.2)\end{pmatrix} & \begin{pmatrix}(0.4,0.8),\\(0.1,0.2)\end{pmatrix} \\ \begin{pmatrix}(0.4,0.7),\\(0.1,0.3)\end{pmatrix} & \begin{pmatrix}(0.4,0.7),\\(0.1,0.3)\end{pmatrix} & \begin{pmatrix}(0.4,0.6),\\(0.3,0.4)\end{pmatrix} & \begin{pmatrix}(0.4,0.7),\\(0.1,0.3)\end{pmatrix} & \begin{pmatrix}(0.3,0.5),\\(0.2,0.4)\end{pmatrix} \end{pmatrix}.$$

[Step 7] The weights of various stages of heart disease, $W(d_i), i = 1,2,...,5$ are calculated as follows:

$$W(d_1) = \begin{bmatrix} [0.5+0.5+0.3+0.5+0.5, \\ 0.7+0.7+0.7+0.7+0.7], \\ [0.1+0.1+0.1+0.2+0.1, \\ 0.2+0.2+0.3+0.3+0.2] \end{bmatrix} = \begin{bmatrix} [2.3,3.5], \\ [0.6,1.2] \end{bmatrix}.$$

Similarly,

$$W(d_2) = \begin{bmatrix} [2.9,4.5], \\ [0.6,1.1] \end{bmatrix},\ W(d_3) = \begin{bmatrix} [3.0,3.7], \\ [0.6,1.2] \end{bmatrix},$$
$$W(d_4) = \begin{bmatrix} [2.3,3.8], \\ [0.6,1.1] \end{bmatrix},\ W(d_5) = \begin{bmatrix} [1.9,3.2], \\ [0.8,1.7] \end{bmatrix}.$$

[Step 8] Scores of various stages of heart disease are computed and given below.

$$S(d_1) = \{(2.3-0.6)+(3.5-1.2)\}/2 = 2$$
$$S(d_2) = \{(2.9-0.6)+(4.5-1.1)\}/2 = 2.85$$
$$S(d_3) = \{(3.0-0.6)+(3.7-1.2)\}/2 = 2.45$$
$$S(d_4) = \{(2.3-0.6)+(3.8-1.1)\}/2 = 2.2$$
$$S(d_5) = \{(1.9-0.8)+(3.2-1.7)\}/2 = 1.3$$

[Step 9] Since score of d_2 is maximum, the patient under consideration belongs to Stage 'II', i.e., initial stage of heart disease as per the collective opinions of the group of experts.

Case 2: It uses normalized IVIFSMs.

[Step 2]: Cardinal matrix of each IVIFSM and the corresponding cardinal score are given below.

Cardinal matrix $[p_1(i,j)]_{1\times5}$ for IFSM $[p_1(i,j)]$ is (as shown in Example 8)

$$[p_1(i,j)]_{1\times5} = \left[\begin{pmatrix} (0.34,0.62) \\ (0.2,0.34) \end{pmatrix} \begin{pmatrix} (0.28,0.5) \\ (0.3,0.44) \end{pmatrix} \begin{pmatrix} (0.32,0.52) \\ (0.26,0.42) \end{pmatrix} \begin{pmatrix} (0.3,0.56) \\ (0.24,0.38) \end{pmatrix} \begin{pmatrix} (0,0) \\ (0,0) \end{pmatrix} \right]$$

and the corresponding cardinal score is$S(c\widehat{F}_{\{p_1\}}) = 0.43$.

Similarly, cardinal matrices for $[p_2(i,j)]$ and $[p_3(i,j)]$and the corresponding cardinal scores are respectively,

$$[p_2(i,j)]_{1\times5} = \left[\begin{pmatrix} (0.32,0.54) \\ (0.24,0.38) \end{pmatrix} \begin{pmatrix} (0.46,0.62) \\ (0.22,0.32) \end{pmatrix} \begin{pmatrix} (0.28,0.54) \\ (0.22,0.36) \end{pmatrix} \begin{pmatrix} (0,0) \\ (0,0) \end{pmatrix} \begin{pmatrix} (0.44,0.54) \\ (0.26,0.38) \end{pmatrix} \right]$$

$$[p_3(i,j)]_{1\times5} = \left[\begin{pmatrix} (0.32,0.54) \\ (0.26,0.4) \end{pmatrix} \begin{pmatrix} (0.36,0.58) \\ (0.26,0.4) \end{pmatrix} \begin{pmatrix} (0,0) \\ (0,0) \end{pmatrix} \begin{pmatrix} (0.34,0.6) \\ (0.28,0.38) \end{pmatrix} \begin{pmatrix} (0.34,0.52) \\ (0.34,0.44) \end{pmatrix} \right]$$

$S(c\widehat{F}_{\{p_2\}}) = 0.68$ and $S(c\widehat{F}_{\{p_3\}}) = 0.42$.

[Step 3]: The normalized IVIFMSs are given below.

$$N_{IFSM}[p_1(i,j)] = [0.43 * p_1(i,j)]_{5\times 5}$$

$$= \begin{pmatrix} \begin{pmatrix}(0.13.0.30),\\(0.04,0.09)\end{pmatrix} & \begin{pmatrix}(0.13,0.17),\\(0.17,0.22)\end{pmatrix} & \begin{pmatrix}(0.22,0.26),\\(0.09,0.17)\end{pmatrix} & \begin{pmatrix}(0.22,0.26),\\(0.09,0.17)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} \\ \begin{pmatrix}(0.17,0.22),\\(0.13,0.22)\end{pmatrix} & \begin{pmatrix}(0.13,0.30),\\(0.09,0.13)\end{pmatrix} & \begin{pmatrix}(0.13,0.26),\\(0.04,0.13)\end{pmatrix} & \begin{pmatrix}(0.13,0.30),\\(0.04,0.09)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} \\ \begin{pmatrix}(0.13,0.29),\\(0.13,0.17)\end{pmatrix} & \begin{pmatrix}(0.17,0.26),\\(0.09,0.13)\end{pmatrix} & \begin{pmatrix}(0.17,0.26),\\(0.09,0.17)\end{pmatrix} & \begin{pmatrix}(0.17,0.26),\\(0.09,0.17)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} \\ \begin{pmatrix}(0.17,0.34),\\(0.04,0.09)\end{pmatrix} & \begin{pmatrix}(0.09,0.22),\\(0.13,0.17)\end{pmatrix} & \begin{pmatrix}(0.09,0.22),\\(0.13,0.17)\end{pmatrix} & \begin{pmatrix}(0.09,0.22),\\(0.13,0.17)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} \\ \begin{pmatrix}(0.13,0.22),\\(0.09,0.17)\end{pmatrix} & \begin{pmatrix}(0.09,0.13),\\(0.17,0.30)\end{pmatrix} & \begin{pmatrix}(0.09,0.13),\\(0.22,0.26)\end{pmatrix} & \begin{pmatrix}(0.13,0.17),\\(0.17,0.22)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} \end{pmatrix},$$

$$N_{IVIFSM}[p_2(i,j)] = [0.68 * p_2(i,j)]_{5\times 5}$$

$$= \begin{pmatrix} \begin{pmatrix}(0.20,0.34),\\(0.20,0.27)\end{pmatrix} & \begin{pmatrix}(0.34,0.41),\\(0.20,0.27)\end{pmatrix} & \begin{pmatrix}(0.14,0.34),\\(0.20,0.27)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.34,0.48),\\(0.14,0.20)\end{pmatrix} \\ \begin{pmatrix}(0.27,0.34),\\(0.20,0.34)\end{pmatrix} & \begin{pmatrix}(0.27,0.48),\\(0.14,0.20)\end{pmatrix} & \begin{pmatrix}(0.20,0.34),\\(0.14,0.20)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.27,0.34),\\(0.20,0.27)\end{pmatrix} \\ \begin{pmatrix}(0.20,0.41),\\(0.20,0.27)\end{pmatrix} & \begin{pmatrix}(0.48,0.54),\\(0.07,0.14)\end{pmatrix} & \begin{pmatrix}(0.14,0.27),\\(0.20,0.34)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.27,0.41),\\(0.14,0.27)\end{pmatrix} \\ \begin{pmatrix}(0.14,0.41),\\(0.07,0.14)\end{pmatrix} & \begin{pmatrix}(0.34,0.48),\\(0.07,0.14)\end{pmatrix} & \begin{pmatrix}(0.20,0.41),\\(0.14,0.20)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.34,0.41),\\(0.20,0.27)\end{pmatrix} \\ \begin{pmatrix}(0.27,0.34),\\(0.14,0.27)\end{pmatrix} & \begin{pmatrix}(0.14,0.20),\\(0.27,0.34)\end{pmatrix} & \begin{pmatrix}(0.27,0.48),\\(0.07,0.20)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.27,0.34),\\(0.20,0.27)\end{pmatrix} \end{pmatrix},$$

$$N_{IVIFSM}[p_3(i,j)] = [0.42 * p_3(i,j)]_{5\times 5}$$

$$= \begin{pmatrix} \begin{pmatrix}(0.13,0.21),\\(0.13,0.17)\end{pmatrix} & \begin{pmatrix}(0.08,0.29),\\(0.04,0.13)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.13,0.25),\\(0.13,0.17)\end{pmatrix} & \begin{pmatrix}(0.08,0.17),\\(0.21,0.25)\end{pmatrix} \\ \begin{pmatrix}(0.13,0.29),\\(0.08,0.13)\end{pmatrix} & \begin{pmatrix}(0.13,0.21),\\(0.13,0.21)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.17,0.34),\\(0.04,0.08)\end{pmatrix} & \begin{pmatrix}(0.29,0.34),\\(0.04,0.08)\end{pmatrix} \\ \begin{pmatrix}(0.21,0.29),\\(0.04,0.13)\end{pmatrix} & \begin{pmatrix}(0.21,0.25),\\(0.13,0.17)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.21,0.25),\\(0.13,0.17)\end{pmatrix} & \begin{pmatrix}(0.17,0.25),\\(0.13,0.17)\end{pmatrix} \\ \begin{pmatrix}(0.17,0.25),\\(0.08,0.13)\end{pmatrix} & \begin{pmatrix}(0.17,0.21),\\(0.13,0.17)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.13,0.29),\\(0.08,0.13)\end{pmatrix} & \begin{pmatrix}(0.13,0.17),\\(0.17,0.21)\end{pmatrix} \\ \begin{pmatrix}(0.04,0.08),\\(0.21,0.29)\end{pmatrix} & \begin{pmatrix}(0.17,0.25),\\(0.13,0.17)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.08,0.13),\\(0.21,0.25)\end{pmatrix} & \begin{pmatrix}(0.04,0.17),\\(0.17,0.21)\end{pmatrix} \end{pmatrix}.$$

[Step 4]: Combined choice matrices for experts p_1, p_2, and p_3 are computed as in Case 1.

[Step 5]: Now the corresponding product of normalized IVIFSM and combined choice matrices are as follows.

$$N_{IVIFSM}\{p_1(i,j)\} \begin{pmatrix} \begin{pmatrix}(0.13.0.30),\\(0.04,0.09)\end{pmatrix} & \begin{pmatrix}(0.13,0.17),\\(0.17,0.22)\end{pmatrix} & \begin{pmatrix}(0.22,0.26),\\(0.09,0.17)\end{pmatrix} & \begin{pmatrix}(0.22,0.26),\\(0.09,0.17)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} \\ \begin{pmatrix}(0.17,0.22),\\(0.13,0.22)\end{pmatrix} & \begin{pmatrix}(0.13,0.30),\\(0.09,0.13)\end{pmatrix} & \begin{pmatrix}(0.13,0.26),\\(0.04,0.13)\end{pmatrix} & \begin{pmatrix}(0.13,0.30),\\(0.04,0.09)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} \\ \begin{pmatrix}(0.13,0.29),\\(0.13,0.17)\end{pmatrix} & \begin{pmatrix}(0.17,0.26),\\(0.09,0.13)\end{pmatrix} & \begin{pmatrix}(0.17,0.26),\\(0.09,0.17)\end{pmatrix} & \begin{pmatrix}(0.17,0.26),\\(0.09,0.17)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} \\ \begin{pmatrix}(0.17,0.34),\\(0.04,0.09)\end{pmatrix} & \begin{pmatrix}(0.09,0.22),\\(0.13,0.17)\end{pmatrix} & \begin{pmatrix}(0.09,0.22),\\(0.13,0.17)\end{pmatrix} & \begin{pmatrix}(0.09,0.22),\\(0.13,0.17)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} \\ \begin{pmatrix}(0.13,0.22),\\(0.09,0.17)\end{pmatrix} & \begin{pmatrix}(0.09,0.13),\\(0.17,0.30)\end{pmatrix} & \begin{pmatrix}(0.09,0.13),\\(0.22,0.26)\end{pmatrix} & \begin{pmatrix}(0.13,0.17),\\(0.17,0.22)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} \end{pmatrix} \otimes s_{\{p_1\}} \overset{s_{\{p_2 \wedge p_3\}}}{\begin{pmatrix} \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} \\ \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} \\ \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} \\ \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} \\ \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} \end{pmatrix}}$$

$$= \begin{pmatrix} \begin{pmatrix}(0.22.0.30)\\(0.04,0.09)\end{pmatrix} & \begin{pmatrix}(0.22.0.30)\\(0.04,0.09)\end{pmatrix} & \begin{pmatrix}(0.0,0.0)\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.0,0.0)\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.22.0.30)\\(0.04,0.09)\end{pmatrix} \\ \begin{pmatrix}(0.17,0.30)\\(0.04,0.09)\end{pmatrix} & \begin{pmatrix}(0.17,0.30)\\(0.04,0.09)\end{pmatrix} & \begin{pmatrix}(0.0,0.0)\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.0,0.0)\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.17,0.30)\\(0.04,0.09)\end{pmatrix} \\ \begin{pmatrix}(0.17,0.29)\\(0.09,0.13)\end{pmatrix} & \begin{pmatrix}(0.17,0.29)\\(0.09,0.13)\end{pmatrix} & \begin{pmatrix}(0.0,0.0)\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.0,0.0)\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.17,0.29)\\(0.09,0.13)\end{pmatrix} \\ \begin{pmatrix}(0.17,0.34)\\(0.04,0.09)\end{pmatrix} & \begin{pmatrix}(0.17,0.34)\\(0.04,0.09)\end{pmatrix} & \begin{pmatrix}(0.0,0.0)\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.0,0.0)\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.17,0.34)\\(0.04,0.09)\end{pmatrix} \\ \begin{pmatrix}(0.13,0.22)\\(0.09,0.17)\end{pmatrix} & \begin{pmatrix}(0.13,0.22)\\(0.09,0.17)\end{pmatrix} & \begin{pmatrix}(0.0,0.0)\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.0,0.0)\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.13,0.22)\\(0.09,0.17)\end{pmatrix} \end{pmatrix}$$

$$N_{IVIFSM}\{p_2(i,k)\} \begin{pmatrix} \begin{pmatrix}(0.20,0.34),\\(0.20,0.27)\end{pmatrix} & \begin{pmatrix}(0.34,0.41),\\(0.20,0.27)\end{pmatrix} & \begin{pmatrix}(0.14,0.34),\\(0.20,0.27)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.34,0.48),\\(0.14,0.20)\end{pmatrix} \\ \begin{pmatrix}(0.27,0.34),\\(0.20,0.34)\end{pmatrix} & \begin{pmatrix}(0.27,0.48),\\(0.14,0.20)\end{pmatrix} & \begin{pmatrix}(0.20,0.34),\\(0.14,0.20)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.27,0.34),\\(0.20,0.27)\end{pmatrix} \\ \begin{pmatrix}(0.20,0.41),\\(0.20,0.27)\end{pmatrix} & \begin{pmatrix}(0.48,0.54),\\(0.07,0.14)\end{pmatrix} & \begin{pmatrix}(0.14,0.27),\\(0.20,0.34)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.27,0.41),\\(0.14,0.27)\end{pmatrix} \\ \begin{pmatrix}(0.14,0.41),\\(0.07,0.14)\end{pmatrix} & \begin{pmatrix}(0.34,0.48),\\(0.07,0.14)\end{pmatrix} & \begin{pmatrix}(0.20,0.41),\\(0.14,0.20)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.34,0.41),\\(0.20,0.27)\end{pmatrix} \\ \begin{pmatrix}(0.27,0.34),\\(0.14,0.27)\end{pmatrix} & \begin{pmatrix}(0.14,0.20),\\(0.27,0.34)\end{pmatrix} & \begin{pmatrix}(0.27,0.48),\\(0.07,0.20)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.27,0.34),\\(0.20,0.27)\end{pmatrix} \end{pmatrix} \otimes s_{\{p_2\}} \overset{s_{\{p_1 \wedge p_3\}}}{\begin{pmatrix} \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} \\ \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} \\ \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} \\ \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} \\ \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} \end{pmatrix}}$$

$$= \begin{pmatrix} \begin{pmatrix}(0.34,0.48)\\(0.14,0.20)\end{pmatrix} & \begin{pmatrix}(0.34,0.48)\\(0.14,0.20)\end{pmatrix} & \begin{pmatrix}(0.0,0.0)\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.34,0.48)\\(0.14,0.20)\end{pmatrix} & \begin{pmatrix}(0.0,0.0)\\(0.0,0.0)\end{pmatrix} \\ \begin{pmatrix}(0.27,0.48)\\(0.14,0.20)\end{pmatrix} & \begin{pmatrix}(0.27,0.48)\\(0.14,0.20)\end{pmatrix} & \begin{pmatrix}(0.0,0.0)\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.27,0.48)\\(0.14,0.20)\end{pmatrix} & \begin{pmatrix}(0.0,0.0)\\(0.0,0.0)\end{pmatrix} \\ \begin{pmatrix}(0.48,0.54)\\(0.07,0.14)\end{pmatrix} & \begin{pmatrix}(0.48,0.54)\\(0.07,0.14)\end{pmatrix} & \begin{pmatrix}(0.0,0.0)\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.48,0.54)\\(0.07,0.14)\end{pmatrix} & \begin{pmatrix}(0.0,0.0)\\(0.0,0.0)\end{pmatrix} \\ \begin{pmatrix}(0.34,0.48)\\(0.07,0.14)\end{pmatrix} & \begin{pmatrix}(0.34,0.48)\\(0.07,0.14)\end{pmatrix} & \begin{pmatrix}(0.0,0.0)\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.34,0.48)\\(0.07,0.14)\end{pmatrix} & \begin{pmatrix}(0.0,0.0)\\(0.0,0.0)\end{pmatrix} \\ \begin{pmatrix}(0.27,0.48)\\(0.07,0.20)\end{pmatrix} & \begin{pmatrix}(0.27,0.48)\\(0.07,0.20)\end{pmatrix} & \begin{pmatrix}(0.0,0.0)\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.27,0.48)\\(0.07,0.20)\end{pmatrix} & \begin{pmatrix}(0.0,0.0)\\(0.0,0.0)\end{pmatrix} \end{pmatrix}$$

$$
N_{IVIFSM}\{p_3(i,l)\}\begin{pmatrix}
\begin{pmatrix}(0.13,0.21),\\(0.13,0.17)\end{pmatrix} & \begin{pmatrix}(0.08,0.29),\\(0.04,0.13)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.13,0.25),\\(0.13,0.17)\end{pmatrix} & \begin{pmatrix}(0.08,0.17),\\(0.21,0.25)\end{pmatrix}\\
\begin{pmatrix}(0.13,0.29),\\(0.08,0.13)\end{pmatrix} & \begin{pmatrix}(0.13,0.21),\\(0.13,0.21)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.17,0.34),\\(0.04,0.08)\end{pmatrix} & \begin{pmatrix}(0.29,0.34),\\(0.04,0.08)\end{pmatrix}\\
\begin{pmatrix}(0.21,0.29),\\(0.04,0.13)\end{pmatrix} & \begin{pmatrix}(0.21,0.25),\\(0.13,0.17)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.21,0.25),\\(0.13,0.17)\end{pmatrix} & \begin{pmatrix}(0.17,0.25),\\(0.13,0.17)\end{pmatrix}\\
\begin{pmatrix}(0.17,0.25),\\(0.08,0.13)\end{pmatrix} & \begin{pmatrix}(0.17,0.21),\\(0.13,0.17)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.13,0.29),\\(0.08,0.13)\end{pmatrix} & \begin{pmatrix}(0.13,0.17),\\(0.17,0.21)\end{pmatrix}\\
\begin{pmatrix}(0.04,0.08),\\(0.21,0.29)\end{pmatrix} & \begin{pmatrix}(0.17,0.25),\\(0.13,0.17)\end{pmatrix} & \begin{pmatrix}(0.0,0.0),\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.08,0.13),\\(0.21,0.25)\end{pmatrix} & \begin{pmatrix}(0.04,0.17),\\(0.17,0.21)\end{pmatrix}
\end{pmatrix}
\otimes s_{\{p_3\}}
\overset{s_{\{p_1\wedge p_2\}}}{\begin{pmatrix}
\begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix}\\
\begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix}\\
\begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix}\\
\begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix}\\
\begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(1,1),\\(1,1)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix} & \begin{pmatrix}(0,0),\\(0,0)\end{pmatrix}
\end{pmatrix}}
$$

$$
=\begin{pmatrix}
\begin{pmatrix}(0.13,0.29)\\(0.04,0.13)\end{pmatrix} & \begin{pmatrix}(0.13,0.29)\\(0.04,0.13)\end{pmatrix} & \begin{pmatrix}(0.13,0.29)\\(0.04,0.13)\end{pmatrix} & \begin{pmatrix}(0.0,0.0)\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.0,0.0)\\(0.0,0.0)\end{pmatrix}\\
\begin{pmatrix}(0.29,0.34)\\(0.04,0.08)\end{pmatrix} & \begin{pmatrix}(0.29,0.34)\\(0.04,0.08)\end{pmatrix} & \begin{pmatrix}(0.29,0.34)\\(0.04,0.08)\end{pmatrix} & \begin{pmatrix}(0.0,0.0)\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.0,0.0)\\(0.0,0.0)\end{pmatrix}\\
\begin{pmatrix}(0.21,0.29)\\(0.04,0.13)\end{pmatrix} & \begin{pmatrix}(0.21,0.29)\\(0.04,0.13)\end{pmatrix} & \begin{pmatrix}(0.21,0.29)\\(0.04,0.13)\end{pmatrix} & \begin{pmatrix}(0.0,0.0)\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.0,0.0)\\(0.0,0.0)\end{pmatrix}\\
\begin{pmatrix}(0.17,0.29)\\(0.08,0.13)\end{pmatrix} & \begin{pmatrix}(0.17,0.29)\\(0.08,0.13)\end{pmatrix} & \begin{pmatrix}(0.17,0.29)\\(0.08,0.13)\end{pmatrix} & \begin{pmatrix}(0.0,0.0)\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.0,0.0)\\(0.0,0.0)\end{pmatrix}\\
\begin{pmatrix}(0.17,0.25)\\(0.13,0.17)\end{pmatrix} & \begin{pmatrix}(0.17,0.25)\\(0.13,0.17)\end{pmatrix} & \begin{pmatrix}(0.17,0.25)\\(0.13,0.17)\end{pmatrix} & \begin{pmatrix}(0.0,0.0)\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.0,0.0)\\(0.0,0.0)\end{pmatrix}
\end{pmatrix}
$$

[Step 6] The sum of these product IVIFSMs is

$$
\begin{pmatrix}
\begin{pmatrix}(0.22.0.30)\\(0.04,0.09)\end{pmatrix} & \begin{pmatrix}(0.22.0.30)\\(0.04,0.09)\end{pmatrix} & \begin{pmatrix}(0.0,0.0)\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.0,0.0)\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.22.0.30)\\(0.04,0.09)\end{pmatrix}\\
\begin{pmatrix}(0.17,0.30)\\(0.04,0.09)\end{pmatrix} & \begin{pmatrix}(0.17,0.30)\\(0.04,0.09)\end{pmatrix} & \begin{pmatrix}(0.0,0.0)\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.0,0.0)\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.17,0.30)\\(0.04,0.09)\end{pmatrix}\\
\begin{pmatrix}(0.17,0.29)\\(0.09,0.13)\end{pmatrix} & \begin{pmatrix}(0.17,0.29)\\(0.09,0.13)\end{pmatrix} & \begin{pmatrix}(0.0,0.0)\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.0,0.0)\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.17,0.29)\\(0.09,0.13)\end{pmatrix}\\
\begin{pmatrix}(0.17,0.34)\\(0.04,0.09)\end{pmatrix} & \begin{pmatrix}(0.17,0.34)\\(0.04,0.09)\end{pmatrix} & \begin{pmatrix}(0.0,0.0)\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.0,0.0)\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.17,0.34)\\(0.04,0.09)\end{pmatrix}\\
\begin{pmatrix}(0.13,0.22)\\(0.09,0.17)\end{pmatrix} & \begin{pmatrix}(0.13,0.22)\\(0.09,0.17)\end{pmatrix} & \begin{pmatrix}(0.0,0.0)\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.0,0.0)\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.13,0.22)\\(0.09,0.17)\end{pmatrix}
\end{pmatrix}
\oplus
\begin{pmatrix}
\begin{pmatrix}(0.34,0.48)\\(0.14,0.20)\end{pmatrix} & \begin{pmatrix}(0.34,0.48)\\(0.14,0.20)\end{pmatrix} & \begin{pmatrix}(0.0,0.0)\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.34,0.48)\\(0.14,0.20)\end{pmatrix} & \begin{pmatrix}(0.0,0.0)\\(0.0,0.0)\end{pmatrix}\\
\begin{pmatrix}(0.27,0.48)\\(0.14,0.20)\end{pmatrix} & \begin{pmatrix}(0.27,0.48)\\(0.14,0.20)\end{pmatrix} & \begin{pmatrix}(0.0,0.0)\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.27,0.48)\\(0.14,0.20)\end{pmatrix} & \begin{pmatrix}(0.0,0.0)\\(0.0,0.0)\end{pmatrix}\\
\begin{pmatrix}(0.48,0.54)\\(0.07,0.14)\end{pmatrix} & \begin{pmatrix}(0.48,0.54)\\(0.07,0.14)\end{pmatrix} & \begin{pmatrix}(0.0,0.0)\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.48,0.54)\\(0.07,0.14)\end{pmatrix} & \begin{pmatrix}(0.0,0.0)\\(0.0,0.0)\end{pmatrix}\\
\begin{pmatrix}(0.34,0.48)\\(0.07,0.14)\end{pmatrix} & \begin{pmatrix}(0.34,0.48)\\(0.07,0.14)\end{pmatrix} & \begin{pmatrix}(0.0,0.0)\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.34,0.48)\\(0.07,0.14)\end{pmatrix} & \begin{pmatrix}(0.0,0.0)\\(0.0,0.0)\end{pmatrix}\\
\begin{pmatrix}(0.27,0.48)\\(0.07,0.20)\end{pmatrix} & \begin{pmatrix}(0.27,0.48)\\(0.07,0.20)\end{pmatrix} & \begin{pmatrix}(0.0,0.0)\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.27,0.48)\\(0.07,0.20)\end{pmatrix} & \begin{pmatrix}(0.0,0.0)\\(0.0,0.0)\end{pmatrix}
\end{pmatrix}
\oplus
$$

$$
\begin{pmatrix}
\begin{pmatrix}(0.13,0.29)\\(0.04,0.13)\end{pmatrix} & \begin{pmatrix}(0.13,0.29)\\(0.04,0.13)\end{pmatrix} & \begin{pmatrix}(0.13,0.29)\\(0.04,0.13)\end{pmatrix} & \begin{pmatrix}(0.0,0.0)\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.0,0.0)\\(0.0,0.0)\end{pmatrix}\\
\begin{pmatrix}(0.29,0.34)\\(0.04,0.08)\end{pmatrix} & \begin{pmatrix}(0.29,0.34)\\(0.04,0.08)\end{pmatrix} & \begin{pmatrix}(0.29,0.34)\\(0.04,0.08)\end{pmatrix} & \begin{pmatrix}(0.0,0.0)\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.0,0.0)\\(0.0,0.0)\end{pmatrix}\\
\begin{pmatrix}(0.21,0.29)\\(0.04,0.13)\end{pmatrix} & \begin{pmatrix}(0.21,0.29)\\(0.04,0.13)\end{pmatrix} & \begin{pmatrix}(0.21,0.29)\\(0.04,0.13)\end{pmatrix} & \begin{pmatrix}(0.0,0.0)\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.0,0.0)\\(0.0,0.0)\end{pmatrix}\\
\begin{pmatrix}(0.17,0.29)\\(0.08,0.13)\end{pmatrix} & \begin{pmatrix}(0.17,0.29)\\(0.08,0.13)\end{pmatrix} & \begin{pmatrix}(0.17,0.29)\\(0.08,0.13)\end{pmatrix} & \begin{pmatrix}(0.0,0.0)\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.0,0.0)\\(0.0,0.0)\end{pmatrix}\\
\begin{pmatrix}(0.17,0.25)\\(0.13,0.17)\end{pmatrix} & \begin{pmatrix}(0.17,0.25)\\(0.13,0.17)\end{pmatrix} & \begin{pmatrix}(0.17,0.25)\\(0.13,0.17)\end{pmatrix} & \begin{pmatrix}(0.0,0.0)\\(0.0,0.0)\end{pmatrix} & \begin{pmatrix}(0.0,0.0)\\(0.0,0.0)\end{pmatrix}
\end{pmatrix}
=
\begin{pmatrix}
\begin{pmatrix}(0.34,0.48)\\(0.04,0.09)\end{pmatrix} & \begin{pmatrix}(0.34,0.48),\\(0.04,0.09)\end{pmatrix} & \begin{pmatrix}(0.13,0.29)\\(0.04,0.13)\end{pmatrix} & \begin{pmatrix}(0.34,0.48)\\(0.14,0.20)\end{pmatrix} & \begin{pmatrix}(0.22,0.30)\\(0.04,0.09)\end{pmatrix}\\
\begin{pmatrix}(0.29,0.48)\\(0.04,0.08)\end{pmatrix} & \begin{pmatrix}(0.29,0.48)\\(0.04,0.08)\end{pmatrix} & \begin{pmatrix}(0.29,0.34)\\(0.04,0.08)\end{pmatrix} & \begin{pmatrix}(0.27,0.48)\\(0.14,0.20)\end{pmatrix} & \begin{pmatrix}(0.17,0.30)\\(0.04,0.09)\end{pmatrix}\\
\begin{pmatrix}(0.48,0.54)\\(0.04,0.13)\end{pmatrix} & \begin{pmatrix}(0.48,0.54)\\(0.04,0.13)\end{pmatrix} & \begin{pmatrix}(0.21,0.29)\\(0.04,0.13)\end{pmatrix} & \begin{pmatrix}(0.48,0.54),\\(0.07,0.14)\end{pmatrix} & \begin{pmatrix}(0.17,0.29)\\(0.09,0.13)\end{pmatrix}\\
\begin{pmatrix}(0.34,0.48)\\(0.04,0.09)\end{pmatrix} & \begin{pmatrix}(0.34,0.48)\\(0.04,0.09)\end{pmatrix} & \begin{pmatrix}(0.17,0.29)\\(0.08,0.13)\end{pmatrix} & \begin{pmatrix}(0.34,0.48)\\(0.07,0.14)\end{pmatrix} & \begin{pmatrix}(0.17,0.34)\\(0.04,0.09)\end{pmatrix}\\
\begin{pmatrix}(0.27,0.48)\\(0.07,0.17)\end{pmatrix} & \begin{pmatrix}(0.27,0.48)\\(0.07,0.17)\end{pmatrix} & \begin{pmatrix}(0.17,0.25)\\(0.13,0.17)\end{pmatrix} & \begin{pmatrix}(0.27,0.48)\\(0.07,0.20)\end{pmatrix} & \begin{pmatrix}(0.13,0.22)\\(0.09,0.17)\end{pmatrix}
\end{pmatrix}.
$$

[Step 7] Now the weights of various stages of heart disease $W(d_i), i = 1,2,...,5$ are calculated as follows:

$$W(d_1) = \begin{bmatrix} [0.34+0.34+0.13+0.34+0.22, \\ 0.48+0.48+0.29+0.48+0.30], \\ [0.04+0.04+0.04+0.14+0.04, \\ 0.09+0.09+0.13+0.20+0.09] \end{bmatrix} =$$

$$= \begin{bmatrix} [1.37, 2.03], \\ [0.3, 0.6] \end{bmatrix}.$$

Similarly,

$$W(d_2) = \begin{bmatrix} [1.31, 2.08], \\ [0.3, 0.53] \end{bmatrix},$$
$$W(d_3) = \begin{bmatrix} [1.82, 2.2], \\ [0.28, 0.66] \end{bmatrix},$$
$$W(d_4) = \begin{bmatrix} [1.36, 2.07], \\ [0.27, 0.54] \end{bmatrix},$$
$$W(d_5) = \begin{bmatrix} [1.11, 1.91], \\ [0.43, 0.88] \end{bmatrix}.$$

[Step 8] Scores of various stages of heart disease are computed as follows:

$$S(d_1) = \{(1.37-0.3)+(2.03-0.6)\}/2 = 1.25$$
$$S(d_2) = \{(1.31-0.3)+(2.08-0.53)\}/2 = 1.28$$
$$S(d_3) = \{(1.82-0.28)+(2.2-0.66)\}/2 = 1.54$$
$$S(d_4) = \{(1.36-0.27)+(2.07-0.54)\}/2 = 1.31$$
$$S(d_5) = \{(1.11-0.43)+(1.91-0.88)\}/2 = 0.86$$

[Step 9] Since score of d_3 is maximum, the patient under consideration belongs to Stage 'III' as per the collective opinions of the group of experts.

7 Discussion of Results

This study proposes a decision making methodology under interval-valued intuitionistic fuzzy environment, where a confident weight is assigned to each of the experts based on prescribed opinion. The confident weight for each expert is computed by deriving the cardinal score of their corresponding IVIFSSs. The opinion of an expert with more cardinal score is more important than others and consequently the opinion of that expert contributes a vital role in decision making process. When an expert is more confident about her opinion, more cardinal score is assigned to that expert. Due to lack of information or limited domain knowledge, experts often prefer to express their opinions only for a subset of attributes instead of the entire attribute set. In that case, cardinal score will be more when an expert provides opinion about more number of attributes and less when she provides opinion about less number of attributes. This is very much similar to our real life situations. This also removes the biasness and adds more credibility to the final decision. Our algorithm does not use any medical knowledgebase, so it does not depend on the correctness of the knowledgebase. We give more importance on the parameter selection of experts by finding initial choice matrices and then combined choice matrix. We rely on experts' opinions and then investigate the collective opinion by a systematic procedure based on cardinal matrix. As our approach is guided by a confident weight assigning mechanism, so chances of biasing is less in our model. Among two cases, Case 1 does not use the confident weight, while Case 2 uses it. As per collective opinion of a group of experts, the ordering of the diseases, i.e., stages of heart disease is given below in Table 6. The result shows different ordering in second case as we have considered the experts' confident weights. Case I produces the final outcome as d_2, i.e., stage 'II' of heart disease and Case II produces d_3, i.e., stage 'III' of heart disease as per the collective opinion.

Table 6. Ordering of diseases in different cases

Case I	$d_2 > d_3 > d_4 > d_1 > d_5$
Case II	$d_3 > d_4 > d_2 > d_1 > d_5$

8 Conclusion

This article has proposed an algorithmic approach for multiple attribute group decision making using IVIFSM based on confident weight assigning mechanism of experts. Firstly, we have presented IVIFSM and some of its relevant operations. Next we have proposed the confident weight assigning mechanism of experts using cardinal score in the context of interval-valued intuitionistic fuzzy environment. Our proposed algorithm is based on combined choice matrix, product IVIFSM, cardinal matrix, score function, and accuracy function, which yields the collective opinion of a group of decision makers. Another important aspect of this study is that we have not used the concept of medical knowledgebase in our decision making process. The case study is related with medical diagnosis, where we have used opinions of a group of experts about a common set of symptoms. IVIFSM has been used to represent the opinions. We have performed a comparative analysis in the case study to demonstrate the effectiveness of using weight assigning procedure. Future scope of this research work might be to investigate the application of robustness in MAGDM in the framework of IVIFSS. Also researchers might focus on various properties of interval-valued intuitionistic fuzzy soft matrix and then apply them to suitable uncertain decision making problems.

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