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DEGENERATE SINGULAR PARABOLIC PROBLEMS WITH NATURAL GROWTH

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Abstract. In this paper, we study the existence and regularity results for nonlinear singular parabolic problems with a natural growth gradient term

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div}((a(x,t)+u^q)|\nabla u|^{p-2}\nabla u) + d(x,t)\frac{|\nabla u|^p}{u^\gamma} = f & \text{in } Q, \\ u(x,t) = 0 & \text{on } \Gamma, \\ u(x,t=0) = u_0(x) & \text{in } \Omega, \end{cases}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, $N > 2$, Q is the cylinder $\Omega \times (0,T)$, $T > 0$, Γ the lateral surface $\partial\Omega \times (0,T)$, $2 \leq p < N$, $a(x,t)$ and $b(x,t)$ are positive measurable bounded functions, $q \geq 0$, $0 \leq \gamma < 1$, and f non-negative function belongs to the Lebesgue space $L^m(Q)$ with $m > 1$, and $u_0 \in L^\infty(\Omega)$ such that

$$\forall \omega \subset\subset \Omega \, \exists D_\omega > 0 : \, u_0 \geq D_\omega \text{ in } \omega.$$

More precisely, we study the interaction between the term u^q $(q > 0)$ and the singular lower order term $d(x,t)|\nabla u|^p u^{-\gamma}$ $(0 < \gamma < 1)$ in order to get a solution to the above problem. The regularizing effect of the term u^q on the regularity of the solution and its gradient is also analyzed.

Keywords: degenerate parabolic equations, singular parabolic equations, natural growth term.

Mathematics Subject Classification: 35A25, 35B45, 35B09, 35D30, 35K65, 35K67.

1. INTRODUCTION

In this work, we restrict our attention to the study of a class of singular nonlinear parabolic problems having natural growth with respect to the gradient. The problem is the following

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div}((a(x,t)+u^q)|\nabla u|^{p-2}\nabla u) + d(x,t)\frac{|\nabla u|^p}{u^\gamma} = f & \text{in } Q, \\ u(x,t) = 0 & \text{on } \Gamma, \\ u(x,t=0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, $N > 2$, and Q is the cylinder $\Omega \times (0,T)$, $T > 0$, Γ the lateral surface $\partial\Omega \times (0,T)$, $2 \le p < N$, $q \ge 0, 0 < \gamma < 1$, $a(x,t)$ and $d(x,t)$ are measurable functions satisfying

$$0 < \alpha_1 \le a(x,t) \le \alpha_2, \tag{1.2}$$

$$0 < \beta_1 \le d(x,t) \le \beta_2, \tag{1.3}$$

where α_1, α_2, β_1 and β_2 are fixed real numbers such that $\alpha_1 < \alpha_2$ and $\beta_1 < \beta_2$. On the function f, we assume that it is non-negative and not identically zero, and that it belongs to some Lebesgue space $L^m(Q)$, $m > 1$. Moreover, the initial data $u_0 \in L^\infty(\Omega)$ satisfies the following condition of strict positivity

$$\exists D_\omega > 0 \, \forall \omega \subset\subset \Omega \, : \, u_0 \ge D_\omega. \tag{1.4}$$

The study of singular nonlinear parabolic problems of this kind is influenced by their connection with the theory of non-Newtonian fluids and heat conduction in electrically active materials (see, for instance, [29, 34] and references therein).

From a purely mathematical view, interest in studying this type of problem like (1.1) naturally arises in the presence of the following two terms: The first term, $a(x,t) + u^q$ appears in the coefficient of the p-Laplace operator. It has a very significant impact on the boundary and growth conditions, where the operator becomes unbounded and has a more general growth condition than the operator discussed in the papers [5, 13, 22]. The second term that appears in problem (1.1) is $d(x,t)\frac{|\nabla u|^p}{u^\gamma}, \gamma > 0$ having natural growth depend on the gradient, which becomes singular where the solution is zero since it depends on a negative power of the solution.

In the non-singular case (i.e. $\gamma = 0$), the problem(1.1) has been extensively studied in the past under different assumptions on the data f; see [11, 12, 20, 30, 43]. More recently, Abdellaoui and Redwane in [1] studied the general non-singular case with $f \in M(Q)$ (the space of Radon measures on Q with total bounded variation), proving the existence of a weak solution of (1.1).

In the elliptic case, when the singular term exists (i.e. $\gamma > 0$) the problem (1.1) has been studied in the literature. If $a(x,t) = 0, q = 0$ and $p = 2$, the authors in [3] have studied the existence and non-existence of the solution to the following problem

$$\begin{cases} -\operatorname{div}(M(x,u)\nabla u) + g(x,u)|\nabla u|^2 = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $|M(x,s)| \le \beta$, $M(x,s)\xi.\xi \ge \alpha|\xi|^2$ for a.e. $x \in \Omega$, for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$, and α, β are real numbers such that $0 < \alpha < \beta$. $g : \Omega \times (0,+\infty) \to \mathbb{R}$ is a Carathéodory function possibly singular at $s = 0$ satisfying $g(x,s) \ge 0$ for a.e. $x \in \Omega$, for all $s > 0$ and $0 < f \in L^{\frac{2N}{N+2}}(\Omega)$. This result has been extended in [44], see also [2]. When the term u^q exist (i.e. $q > 0$) and $p = 2$, Boccardo *et al.* in [7] proved that there exists a non-negative solution to the following singular elliptic problem

$$\begin{cases} -\operatorname{div}((a(x) + u^q)\nabla u) + b(x)\frac{|\nabla u|^2}{u^\gamma} = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega, \end{cases}$$

where $q > 0$, $0 < \gamma < 1$ and $b(x)$ is a measurable bounded function and $0 < f \in L^m(\Omega)$, $m \geq 1$. For more and different aspects concerning singular elliptic problems, we refer to [8, 9, 27, 32, 37–40, 42].

In the last few years, great attention has been paid to the study of singular parabolic problems. Here, we limit ourselves to giving a very brief description of some papers that mostly influenced us. The authors in [6] proved the existence of non-negative solutions to the following singular parabolic problem

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div}(M(x,t)\nabla u) + B\frac{|\nabla u|^2}{u^\gamma} = u^r & \text{in } Q, \\ u(x,t) = 0 & \text{on } \Gamma, \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases}$$

where $B > 0$, $0 < \gamma < 1, 0 < r < 2 - \gamma$ and M is a bounded and measurable uniformly elliptic matrix, u_0 is a strictly positive function in $L^\infty(\Omega)$. In the same kind, Martínez-Aparicio and Petitta in [33] studied the singular parabolic problem

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div}(M(x,t,u)\nabla u) + g(x,t,u)|\nabla u|^2 = f(x,t) & \text{in } Q, \\ u(x,t) = 0 & \text{on } \Gamma, \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases}$$

where $f \in L^r(0,T;L^q(\Omega))$, with $\frac{1}{r} + \frac{2}{Nq} < 1$, $q \geq 1, r > 1$, and $u_0 \in L^\infty(\Omega)$, and the function $g(x,t,s) : Q \times (0,+\infty) \to \mathbb{R}$ is a Carathéodory function that is singular at $s = 0$, and possibly negative (see also [17]). If $p \geq 2$ and $q = 0$, the authors in [13] have proved the existence of weak solutions to homogeneous nonlinear and singular parabolic problems

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta_p u + B\frac{|\nabla u|^p}{u} = 0 & \text{in } Q, \\ u(x,t) = 0 & \text{on } \Gamma, \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases}$$

with $p > 1$, $B > 0$, and u_0 is a positive function in $L^\infty(\Omega)$ such that $u_0 \geq c > 0$ a.e. on Ω; see [5] concerning the non-homogeneous case. Finally, more recently, if $a(x,t) = 0$ and $q = 0$, the problem (1.1) has been studied in [22]. For more different aspects concerning singular parabolic problems, we refer to [14–16, 18, 21–25, 31, 35].

The difficulties in studying problem (1.1) arise from the presence of the term u^q ($q > 0$) and from the lower-order term: The natural growth term depends on the gradient, and singularity depends on u, as well as the proof of the strict positivity of the solution in the interior of the parabolic cylinder. To overcome these difficulties, we must approximate the singular problem (1.1) by another non-singular one, and we show that this problem admits a non-negative solution (the proof is based on the application of Schauder's fixed point theorem) and that this solution is strictly positive in the interior of the parabolic cylinder (the proof is based on the use of the intrinsic Harnack inequality).

Let us introduce some notation. We will use meas(E) and $|E|$ to denote the Lebesgue measure of a subset E of $\mathbb{R}^N$. The Hölder conjugate exponent of $q > 1$ is $q' = q/(q-1)$, while the Sobolev conjugate exponent of p for $1 \leq p < N$ is $p^* = Np/(N-p)$. For a fixed $k > 0$, we define the truncation functions T_k and G_k as follows:

$$\begin{aligned} T_k(s) &= \max(-k, \min(s, k)), \\ G_k(s) &= s - T_k(s) = (|s| - k)^+ \operatorname{sign}(s). \end{aligned}$$

To simplify notation, we will use $\int_Q f$ to denote $\int_Q f(x,t)\,dx\,dt$, when there is no ambiguity in the integration variables. We use C to denote constants whose values may change from line to line and even within the same line, depending on the parameters, such as N, p, B, θ, m, T, Ω, or Q, but not on the indexes of the sequences we introduce.

Next, we state a lemma we will use, which is the Gagliardo–Nirenberg inequality.

Lemma 1.1 ([19, Theorem 2.1])**.** *Let v be a function in $W_0^{1,h}(\Omega) \cap L^\rho(\Omega)$, with $h \geq 1$, $\rho \geq 1$. Then there exists a positive constant C, depending on N, h, ρ and σ such that*

$$\|v\|_{L^\sigma(\Omega)} \leq C \|\nabla v\|^{\eta}_{(L^h(\Omega))^N} \|v\|^{1-\eta}_{L^\rho(\Omega)},$$

for every η and σ satisfying

$$0 < \eta < 1, \quad \frac{1}{\sigma} = \eta\left(\frac{1}{h} - \frac{1}{N}\right) + \frac{1-\eta}{\rho}.$$

An immediate consequence of the previous lemma is the following embedding result:

$$\int_Q |v|^\sigma dxdt \leq C \|v\|^{\frac{\rho h}{N}}_{L^\infty(0,T;L^\rho(\Omega))} \int_Q |\nabla v|^h dxdt,$$

which holds for every function $v \in L^h(0,T; W_0^{1,h}(\Omega)) \cap L^\infty(0,T; L^\rho(\Omega))$ with $h \geq 1$, $\rho \geq 1$ and $\sigma = \frac{h(N+\rho)}{N}$ (see, for instance, [19, Proposition 3.1]).

Now we give the definition of a weak solution to the problem (1.1).

Definition 1.2. A weak solution to problem (1.1) is a function u in $L^1(0,T; W_0^{1,1}(\Omega))$ such that for every $w \subset\subset \Omega$ there exists c_w such that $u \geq c_w > 0$ in $w \times (0,T)$, $(a(x,t) + u^q)|\nabla u|^{p-1} \in L^1(Q)$, $\frac{|\nabla u|^p}{u^\gamma} \in L^1(0,T; L^1_{\mathrm{loc}}(\Omega))$. Furthermore, we have that

$$\begin{aligned} &-\int_Q u \frac{\partial \phi}{\partial t}\,dx\,dt + \int_Q (a(x,t) + u^q)|\nabla u|^{p-2} \nabla u \cdot \nabla \phi\,dx\,dt \\ &+ \int_Q d(x,t) \frac{|\nabla u|^p}{u^\gamma} \phi\,dx\,dt = \int_Q f\phi\,dx\,dt + \int_\Omega u_0(x)\phi(x,0), \end{aligned} \tag{1.5}$$

for every $\phi \in C^1_c(\Omega \times [0,T))$.

Here, we give the main results of this paper.

Theorem 1.3. *Let* $0 < \gamma < 1, 0 < q < 1$ *and* $\delta = \min(\gamma, 1-q)$. *Assume that* a *satisfy* (1.2), d *satisfy* (1.3) *and* f *is a non-negative function belonging to* $L^m(Q)$ *with* $1 < m < N/p + 1$. *Then there exists a solution* u *of the problem* (1.1) *in the sense of Definition 1.2 verify the following regularity:*

(i) *If* $\frac{p(N+1+\delta)}{p(N+1+\delta)-N\delta} \leq m < N/p+1$, *then* u *belongs to* $L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\sigma(Q)$, *with* $\sigma = m\frac{N(p-\delta)+p}{N-pm+p}$.

(ii) *If* $1 < m < \frac{p(N+1+\delta)}{p(N+1+\delta)-N\delta}$, *then* u *belongs to* $L^s(0,T;W_0^{1,s}(\Omega)) \cap L^\sigma(Q)$, *where*

$$s = m\frac{N(p-\delta)+p}{N+1-\delta(m-1)} \quad \textit{and} \quad \sigma = m\frac{N(p-\delta)+p}{N-pm+p}.$$

Moreover, we have the following summability:

$$u^q|\nabla u|^{p-1} \in L^\rho(Q), \quad \textit{with } 1 \leq \rho < \frac{s}{p-1}.$$

Remark 1.4. Observe that by the fact that $m > 1$, we have $s > p-1$, then the interval $[1, \frac{s}{p-1})$ is not empty.

Theorem 1.5. *Let* $0 < \gamma < 1, q \geq 1$ *and* $\delta = \min(\gamma, 1-q)$. *Assume that* a *satisfy* (1.2), d *satisfy* (1.3) *and* f *is non-negative function belongs to* $L^m(Q)$ *with* $1 < m < N/p+1$. *Then there exists a solution* u *of the problem* (1.1) *in the sense of Definition 1.2 satisfy the following regularity:* $u \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\sigma(Q)$, *with*

$$\sigma = m\frac{N(p-\delta)+p}{N-pm+p}.$$

Moreover, we have the following summability:

$$u^q|\nabla u|^{p-1} \in L^\rho(Q), \quad \textit{with } 1 \leq \rho \leq p'.$$

Theorem 1.6. *Let* $0 < \gamma < 1$, $\max\left(0, \frac{N+p-p(N-\gamma)}{N}\right) \leq q \leq \frac{p+\gamma}{p-1}$ *and* f *be a non-negative function belongs to* $L^m(Q)$, *with* $m = N/p+1$. *Assume that* (1.2) *and* (1.3) *hold true. Then there exists a solution* u *of the problem* (1.1) *in the sense of Definition 1.2 verify the following regularity:*

$$u \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^{q+\gamma+p}(0,T;L^{\frac{N(q+\gamma+p)}{N-p}}(\Omega)).$$

Moreover,

$$u^q|\nabla u|^{p-1} \in L^\rho(Q), \quad \textit{with } \rho = \frac{p(p+\gamma+q)}{qp+(p-1)(p+\gamma+q)}.$$

Remark 1.7.

(i) The condition $q \leq \frac{p+\gamma}{p-1}$ is due to the fact $\rho \geq 1$, and

$$\max\left(0, \frac{N+p-p(N-\gamma)}{N}\right) \leq q$$

is due to the fact of the regularity of u in the space $L^{q+\gamma+p}(0,T; L^{\frac{N(q+\gamma+p)}{N-p}}(\Omega))$ and the choice of $q \geq 0$ in problem (1.1).

(ii) By the fact $\frac{N+p-p(N-\gamma)}{N} < \frac{p+\gamma}{p-1}$, then the interval $\left[\max\left(0, \frac{N+p-p(N-\gamma)}{N}\right), \frac{p+\gamma}{p-1}\right]$ is not empty.

Theorem 1.8. *Let* $0 < \gamma < 1, q > 0$ *and* f *be a non-negative function belongs to* $L^m(Q)$*, with* $m > N/p + 1$*. Assume that* (1.2) *and* (1.3) *hold true. Then there exists a solution* u *of the problem* (1.1) *in the sense of Definition 1.2 verify the following regularity:* $u \in L^p(0,T; W_0^{1,p}(\Omega)) \cap L^\infty(Q)$*. Moreover,*

$$u^q |\nabla u|^{p-1} \in L^{p'}(Q).$$

Remark 1.9. The above results extended the results contained in the paper [7] in the evolution case and improved the results contained in the paper [22].

2. APPROXIMATION OF (1.1), POSITIVITY AND A PRIORI ESTIMATES

Let $0 < \varepsilon < 1$. We approximate the problem (1.1) by the following nonlinear and non-singular problems

$$\begin{cases} \frac{\partial u_\varepsilon}{\partial t} - \operatorname{div}((a(x,t) + |u_\varepsilon|^q)|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon) + d(x,t)\frac{u_\varepsilon |\nabla u_\varepsilon|^p}{(|u_\varepsilon|+\varepsilon)^{\gamma+1}} = f_\varepsilon & \text{in } Q, \\ u_\varepsilon(x,t) = 0 & \text{on } \Gamma, \\ u_\varepsilon(x,0) = u_{\varepsilon 0}(x) & \text{in } \Omega, \end{cases} \tag{2.1}$$

where $f_\varepsilon = \frac{f}{1+\varepsilon f}$ and $f_\varepsilon \in L^\infty(Q)$, such that

$$\|f_\varepsilon\|_{L^m(Q)} \leq \|f\|_{L^m(Q)} \text{ and } f_\varepsilon \to f \text{ strongly in } L^m(Q),\ m > 1, \tag{2.2}$$

and $u_{\varepsilon 0}(x) = \frac{u_0(x)}{1+\varepsilon u_0(x)} \in L^\infty(\Omega)$ such that

$$\|u_{\varepsilon 0}(x)\|_{L^\infty(\Omega)} \leq \|u_0\|_{L^\infty(\Omega)} \text{ and } u_{\varepsilon 0}(x) \to u_0(x) \text{ strongly in } L^1(\Omega). \tag{2.3}$$

The problem (2.1) admits weak solutions u_ε that belong to $L^p(0,T; W_0^{1,p}(\Omega)) \cap L^\infty(Q)$, see [10, 13, 28, 36]. Also, the solution of problem (2.1) is to continue in time, that is, $u_\varepsilon \in C([0,T]; L^1_{loc}(\Omega))$, as can be easily proved using a method similar to Theorem 1.1 of [36]. Because the right-hand side of (2.1) is non-negative, u_ε is also non-negative.

In the next lemma, we will prove the strict positivity of u_ε in $\omega \times (0,T)$, for every $\omega \subset\subset \Omega$.

Lemma 2.1. *Let u_ε be a solution to* (2.1)*. Then for every $\omega \subset\subset \Omega$, there exists a positive constant c_ω such that*

$$u_\varepsilon \geq c_w > 0, \text{ in } \omega \times (0,T), \text{ for all } \varepsilon \in (0,1).$$

Proof. Let us define the functions

$$H_\varepsilon(s) = \frac{(s+\varepsilon)^{1-\gamma}}{1-\gamma}, \quad H_0(s) = \frac{s^{1-\gamma}}{1-\gamma},$$

for $s \geq 0$, and take $\varphi(u_\varepsilon) = e^{-\frac{\beta_2}{\alpha_1} H_\varepsilon(u_\varepsilon)}\phi$, with ϕ in $L^p(0,T; W_0^{1,p}(\Omega)) \cap L^\infty(Q)$, $\phi \geq 0$ as a test function in (2.1). We obtain

$$\begin{aligned}
&\int_0^T \int_\Omega \frac{\partial u_\varepsilon}{\partial t} e^{\frac{-\beta_2}{\alpha_1} H_\varepsilon(u_\varepsilon)} \phi \\
&+ \int_Q [a(x,t) + u_\varepsilon^q] |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \cdot \nabla \phi \, e^{\frac{-\beta_2}{\alpha_1} H_\varepsilon(u_\varepsilon)} \\
&- \frac{\beta_2}{\alpha_1} \int_Q [a(x,t) + u_\varepsilon^q] \frac{|\nabla u_\varepsilon|^p}{(u_\varepsilon + \varepsilon)^\gamma} e^{\frac{-\beta_2}{\alpha_1} H_\varepsilon(u_\varepsilon)} \phi \\
&+ \int_Q d(x,t) \frac{u_\varepsilon |\nabla u_\varepsilon|^p}{(u_\varepsilon + \varepsilon)^{\theta+1}} e^{\frac{-\beta_2}{\alpha_1} H_\varepsilon(u_\varepsilon)} \phi = \int_Q f_\varepsilon e^{\frac{-\beta_2}{\alpha_1} H_\varepsilon(u_\varepsilon)} \phi.
\end{aligned}$$

Recalling the conditions (1.2) and (1.3), we can write

$$\begin{aligned}
&\int_0^T \int_\Omega \frac{\partial u_\varepsilon}{\partial t} e^{\frac{-\beta_2}{\alpha_1} H_\varepsilon(u_\varepsilon)} \phi \\
&+ \int_Q [a(x,t) + u_\varepsilon^q] |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \cdot \nabla \phi \, e^{\frac{-\beta_2}{\alpha_1} H_\varepsilon(u_\varepsilon)} \\
&- \frac{\beta_2}{\alpha_1} \int_Q \alpha_1 \frac{|\nabla u_\varepsilon|^p}{(u_\varepsilon + \varepsilon)^\gamma} e^{\frac{-\beta_2}{\alpha_1} H_\varepsilon(u_\varepsilon)} \phi \\
&+ \int_Q \beta_2 \frac{|\nabla u_\varepsilon|^p}{(u_\varepsilon + \varepsilon)^\gamma} e^{\frac{-\beta_2}{\alpha_1} H_\varepsilon(u_\varepsilon)} \phi \geq \int_Q f_\varepsilon e^{\frac{-\beta_2}{\alpha_1} H_\varepsilon(u_\varepsilon)} \phi.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
&\int_0^T \int_\Omega \frac{\partial u_\varepsilon}{\partial t} e^{\frac{-\beta_2}{\alpha_1} H_\varepsilon(u_\varepsilon)} \phi \\
&+ \int_Q [a(x,t) + u_\varepsilon^q] |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \cdot \nabla \phi e^{\frac{-\beta_2}{\alpha_1} H_\varepsilon(u_\varepsilon)} \geq \int_Q f_\varepsilon e^{\frac{-\beta_2}{\alpha_1} H_\varepsilon(u_\varepsilon)} \phi,
\end{aligned} \tag{2.4}$$

for all $\phi \in L^p(0,T : W_0^{1,p}(\Omega)) \cap L^\infty(Q)$, $\phi \geq 0$.

Now, given $\delta > 0$, define the function

$$\psi_\delta(s) = \begin{cases} 1 & \text{if } 0 \leq s < 1, \\ \frac{-1}{\delta}(s - 1 - \delta) & \text{if } \ 1 \leq s < \delta + 1, \\ 0 & \text{if } s \geq \delta + 1, \end{cases}$$

and fix a function v in $L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q)$ with $v \geq 0$.

Taking $\phi = \psi_\delta(u_\varepsilon)v$ in (2.4) we have

$$\begin{aligned} &\int_0^T\int_\Omega \frac{\partial u_\varepsilon}{\partial t}\psi_\delta(u_\varepsilon)e^{\frac{-\beta_2}{\alpha_1}H_\varepsilon(u_\varepsilon)}v \\ &+ \int_Q [a(x,t) + u_\varepsilon^q]\psi_\delta(u_\varepsilon)e^{\frac{-\beta_2}{\alpha_1}H_\varepsilon(u_\varepsilon)}|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \cdot \nabla v \\ &\geq \int_Q f_\varepsilon e^{\frac{-\beta_2}{\alpha_1}H_\varepsilon(u_\varepsilon)}\psi_\delta(u_\varepsilon)v + \frac{1}{\delta}\int_{\{1\leq u_\varepsilon\leq\delta+1\}} [a(x,t) + u_\varepsilon^q]|\nabla u_\varepsilon|^p e^{\frac{-\beta_2}{\alpha_1}H_\varepsilon(u_\varepsilon)}v. \end{aligned}$$

Then, dropping the positive term, we get

$$\begin{aligned} &\int_0^T\int_\Omega \frac{\partial u_\varepsilon}{\partial t}\psi_\delta(u_\varepsilon)e^{\frac{-\beta_2}{\alpha_1}H_\varepsilon(u_\varepsilon)}v \\ &+ \int_Q [a(x,t) + u_\varepsilon^q]\psi_\delta(u_\varepsilon)e^{\frac{-\beta_2}{\alpha_1}H_\varepsilon(u_\varepsilon)}|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \cdot \nabla v \\ &\geq \int_Q f_\varepsilon e^{\frac{-\beta_2}{\alpha_1}H_\varepsilon(u_\varepsilon)}\psi_\delta(u_\varepsilon)v. \end{aligned}$$

Then, passing to the limit as δ tends to zero, we obtain

$$\begin{aligned} &\int_0^T\int_\Omega \frac{\partial u_\varepsilon}{\partial t}e^{\frac{-\beta_2}{\alpha_1}H_\varepsilon(u_\varepsilon)}v\chi_{\{0\leq u_\varepsilon<1\}} \\ &+ \int_Q [a(x,t) + u_\varepsilon^q]e^{\frac{-\beta_2}{\alpha_1}H_\varepsilon(u_\varepsilon)}|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \cdot \nabla v\chi_{\{0\leq u_\varepsilon<1\}} \\ &\geq \int_Q f_\varepsilon e^{\frac{-\beta_2}{\alpha_1}H_\varepsilon(u_\varepsilon)}v\chi_{\{0\leq u_\varepsilon<1\}}. \end{aligned}$$

As a result, we can formulate the inequality above as follows:

$$\int_0^T \int_\Omega \frac{\partial T_1(u_\varepsilon)}{\partial t} e^{\frac{-\beta_2}{\alpha_1} H_\varepsilon(T_1(u_\varepsilon))} v$$
$$+ \int_Q [a(x,t) + T_1(u_\varepsilon)^q] e^{\frac{-\beta_2}{\alpha_1} H_\varepsilon(T_1(u_\varepsilon))} |\nabla T_1(u_\varepsilon)|^{p-2} \nabla T_1(u_\varepsilon) \cdot \nabla v$$
$$\geq \int_Q f_\varepsilon e^{\frac{-\beta_2}{\alpha_1} H_\varepsilon(u_\varepsilon)} v \chi_{\{0 \leq u_\varepsilon < 1\}}.$$

From (1.2) we have $\alpha_1 \leq a(x,t) + T_1(u_\varepsilon)^q \leq \alpha_2 + 1$, then the last inequality becomes

$$\int_0^T \int_\Omega \frac{\partial T_1(u_\varepsilon)}{\partial t} e^{\frac{-\beta_2}{\alpha_1} H_\varepsilon(T_1(u_\varepsilon))} v$$
$$+ (\alpha_2 + 1) \int_Q e^{\frac{-\beta_2}{\alpha_1} H_\varepsilon(T_1(u_\varepsilon))} |\nabla T_1(u_\varepsilon)|^{p-2} \nabla T_1(u_\varepsilon) \cdot \nabla v$$
$$\geq \int_Q f_\varepsilon e^{\frac{-\beta_2}{\alpha_1} H_\varepsilon(u_\varepsilon)} v \chi_{\{0 \leq u_\varepsilon < 1\}}.$$

Since $f_\varepsilon e^{\frac{-\beta_2}{\alpha_1} H_\varepsilon(u_\varepsilon)} \chi_{\{0 \leq u_\varepsilon < 1\}}$ is not identically zero for every $0 < \varepsilon < 1$, therefore we obtain

$$\int_0^T \int_\Omega \frac{\partial T_1(u_\varepsilon)}{\partial t} e^{\frac{-\beta_2}{\alpha_1} H_\varepsilon(T_1(u_\varepsilon))} v$$
$$+ (\alpha_2 + 1) \int_Q e^{\frac{-\beta_2}{\alpha_1} H_\varepsilon(T_1(u_\varepsilon))} |\nabla T_1(u_\varepsilon)|^{p-2} \nabla T_1(u_\varepsilon) \cdot \nabla v \geq 0.$$

If we define

$$w_\varepsilon(x,t) = \int_0^{T_1(u_\varepsilon)} e^{-\frac{\beta_2}{\alpha_1} H_\varepsilon(\ell)} \, d\ell, \tag{2.5}$$

then the above inequality becomes

$$\int_0^T \int_\Omega \frac{\partial w_\varepsilon}{\partial t} v + (\alpha_2 + 1) \int_Q e^{\frac{\beta_2}{\alpha_1}(p-2) H_\varepsilon(T_1(u_\varepsilon))} |\nabla w_\varepsilon|^{p-2} \nabla w_\varepsilon \cdot \nabla v \geq 0.$$

By the fact that

$$e^{\frac{\beta_2 (p-2) 2^{1-\gamma}}{\alpha_1 (1-\gamma)}} \geq e^{\frac{\beta_2}{\alpha_1}(p-2) H_\varepsilon(T_1(u_\varepsilon))},$$

it follows that

$$\int_0^T \int_\Omega \frac{\partial w_\varepsilon}{\partial t} v + M \int_Q |\nabla w_\varepsilon|^{p-2} \nabla w_\varepsilon \cdot \nabla v \geq 0, \tag{2.6}$$

where $M = (\alpha_2 + 1)e^{\frac{\beta_2 (p-2)2^{1-\gamma}}{\alpha_1(1-\gamma)}} > 0$, for every $v \in L^p(0,T; W_0^{1,p}(\Omega)) \cap L^\infty(Q)$, with $v \geq 0$. This yields that w_ε is a weak solution to the variational inequality

$$\begin{cases} \frac{1}{M}\frac{\partial w_\varepsilon}{\partial t} - \Delta_p w_\varepsilon \geq 0 & \text{in } Q, \\ w_\varepsilon(x,t) = 0 & \text{on } \Gamma, \\ w_\varepsilon(x,0) = \int_0^{T_1(u_0)} e^{-\frac{\beta_2}{\alpha_1} H_\varepsilon(\ell)}\, d\ell & \text{in } \Omega. \end{cases}$$

We are going to prove that

$$\forall \omega \subset\subset \Omega\, \exists\, c_\omega > 0 : \; w_\varepsilon(x,t) \geq c_\omega \,\text{in}\, \omega \times (0,T), \forall\, \varepsilon \in (0,1). \tag{2.7}$$

Let v_ε be the solution of the following problem

$$\begin{cases} \frac{1}{M}\frac{\partial v_\varepsilon}{\partial t} - \Delta_p v_\varepsilon = 0 & \text{in } Q, \\ v_\varepsilon(x,t) = 0 & \text{on } \Gamma, \\ v_\varepsilon(x,0) = w_\varepsilon(x,0) & \text{in } \Omega. \end{cases} \tag{2.8}$$

From (2.6), w_ε is a super-solution of (2.8), we have $w_\varepsilon \geq v_\varepsilon$, so that we only need to prove that

$$\forall \omega \subset\subset \Omega\, \exists\, c_\omega > 0 : \; v_\varepsilon(x,t) \geq c_\omega \text{ in } \omega \times (0,T), \forall\, \varepsilon \in (0,1). \tag{2.9}$$

Observe that, by definition of w_ε (in (2.5)), we have

$$v_\varepsilon(x,0) = w_\varepsilon(x,0) = \int_0^{T_1(u_0)} e^{-\frac{\beta_2}{\alpha_1} H_\varepsilon(\ell)}\, d\ell \geq e^{-\frac{\beta_2}{\alpha_1} H_\varepsilon(1)} T_1(u_0) > 0,$$

so that, by (1.4),

$$\forall \omega \subset\subset \Omega\, \exists\, D'_\omega > 0 : \; v_\varepsilon(x,0) \geq D'_\omega \text{ in } \omega \times (0,T), \forall\, \varepsilon \in (0,1). \tag{2.10}$$

For the rest of the proof we can argue as De Bonis and De Cave in [16, pp. 957–962], (also see [25]). We deduce that there exists $c_\omega > 0$ such that $v_\varepsilon \geq c_\omega$ in $\omega \times (0,T)$ for all $\omega \subset\subset \Omega$. Since $w_\varepsilon \geq v_\varepsilon$, then $w_\varepsilon \geq c_\omega$ in $\omega \times (0,T)$ for all $\omega \subset\subset \Omega$.

As $u_\varepsilon \geq T_1(u_\varepsilon) \geq w_\varepsilon$, then we obtain

$$u_\varepsilon \geq c_\omega \,\text{in}\, \omega \times (0,T), \forall\, \omega \subset\subset \Omega, \forall\, \varepsilon \in (0,1). \qquad \square$$

We will now show some a priori estimates of the u_ε solution to the approximation problem (2.1). The following lemma gives control over the natural growth term.

Lemma 2.2. *Let u_ε be solutions to problem* (2.1)*. Then*

$$\int_Q d(x,t)\frac{u_\varepsilon|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\gamma+1}} \le |Q|^{1-\frac{1}{m}}\|f\|_{L^m(Q)} + \|u_0\|_{L^1(\Omega)}. \tag{2.11}$$

Proof. For any fixed $h>0$, let us consider $\frac{T_h(u_\varepsilon)}{h}$ as a test function in the approximation problem (2.1). Then, it results

$$\int_0^T\int_\Omega \frac{\partial u_\varepsilon}{\partial t}\frac{T_h(u_\varepsilon)}{h} + \frac{1}{h}\int_Q (a(x,t)+u_\varepsilon^q)|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \nabla T_h(u_\varepsilon)$$
$$+\int_Q d(x,t)\frac{u_\varepsilon|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\gamma+1}}\frac{T_h(u_\varepsilon)}{h} = \int_Q f_\varepsilon \frac{T_h(u_\varepsilon)}{h}.$$

Recalling (1.2), we obtain

$$\int_\Omega S_h(u_\varepsilon(x,T)) + \frac{1}{h}\int_{\{u_\varepsilon\le h\}} (\alpha_1+u_\varepsilon^q)|\nabla T_h(u_\varepsilon)|^p$$
$$+\int_Q d(x,t)\frac{u_\varepsilon|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\gamma+1}}\frac{T_h(u_\varepsilon)}{h} \le \int_Q f_\varepsilon\frac{T_h(u_\varepsilon)}{h} + \frac{1}{h}\int_\Omega S_h(u_0),$$

where $S_h(y)=\int_0^y T_h(\ell)\,d\ell$. Observe that $S_h(y)\ge \frac{T_h(y)^2}{2}$ and $S_h(y)\le yh$ for every $y\ge 0$. We may now remove the first and second non-negative terms from the previous inequality, yielding

$$\int_Q d(x,t)\frac{u_\varepsilon|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\gamma+1}}\frac{T_h(u_\varepsilon)}{h} \le \int_Q f_\varepsilon\frac{T_h(u_\varepsilon)}{h} + \int_\Omega u_0.$$

Since $u_0\in L^\infty(\Omega)$, recalling that $f_\varepsilon\le f$, $\frac{T_h(u_\varepsilon)}{h}\le 1$ and by Hölder's inequality, we find that

$$\int_Q d(x,t)\frac{u_\varepsilon|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\gamma+1}}\frac{T_h(u_\varepsilon)}{h} \le |Q|^{1-\frac{1}{m}}\|f\|_{L^m(Q)} + \|u_0\|_{L^1(\Omega)}.$$

Letting h tend to 0, by Fatou's Lemma, we conclude that (2.11) holds. □

Remark 2.3. According to Lemma 2.2, and since

$$\int_Q d(x,t)\frac{u_\varepsilon|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\gamma+1}} \ge 0, \quad f\in L^1(Q),$$

one has that

$$\int_Q \left|d(x,t)\frac{u_\varepsilon|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\gamma+1}} - f\right| \le \int_Q d(x,t)\frac{u_\varepsilon|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\gamma+1}} + \int_Q f$$
$$\le 2|Q|^{1-\frac{1}{m}}\|f\|_{L^m(Q)} + \|u_0\|_{L^1(\Omega)} < C,$$

where C is independent of ε. Therefore,

$$d(x,t)\frac{u_\varepsilon|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\gamma+1}} - f \in L^1(Q).$$

In the sequel, we will need the following lemma:

Lemma 2.4. *Let $\delta = \min(\gamma,\, 1-q)$, and let $\lambda > 0$. Then there exists $C_0 > 0$ such that*

$$\lambda(s+\varepsilon)^{\delta-1}(\alpha_1 + s^q) + \beta_1 s(s+\varepsilon)^{\lambda-1-\gamma} \ge C_0(s+\varepsilon)^{\lambda-\delta}, \tag{2.12}$$

for every $s \ge 0$.

Proof. Multiplying (2.12) by $(s+\varepsilon)^{\delta-\lambda}$, we have to prove that

$$\lambda(s+\varepsilon)^{\delta-1}(\alpha_1 + s^q) + \beta_1 s(s+\varepsilon)^{\delta-1-\gamma} \ge C_0 > 0.$$

If $\delta = \gamma$, we have to prove

$$\lambda\frac{\alpha_1 + s^q}{(s+\varepsilon)^{1-\gamma}} + \beta_1\frac{s}{s+\varepsilon} \ge C_0 > 0.$$

Clearly, if $s \ge \varepsilon$, we have $\frac{s}{s+\varepsilon} \ge \frac{1}{2}$, while $s < \varepsilon$ we find that

$$\frac{\alpha_1 + s^q}{(s+\varepsilon)^{1-\gamma}} \ge \frac{\alpha_1}{(2\varepsilon)^{1-\gamma}} \ge \frac{\alpha_1}{2^{1-\gamma}},$$

since $\varepsilon < 1$. Therefore, the claim is proved. If, instead, $\delta = 1-q$, we have to prove that

$$\lambda\frac{\alpha_1 + s^q}{(s+\varepsilon)^q} + \beta_1\frac{s}{(s+\varepsilon)^{q+\gamma}} \ge C_0 > 0,$$

which is true since the first term is greater than $\frac{\lambda}{2^q}$ if $s \ge \varepsilon$ and is greater than $\frac{\lambda\alpha_1}{2^q}$ if $s < \varepsilon$. □

Lemma 2.5. *Let the assumptions of Theorem 1.3 be in force. Then the solution u_ε of* (2.1) *satisfy the following estimate:*

(i) *If $\frac{p(N+1+\delta)}{p(N+1+\delta)-N\delta} \le m < N/p+1$, then u_ε is uniformly bounded in the space $L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\sigma(Q)$.*
(ii) *If $1 < m < \frac{p(N+1+\delta)}{p(N+1+\delta)-N\delta}$, then u_ε is uniformly bounded in the space $L^s(0,T;W_0^{1,s}(\Omega)) \cap L^\sigma(Q)$,*

where s and σ are defined in Theorem 1.3. Moreover, the sequence $u_\varepsilon^q|\nabla u_\varepsilon|^{p-1}$ is bounded in $L^\rho(Q)$, with $1 \le \rho < \frac{s}{p-1}$.

Proof. Let u_ε be a solution of (2.1). We choose $\varphi(u_\varepsilon) = (u_\varepsilon + \varepsilon)^\lambda - \varepsilon^\lambda$ (with $\lambda > 0$) as a test function in (2.1), we obtain, using the conditions (1.2) and (1.3),

$$\begin{aligned}
&\int_\Omega \Psi(u_\varepsilon(x,T)) + \lambda \int_0^t \int_\Omega (\alpha_1 + u_\varepsilon^q)(u_\varepsilon + \varepsilon)^{\lambda-1} |\nabla u_\varepsilon|^p \\
&+ \beta_1 \int_0^t \int_\Omega \frac{u_\varepsilon (u_\varepsilon + \varepsilon)^\lambda |\nabla u_\varepsilon|^p}{(u_\varepsilon + \varepsilon)^{\gamma+1}} \\
&\le \int_0^t \int_\Omega f_\varepsilon (u_\varepsilon + \varepsilon)^\lambda + \varepsilon^\lambda \int_0^t \int_\Omega d(x,t) \frac{u_\varepsilon |\nabla u_\varepsilon|^p}{(u_\varepsilon + \varepsilon)^{\gamma+1}} + \int_\Omega \Psi(u_0),
\end{aligned}$$

where $\Psi(s) = \int_0^s \varphi(\ell)\, d\ell$. Since $u_0 \in L^\infty(\Omega)$ and recalling (2.11), we have

$$\begin{aligned}
&\int_\Omega \Psi(u_\varepsilon(x,T)) + \lambda \int_0^t \int_\Omega (\alpha_1 + u_\varepsilon^q)(u_\varepsilon + \varepsilon)^{\lambda-1} |\nabla u_\varepsilon|^p \\
&+ \beta_1 \int_0^t \int_\Omega \frac{u_\varepsilon (u_\varepsilon + \varepsilon)^\lambda |\nabla u_\varepsilon|^p}{(u_\varepsilon + \varepsilon)^{\gamma+1}} \\
&\le \int_0^t \int_\Omega f_\varepsilon (u_\varepsilon + \varepsilon)^\lambda + \varepsilon^\lambda \left(\int_Q f + \|u_0\|_{L^1(\Omega)} \right) + C.
\end{aligned}$$

Therefore, the last inequality becomes

$$\begin{aligned}
&\int_\Omega \Psi(u_\varepsilon(x,T)) + \int_Q [\lambda(\alpha_1 + u_\varepsilon^q)(u_\varepsilon + \varepsilon)^{\lambda-1} + \beta_1 u_\varepsilon (u_\varepsilon + \varepsilon)^{\lambda-1-\gamma}] |\nabla u_\varepsilon|^p \\
&\le \int_0^t \int_\Omega f_\varepsilon (u_\varepsilon + \varepsilon)^\lambda + \varepsilon^\lambda \left(\int_Q f + \|u_0\|_{L^1(\Omega)} \right) + C.
\end{aligned}$$

Recalling (2.12) and by the fact that $\delta = \min(\gamma, 1-q)$, we have

$$\int_\Omega \Psi(u_\varepsilon(x,T)) + C_0 \int_0^t \int_\Omega (u_\varepsilon + \varepsilon)^{\lambda-\delta} |\nabla u_\varepsilon|^p \le \int_0^t \int_\Omega f_\varepsilon (u_\varepsilon + \varepsilon)^\lambda + C. \tag{2.13}$$

By definitions of $\Psi(s)$ and $\varphi(s)$, whenever $\lambda > 1$ we can get

$$\Psi(s) \ge \frac{s^{\lambda+1}}{\lambda+1}, \quad \forall\, s \in \mathbb{R}^+. \tag{2.14}$$

Using (2.14) in (2.13), we can write

$$\begin{aligned}&\frac{1}{\lambda+1}\int_{\Omega} u_\varepsilon(x,t)^{\lambda+1} + C_0\int_0^t\int_{\Omega}(u_\varepsilon+\varepsilon)^{\lambda-\delta}|\nabla u_\varepsilon|^p\\ &\leq \int_0^t\int_{\Omega} f_\varepsilon(u_\varepsilon+\varepsilon)^{\lambda} + C.\end{aligned} \tag{2.15}$$

Since $\lambda > 1 > \delta$, then we can write

$$\int_0^t\int_{\Omega}(u_\varepsilon+\varepsilon)^{\lambda-\delta}|\nabla u_\varepsilon|^p = \frac{p^p}{(\lambda-\delta+p)^p}\int_0^t\int_{\Omega}|\nabla[(u_\varepsilon+\varepsilon)^{\frac{\lambda-\delta+p}{p}} - \varepsilon^{\frac{\lambda-\delta+p}{p}}]|^p,$$

and

$$\frac{1}{\lambda+1}\int_{\Omega} u_\varepsilon(x,t)^{\lambda+1} = \frac{1}{\lambda+1}\int_{\Omega}[u_\varepsilon^{\frac{\lambda-\delta+p}{p}}]^{\frac{p(\lambda+1)}{\lambda-\delta+p}}.$$

From the two last equalities, the inequality (2.15) becomes

$$\begin{aligned}&\frac{1}{\lambda+1}\int_{\Omega}[u_\varepsilon^{\frac{\lambda-\delta+p}{p}}]^{\frac{p(\lambda+1)}{\lambda-\delta+p}} + \frac{C_0p^p}{(\lambda-\delta+p)^p}\int_0^t\int_{\Omega}|\nabla[(u_\varepsilon+\varepsilon)^{\frac{\lambda-\delta+p}{p}} - \varepsilon^{\frac{\lambda-\delta+p}{p}}]|^p\\ &\leq \int_0^t\int_{\Omega} f_\varepsilon(u_\varepsilon+\varepsilon)^{\lambda} + C.\end{aligned}$$

Applying Hölder's inequality with indices $(m,\, m')$, we have

$$\begin{aligned}&\frac{1}{\lambda+1}\int_{\Omega}[u_\varepsilon^{\frac{\lambda-\delta+p}{p}}]^{\frac{p(\lambda+1)}{\lambda-\delta+p}} + \frac{C_0p^p}{(\lambda-\delta+p)^p}\int_0^t\int_{\Omega}|\nabla[(u_\varepsilon+\varepsilon)^{\frac{\lambda-\delta+p}{p}} - \varepsilon^{\frac{\lambda-\delta+p}{p}}]|^p\\ &\leq \|f\|_{L^m(Q)}\left(\int_0^t\int_{\Omega}(u_\varepsilon+\varepsilon)^{\lambda m'}\right)^{\frac{1}{m'}} + C.\end{aligned} \tag{2.16}$$

Now, applying Lemma 1.1 (here $v = u_\varepsilon^{\frac{\lambda-\delta+p}{p}}$, $\rho = \frac{p(\lambda+1)}{\lambda-\delta+p}$, $h = p$) and from (2.16), we have

$$\int\limits_Q [u_\varepsilon^{\frac{\lambda-\delta+p}{p}}]^{p\frac{N+\frac{p(\lambda+1)}{\lambda-\delta+p}}{N}} \le C \left(\|u_\varepsilon^{\frac{\lambda-\delta+p}{p}}\|^{\frac{p(\lambda+1)}{\lambda-\delta+p}}_{L^\infty(0,T;L^{\frac{p(\lambda+1)}{\lambda-\delta+p}}(\Omega))} \right)^{\frac{p}{N}} \int\limits_Q |\nabla u_\varepsilon^{\frac{\lambda-\delta+p}{p}}|^p$$

$$\le C \left(\int\limits_Q (u_\varepsilon + \varepsilon)^{\lambda m'} \right)^{\frac{1}{m'}(\frac{p}{N}+1)} + C$$

$$\le C \left(\int\limits_Q u_\varepsilon^{\lambda m'} \right)^{\frac{1}{m'}(\frac{p}{N}+1)} + C_\varepsilon + C.$$

By a simple simplification, the last inequality becomes

$$\int\limits_Q u_\varepsilon^{\frac{N(\lambda-\delta+p)+p(\lambda+1)}{N}} \le C \left(\int\limits_Q u_\varepsilon^{\lambda m'} \right)^{\frac{1}{m'}(\frac{p}{N}+1)} + C. \tag{2.17}$$

Let now choosing λ such that

$$\frac{N(\lambda-\delta+p)+p(\lambda+1)}{N} = \lambda m', \tag{2.18}$$

this equivalent to $\lambda = (m-1)\frac{N(p-\delta)+p}{N-pm+p}$ and $\lambda m' = m\frac{N(p-\delta)+p}{N-pm+p} = \sigma$, since $m > 1$, then $\lambda > 0$. Combining (2.17) with (2.18), we get

$$\int\limits_Q u_\varepsilon^\sigma \le C \left(\int\limits_Q u_\varepsilon^\sigma \right)^{\frac{1}{m'}(\frac{p}{N}+1)} + C.$$

Since $m < N/p + 1$, then $\frac{1}{m'}(N/p+1)$ and applying Young's inequality with indices $\left(\frac{Nm'}{N+p}, \frac{Nm'}{Nm'-(N+p)}\right)$, it result that

$$\int\limits_Q u_\varepsilon^\sigma \le C. \tag{2.19}$$

Going back to (2.15), we obtain that by eliminating the non-negative term and using Hölder's inequality

$$C_0 \int\limits_Q (u_\varepsilon + \varepsilon)^{\lambda-\delta} |\nabla u_\varepsilon|^p \le \|f\|_{L^m(Q)} \left(\int\limits_Q u_\varepsilon^{\lambda m'} \right)^{\frac{1}{m'}} + C_\varepsilon + C.$$

Using (2.18) and (2.19) in the later inequality, we obtain

$$C_0 \int\limits_Q (u_\varepsilon + \varepsilon)^{\lambda-\delta} |\nabla u_\varepsilon|^p \leq \|f\|_{L^m(Q)} \left(\int\limits_Q u_\varepsilon^\sigma \right)^{\frac{1}{m'}} + C \leq C. \tag{2.20}$$

Now we turn to gradient estimates. If $q < 1$ and $\lambda \geq \delta$, that is, if $m \geq \frac{p(N+1+\delta)}{p(N+1+\delta)-N\delta}$, from (2.20) we obtain

$$\int\limits_Q |\nabla u_\varepsilon|^p \leq C,$$

which gives the boundedness of u_ε in $L^p(0,T;W_0^{1,p}(\Omega))$. Therefore, the proof of item (i) is achieved.

If instead $1 < m < \frac{p(N+1+\delta)}{p(N+1+\delta)-N\delta}$, i.e. if $\lambda < \delta$, we also have by the definitions of $\Psi(s)$ and $\varphi(s)$ that

$$\Psi(s) \geq C_\lambda s^{\lambda+1} - \tilde{C_\lambda}, \quad \forall s \in \mathbb{R}^+. \tag{2.21}$$

Going back to (2.13) and from (2.21), we get

$$\begin{aligned} &C_\lambda \int\limits_\Omega u_\varepsilon(x,t)^{\lambda+1} + C_0 \int\limits_0^t \int\limits_\Omega \frac{|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\delta-\lambda}} \\ &\leq \int\limits_Q f(u_\varepsilon+\varepsilon)^\lambda + \tilde{C_\lambda} \operatorname{meas}(\Omega) + C. \end{aligned}$$

Applying Hölder's inequality with indices (m, m') to the right of the previous inequality allows us to obtain

$$\begin{aligned} &C_\lambda \int\limits_\Omega u_\varepsilon(x,t)^{\lambda+1} + C_0 \int\limits_0^t \int\limits_\Omega \frac{|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\delta-\lambda}} \\ &\leq C \left(\int\limits_Q (u_\varepsilon+\varepsilon)^{\lambda m'} \right)^{\frac{1}{m'}} + C. \end{aligned}$$

Passing to the supremum in time for $t \in (0,T)$, we have

$$\begin{aligned} &C_\lambda \|u_\varepsilon\|^{\lambda+1}_{L^\infty(0,T;L^{\lambda+1}(\Omega))} + C_0 \int\limits_Q \frac{|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\delta-\lambda}} \\ &\leq C \left(\int\limits_Q (u_\varepsilon+\varepsilon)^{\lambda m'} \right)^{\frac{1}{m'}} + C. \end{aligned} \tag{2.22}$$

Let $1 < s < p$. Using Hölder's inequality with indices $\left(\frac{p}{s}, \frac{p}{p-s}\right)$, we may deduce that

$$\begin{aligned}\int_Q |\nabla u_\varepsilon|^s &= \int_Q \frac{|\nabla u_\varepsilon|^s}{(u_\varepsilon+\varepsilon)^{\frac{(\delta-\lambda)s}{p}}}(u_\varepsilon+\varepsilon)^{\frac{(\delta-\lambda)s}{p}} \\ &\le \left(\int_Q \frac{|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\delta-\lambda}}\right)^{\frac{s}{p}} \left(\int_Q (u_\varepsilon+\varepsilon)^{\frac{(\delta-\lambda)s}{p-s}}\right)^{\frac{p-s}{p}}. \end{aligned} \tag{2.23}$$

Combining the inequality (2.22) with (2.23) results in the conclusion that

$$\int_Q |\nabla u_\varepsilon|^s \le C\left(\left(\int_Q (u_\varepsilon+\varepsilon)^{\lambda m'}\right)^{\frac{1}{m'}}+1\right)^{\frac{s}{p}} \left(\int_Q (u_\varepsilon+\varepsilon)^{\frac{(\delta-\lambda)s}{p-s}}\right)^{\frac{p-s}{p}}. \tag{2.24}$$

Recalling Lemma 1.1 (here $v = u_\varepsilon, \rho = \lambda+1, h = s$), we have

$$\int_Q u_\varepsilon^{\frac{s(N+\lambda+1)}{N}} \le \|u_\varepsilon\|_{L^\infty(0,T;L^{\lambda+1}(\Omega))}^{\frac{s(\lambda+1)}{N}} \int_Q |\nabla u_\varepsilon|^s.$$

We improve on the above estimate by applying (2.22) and (2.24), yielding

$$\begin{aligned}\int_Q u_\varepsilon^{\frac{s(N+\lambda+1)}{N}} \le C&\left(\left(\int_Q (u_\varepsilon+\varepsilon)^{\lambda m'}\right)^{\frac{1}{m'}}+1\right)^{\frac{s}{p}+\frac{s}{N}} \\ &\cdot\left(\int_Q (u_\varepsilon+\varepsilon)^{\frac{(\delta-\lambda)s}{p-s}}\right)^{\frac{p-s}{p}}.\end{aligned} \tag{2.25}$$

Let we choose λ such that

$$\sigma = \frac{s(N+\lambda+1)}{N} = \lambda m' = \frac{(\delta-\lambda)s}{p-s}. \tag{2.26}$$

This is equivalent to

$$\begin{aligned}&\lambda = \frac{(m-1)(N(p-\delta)+p)}{N-pm+p}, \quad s = \frac{m(N(p-\delta)+p)}{N+1-\delta(m-1)}, \\ &\sigma = \frac{m(N(p-\delta)+p)}{N-pm+p}.\end{aligned} \tag{2.27}$$

Invoking (2.26) in (2.25), we obtain

$$\int_Q u_\varepsilon^\sigma \le C\left(\int_Q (u_\varepsilon+\varepsilon)^\sigma\right)^{\frac{1}{m'}(\frac{s}{p}+\frac{s}{N})+\frac{p-s}{p}} + C. \tag{2.28}$$

Since $\lambda < \delta$, then $m < \frac{p(N+1+\delta)}{p(N+1+\delta)-N\delta} < N/p+1$, and hence we have

$$\frac{1}{m'}\left(\frac{s}{p}+\frac{s}{N}\right)+\frac{p-s}{p}<1.$$

Applying Young's inequality, we deduce

$$\int_Q u_\varepsilon^\sigma \le C. \tag{2.29}$$

Using (2.26) in (2.24), we get

$$\int_Q |\nabla u_\varepsilon|^s \le C\left(\left(\left(\int_Q (u_\varepsilon+\varepsilon)^\sigma\right)^{\frac{1}{m'}}+1\right)^{\frac{s}{p}}\left(\int_Q (u_\varepsilon+\varepsilon)^\sigma\right)^{\frac{p-s}{p}}\right).$$

Utilizing (2.29) in the above estimate, allow us to conclude

$$\int_Q |\nabla u_\varepsilon|^s \le C. \tag{2.30}$$

The estimates (2.29) and (2.30) prove that the sequence u_ε is bounded in $L^s(0,T;W_0^{1,s}(\Omega))$ and in $L^\sigma(Q)$. Hence, the proof of item (ii) is done.

Now, we prove the estimate of $u_\varepsilon^q|\nabla u_\varepsilon|^{p-1}$ in $L^\rho(Q)$. Let $1 \le \rho < \frac{s}{p-1}$, applying Hölder's inequality with the indices $\left(\frac{s}{\rho(p-1)}, \frac{s}{s-\rho(p-1)}\right)$ and recalling the estimate (2.30), we have

$$\begin{aligned}\int_Q u_\varepsilon^{q\rho}|\nabla u_\varepsilon|^{\rho(p-1)} &\le \left(\int_Q |\nabla u_\varepsilon|^s\right)^{\frac{\rho(p-1)}{s}}\left(\int_Q u_\varepsilon^{\frac{sq\rho}{s-\rho(p-1)}}\right)^{\frac{s-\rho(p-1)}{s}} \\ &\le C\left(\int_Q u_\varepsilon^{\frac{sq\rho}{s-\rho(p-1)}}\right)^{\frac{s-\rho(p-1)}{s}}.\end{aligned} \tag{2.31}$$

Choosing ρ such that $\frac{sq\rho}{s-\rho(p-1)} = \sigma$ and recalling the estimate (2.29), we find that

$$\int_Q u_\varepsilon^{q\rho}|\nabla u_\varepsilon|^{\rho(p-1)} \le C\left(\int_Q u_\varepsilon^\sigma\right)^{\frac{s-\rho(p-1)}{s}} \le C.$$

Hence the sequence $u_\varepsilon^q|\nabla u_\varepsilon|^{p-1}$ is bounded in $L^\rho(Q)$ for every $1 \le \rho < \frac{s}{p-1}$. Therefore, the proof of the lemma is achieved. □

Lemma 2.6. *Let the assumptions of Theorem 1.5 be in force. Then the solution u_ε of* (2.1) *is uniformly bounded in $L^p(0,T;W_0^{1,p}(\Omega))\cap L^\sigma(Q)$, with σ defined in Theorem 1.5.*

Proof. Let u_ε be a solution of (2.1). If $q=1$, we can proceed as in item (i) of Theorem 1.3, we get the boundedness of u_ε in $L^\sigma(Q)$.

Now, we choose $\varphi(u_\varepsilon)=\log(1+u_\varepsilon)$ a test function in (2.1), we have

$$\int\limits_Q \frac{\partial u_\varepsilon}{\partial t}\varphi(u_\varepsilon)+\int\limits_Q (a(x,t)+u_\varepsilon)\frac{|\nabla u_\varepsilon|^p}{1+u_\varepsilon}$$
$$+\int\limits_Q d(x,t)\frac{u_\varepsilon|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\gamma+1}}\log(1+u_\varepsilon)=\int\limits_Q f_\varepsilon\log(1+u_\varepsilon).$$

From (1.3), $u_\varepsilon\geq 0$ and the fact that $\log(1+u_\varepsilon)\geq 0$, may remove the third non-negative and use (1.2), to get

$$\int\limits_Q \frac{\partial u_\varepsilon}{\partial t}\varphi(u_\varepsilon)+\min(\alpha_1,1)\int\limits_Q|\nabla u_\varepsilon|^p\leq\int\limits_Q f\log(1+u_\varepsilon). \tag{2.32}$$

Observe that $\varphi(s)=\frac{\partial\phi(s)}{\partial s}$, with $\phi(s)=s\log(1+s)$, for all $s\geq 0$. Then, we have

$$\int\limits_Q\frac{\partial u_\varepsilon}{\partial t}\varphi(u_\varepsilon)=\int\limits_0^T\frac{\partial}{\partial t}\left(\int\limits_\Omega\phi(u_\varepsilon)\right)=\int\limits_\Omega\phi(u_\varepsilon(T))-\int\limits_\Omega\phi(u_0)\geq-\int\limits_\Omega\phi(u_0).$$

Using this last affirmation in (2.32) and by Hölder's inequality, we reach that

$$\min(\alpha_1,1)\int\limits_Q|\nabla u_\varepsilon|^p\leq\int\limits_Q f\log(1+u_\varepsilon)+\int\limits_\Omega\psi(u_0)$$
$$\leq\|f\|_{L^m(Q)}\|\log(1+u_\varepsilon)\|_{L^{m'}(Q)}+\|u_0\|_{L^\infty(\Omega)}\log(1+\|u_0\|_{L^\infty(\Omega)})$$

and this gives an a priori estimate of u_ε in $L^p(0,T;W_0^{1,p}(\Omega))$ since $u_0\in L^\infty(\Omega)$ and $\log(1+u_\varepsilon)\in L^{m'}(Q)$.

If $q>1$, also we proceed as in item (i) of Theorem 1.3, we get u_ε is bounded in $L^\sigma(Q)$. Now we prove the boundedness of u_ε in $L^p(0,T;W_0^{1,p}(\Omega))$. Taking $\varphi(u_\varepsilon)=(1-(1+u_\varepsilon)^{1-q})$ as the test function in (2.1) and recall the condition (1.2), obtaining

$$\begin{aligned}&\int\limits_\Omega\Psi(u_\varepsilon(x,T))+\int\limits_Q\frac{(\alpha_1+u_\varepsilon^q)}{(u_\varepsilon+1)^q}|\nabla u_\varepsilon|^p\\&+\int\limits_Q d(x,t)\frac{u_\varepsilon|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\gamma+1}}(1-(1+u_\varepsilon)^{1-q})\\&\leq\int\limits_Q f(1-(1+u_\varepsilon)^{1-q})+\int\limits_\Omega\Psi(u_0),\end{aligned} \tag{2.33}$$

where $\Psi(s) = \int_0^s \varphi(\ell)\, d\ell$. Observing that $\Psi(u_\varepsilon(x,T)) \geq 0, f \in L^m(Q)$ and the fact that $u_0 \in L^\infty(\Omega)$, we can remove the first and third non-negative term, we arrive at that

$$\frac{\min(\alpha_1, 1)}{2^{q-1}} \int\limits_Q |\nabla u_\varepsilon|^p \leq \int\limits_Q f + C \leq C,$$

from which the boundedness of u_ε in $L^p(0,T; W_0^{1,p}(\Omega))$ follows.

The proof of the boundedness of $u_\varepsilon^q |\nabla u_\varepsilon|^{p-1}$ in $L^\rho(Q)$ with $1 \leq \rho < p'$ is similar to the one in Lemma 2.5. □

Lemma 2.7. *Let the assumptions of Theorem 1.6 be in force. Then the solution u_ε of (2.1) is bounded in $L^p(0,T; W_0^{1,p}(\Omega)) \cap L^{q+\gamma+p}(0,T; L^{\frac{N(q+\gamma+p)}{N-p}}(\Omega))$. Moreover, the sequence $u_\varepsilon^q |\nabla u_\varepsilon|^{p-1}$ is bounded in $L^\rho(Q)$ with*

$$\rho = \frac{p(p+\gamma+q)}{qp + (p-1)(p+\gamma+q)}.$$

Proof. Let u_ε be a solution of (2.1) such that u_ε converges to a solution of (1.1). Let $0 < \varepsilon < 1$, and using $\varphi(u_\varepsilon) = ((u_\varepsilon + 1)^{\gamma+1} - 1)$ a test function in (2.1), we have

$$\begin{aligned} &\int\limits_\Omega \Psi(u_\varepsilon(x,T)) + (\gamma+1) \int\limits_Q (a(x,t) + u_\varepsilon^q) |\nabla u_\varepsilon|^p (u_\varepsilon + 1)^\gamma \\ &+ \int\limits_Q d(x,t) \frac{u_\varepsilon |\nabla u_\varepsilon|^p}{(u_\varepsilon + \varepsilon)^{\gamma+1}} (u_\varepsilon + 1)^{\gamma+1} \\ &= \int\limits_Q f_\varepsilon \left((u_\varepsilon + 1)^{\gamma+1} - 1 \right) + \int\limits_Q d(x,t) \frac{u_\varepsilon |\nabla u_\varepsilon|^p}{(u_\varepsilon + \varepsilon)^{\gamma+1}} + \int\limits_\Omega \Psi(u_0), \end{aligned}$$

where $\Psi(y) = \int_0^y \varphi(\ell)\, d\ell$. Observing that φ is an increasing and positive function on $[0, +\infty)$, then $\int_\Omega \Psi(u_\varepsilon(x,T)) \geq 0$, therefore we can drop the first non-negative term, from (1.2), (1.3) and (2.11), $u_0 \in L^\infty(\Omega)$ and $c_0(u_\varepsilon + 1)^q \leq a(x,t) + u_\varepsilon^q$, and by the fact that $\frac{1}{(u_\varepsilon+1)^{\gamma+1}} \leq \frac{1}{(u_\varepsilon+\varepsilon)^{\gamma+1}}$, we deduce

$$(\gamma+1) \int\limits_Q |\nabla u_\varepsilon|^p (u_\varepsilon + 1)^{q+\gamma} + \beta_1 \int\limits_Q u_\varepsilon |\nabla u_\varepsilon|^p \leq \int\limits_Q f_\varepsilon (u_\varepsilon + 1)^{\gamma+1} + C. \tag{2.34}$$

Using Hölder's inequality with the indices $\left(m = \frac{N+p}{p}, m' = \frac{N+p}{N}\right)$, we obtain

$$\begin{aligned}(\gamma+1)\int_Q u_\varepsilon^{\gamma+q}|\nabla u_\varepsilon|^p + \beta_1\int_Q u_\varepsilon|\nabla u_\varepsilon|^p \\ \leq C\left(\int_Q (u_\varepsilon+1)^{\frac{(\gamma+1)(N+p)}{N}}\right)^{\frac{N}{N+p}} + C \\ \leq C\left(\int_Q u_\varepsilon^{\frac{(\gamma+1)(N+p)}{N}}\right)^{\frac{N}{N+p}} + C.\end{aligned} \tag{2.35}$$

By applying the Sobolev inequality and (2.35), we have

$$\begin{aligned}\int_0^T\left(\int_\Omega u_\varepsilon^{\frac{\gamma+q+p}{p}p^*}\right)^{\frac{p}{p^*}} &\leq C_s\int_0^T\left(\int_\Omega |\nabla u_\varepsilon^{\frac{q+\gamma+p}{p}}|^p\right) \\ &= C_s\left(\frac{q+\gamma+p}{p}\right)^p\int_0^T\int_\Omega u_\varepsilon^{q+\gamma}|\nabla u_\varepsilon|^p \\ &\leq C\left(\int_0^T\int_\Omega (u_\varepsilon+1)^{\frac{(\gamma+1)(N+p)}{N}}\right)^{\frac{N}{N+p}} + C \\ &\leq C\left(\int_0^T\int_\Omega u_\varepsilon^{\frac{(\gamma+1)(N+p)}{N}}\right)^{\frac{N}{N+p}} + C.\end{aligned}$$

Since $\frac{N(q+\gamma+p)}{N-p} > \frac{(\gamma+1)(N+p)}{N}$, by Hölder's inequality with indices

$$\left(\frac{N^2(q+\gamma+p)}{(N^2-p^2)(\gamma+1)}, \frac{N^2(q+\gamma+p)}{N^2(q+\gamma+p)-(N^2-p^2)(\gamma+1)}\right),$$

we find that

$$\begin{aligned}\int_0^T \|u_\varepsilon\|^{q+\gamma+p}_{L^{\frac{N(\gamma+q+p)}{N-p}}(\Omega)} &\leq C\left[\int_0^T\left(\int_\Omega u_\varepsilon^{\frac{N(q+\gamma+p)}{N-p}}\right)^{\frac{(\gamma+1)(N+p)}{N}\frac{N-p}{N(q+\gamma+p)}}\right]^{\frac{N}{N+p}} + C \\ &= C\left[\int_0^T \|u_\varepsilon\|^{\frac{(\gamma+1)(N+p)}{N}}_{L^{\frac{N(\gamma+q+p)}{N-p}}(\Omega)}\right]^{\frac{N}{N+p}} + C.\end{aligned}$$

Due to the fact that $q \geq \frac{N+p-p(N-\gamma)}{N}$, we have $q+\gamma+p > \frac{(\gamma+1)(N+p)}{N}$, then we apply Hölder's inequality in the right-hand side of the above estimate, we arrive at

$$\int_0^T \|u_\varepsilon\|^{q+\gamma+p}_{L^{\frac{N(\gamma+q+p)}{N-p}}(\Omega)} \leq C \left[\int_0^T \|u_\varepsilon\|^{q+\gamma+p}_{L^{\frac{N(\gamma+q+p)}{N-p}}(\Omega)}\right]^{\frac{\gamma+1}{q+\gamma+p}} + C.$$

Note that $\frac{\gamma+1}{q+\gamma+p} < 1$, and Young's inequality in the above estimate yields

$$\int_0^T \|u_\varepsilon\|^{q+\gamma+p}_{L^{\frac{N(\gamma+q+p)}{N-p}}(\Omega)} \leq C, \tag{2.36}$$

then this last estimate gives the boundedness of the sequence u_ε in the space $L^{q+\gamma+p}(0,T;L^{\frac{N(q+\gamma+p)}{N-p}}(\Omega))$.

Let us assume that $u_\varepsilon \geq 1$. Then, we return to (2.35), and we get that

$$\beta_1 \int_{\{u_\varepsilon \geq 1\}} |\nabla u_\varepsilon|^p \leq C \left(\int_Q u_\varepsilon^{\frac{(\gamma+1)(N+p)}{N}}\right)^{\frac{N}{N+p}} + C.$$

Since $\frac{(\gamma+1)(N+p)}{N} < q+\gamma+p < \frac{N(q+\gamma+p)}{N-p}$, we again apply Hölder's inequality twice combined with (2.36), which yields

$$\begin{aligned}\beta_1 \int_{\{u_\varepsilon \geq 1\}} |\nabla u_\varepsilon|^p &\leq C \left[\int_0^T \|u_\varepsilon\|^{\frac{(\gamma+1)(N+p)}{N}}_{L^{\frac{N(q+\gamma+p)}{N-p}}(\Omega)}\right]^{\frac{N}{N+p}} + C \\ &\leq C \left[\int_0^T \|u_\varepsilon\|^{q+\gamma+p}_{L^{\frac{N(q+\gamma+p)}{N-p}}(\Omega)}\right]^{\frac{N}{N+p}} + C.\end{aligned}$$

Then, from (2.36), it follows that

$$\int_{\{u_\varepsilon \geq 1\}} |\nabla u_\varepsilon|^p \leq C. \tag{2.37}$$

It is still necessary to investigate the behavior of ∇u_ε in $\{u_\varepsilon < 1\}$. Using $T_1(u_\varepsilon)$ as a test function in (2.1), we have

$$\begin{aligned}&\int_0^T \int_\Omega \frac{\partial u_\varepsilon}{\partial t} T_1(u_\varepsilon) + \int_Q (a(x,t) + u_\varepsilon^q)|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \nabla T_1(u_\varepsilon) \\ &+ \int_Q d(x,t)\frac{u_\varepsilon|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\gamma+1}} T_1(u_\varepsilon) = \int_Q f_\varepsilon T_1(u_\varepsilon).\end{aligned}$$

Recalling the conditions (1.2) and (1.3), we can drop the non-negative terms, then we obtain

$$\int_\Omega S_1(u_\varepsilon(x,T)) + \alpha_1 \int_{\{u_\varepsilon<1\}} |\nabla T_1(u_\varepsilon)|^p \le \int_Q f_\varepsilon T_1(u_\varepsilon) + \int_\Omega S_1(u_0),$$

where $S_1(y) = \int_0^y T_1(\ell)\, d\ell$. Observing that $\frac{T_1(y)^2}{2} \le S_1(y) \le y$ for every $y \ge 0$. By the fact that $u_0 \in L^\infty(\Omega)$ and using (2.2), we get

$$\int_{\{u_\varepsilon<1\}} |\nabla u_\varepsilon|^p = \int_Q |\nabla T_1(u_\varepsilon)|^p \le \int_Q f_\varepsilon T_1(u_\varepsilon) \le \int_Q f \le C. \tag{2.38}$$

Combining (2.37) with (2.38), we deduce that

$$\int_Q |\nabla u_\varepsilon|^p = \int_{\{u_\varepsilon\ge 1\}} |\nabla u_\varepsilon|^p + \int_{\{u_\varepsilon<1\}} |\nabla u_\varepsilon|^p \le C. \tag{2.39}$$

Then (2.36) and (2.39) imply that u_ε is uniformly bounded in $L^p(0,T;W_0^{1,p}(\Omega)) \cap L^{q+\gamma+p}(0,T;L^{\frac{N(q+\gamma+p)}{N-p}}(\Omega))$.

Now, we are going to prove the estimate of $u_\varepsilon^q|\nabla u_\varepsilon|^{p-1}$ in $L^\rho(Q)$. Let $1 \le \rho < p'$, applying Hölder's inequality, and from the estimate (2.39), we have

$$\begin{aligned} \int_Q u_\varepsilon^{q\rho}|\nabla u_\varepsilon|^{\rho(p-1)} &\le \left(\int_Q |\nabla u_\varepsilon|^p\right)^{\frac{\rho(p-1)}{p}} \left(\int_Q u_\varepsilon^{\frac{qp\rho}{p-\rho(p-1)}}\right)^{\frac{p-\rho(p-1)}{p}} \\ &\le C\left(\int_Q u_\varepsilon^{\frac{qp\rho}{p-\rho(p-1)}}\right)^{\frac{p-\rho(p-1)}{p}} \\ &= C\left(\int_0^T\int_\Omega u_\varepsilon^{\frac{qp\rho}{p-\rho(p-1)}}\right)^{\frac{p-\rho(p-1)}{p}}. \end{aligned} \tag{2.40}$$

Choosing ρ such that $\frac{qp\rho}{p-\rho(p-1)} = q+\gamma+p$, implies that $\rho = \frac{p(p+\gamma+q)}{pq+(p-1)(p+\gamma+q)}$. Since $\frac{N-p}{p} < q+\gamma+p < \frac{N(q+\gamma+p)}{N-p}$, applying Hölder's inequality twice in the right-hand side of the above inequality and recalling the estimate (2.36), we arrive at

$$\begin{aligned}\int\limits_Q u_\varepsilon^{q\rho}|\nabla u_\varepsilon|^{\rho(p-1)} &\le C\left[\int\limits_0^T\left(\int\limits_\Omega u_\varepsilon^{\frac{N(p+\gamma+q)}{N-p}}\right)^{\frac{N-p}{p}}\right]^{\frac{p-\rho(p-1)}{p}}\\ &\le C\left[\int\limits_0^T\left(\int\limits_\Omega u_\varepsilon^{\frac{N(p+\gamma+q)}{N-p}}\right)^{p+\gamma+q}\right]^{\frac{(N-p)(p-\rho(p-1))}{Np(p+\gamma+q)}}\\ &\le C\left[\int\limits_0^T \|u_\varepsilon\|^{q+\gamma+p}_{L^{\frac{N(q+\gamma+p)}{N-p}}(\Omega)}\right]^{\frac{(N-p)(p-\rho(p-1))}{Np(p+\gamma+q)}} \le C.\end{aligned} \tag{2.41}$$

Therefore the estimate of the sequence $u_\varepsilon^q|\nabla u_\varepsilon|^{p-1}$ in $L^\rho(Q)$ is achieved. □

Lemma 2.8. *Let the assumption of Theorem 1.8 be in force. Then the solution u_ε of (2.1) is bounded in $L^p(0,T;W_0^{1,p}(\Omega))\cap L^\infty(Q)$. Moreover, the sequence $u_\varepsilon^q|\nabla u_\varepsilon|^{p-1}$ is bounded in $L^{p'}(Q)$.*

Proof. For $k>0$, take $G_k(u_\varepsilon)\chi_{(0,s)}(t)$, where $0<s<T$ as test function in the approximation problem (2.1), we have

$$\begin{aligned}&\int\limits_0^s\int\limits_\Omega \frac{\partial u_\varepsilon}{\partial t}G_k(u_\varepsilon)+\int\limits_0^s\int\limits_\Omega (a(x,t)+u_\varepsilon^q)|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon\nabla G_k(u_\varepsilon)\\ &+\int\limits_0^s\int\limits_\Omega d(x,t)\frac{u_\varepsilon|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\gamma+1}}G_k(u_\varepsilon)=\int\limits_0^s\int\limits_\Omega f_\varepsilon G_k(u_\varepsilon).\end{aligned} \tag{2.42}$$

Let

$$A_{k,\varepsilon}=\{(x,t)\in Q: u_\varepsilon(x,t)>k\}.$$

Observe that the function $G_k(u_\varepsilon)$ is different from zero only on the set $A_{k,\varepsilon}$. Using the conditions (1.3), $u_\varepsilon\ge 0$ and the fact that $G_k(u_\varepsilon)\ge 0$ in $A_{k,\varepsilon}$, we have

$$\begin{aligned}\int\limits_0^s\int\limits_\Omega d(x,t)\frac{u_\varepsilon|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\gamma+1}}G_k(u_\varepsilon)&=\int\limits_{A_{k,\varepsilon}} d(x,t)\frac{u_\varepsilon|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\gamma+1}}(u_\varepsilon-k)\\ &\ge \beta_1\int\limits_{A_{k,\varepsilon}}\frac{u_\varepsilon|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\gamma+1}}(u_\varepsilon-k)\ge 0,\end{aligned}$$

$$\int_0^s \int_\Omega (a(x,t)+u_\varepsilon^q)|\nabla u_\varepsilon|^p \nabla u_\varepsilon \nabla G_k(u_\varepsilon) = \int_{A_{k,\varepsilon}(t)} (a(x,t)+u_\varepsilon^q)|\nabla u_\varepsilon|^p$$

$$\geq \alpha_1 \int_{A_{k,\varepsilon}} |\nabla u_\varepsilon|^p = \alpha_1 \int_0^s \int_\Omega |\nabla G_k(u_\varepsilon)|^p$$

and

$$\begin{aligned}\int_0^s \int_\Omega \frac{\partial u_\varepsilon}{\partial t} G_k(u_\varepsilon) &= \frac{1}{2} \int_{A_{k,\varepsilon}} \frac{d}{dt}(u_\varepsilon - k)^2 \\ &= \frac{1}{2} \int_0^t \int_\Omega \frac{d}{dt}((u_\varepsilon - k)^+)^2 \\ &= \frac{1}{2} \int_\Omega G_k^2(u_\varepsilon(x,s)) - \frac{1}{2} \int_\Omega G_k^2(u_0(x)).\end{aligned}$$

Now we can drop the third non-negative term. Then inequality (2.42) becomes

$$\frac{1}{2} \int_\Omega G_k^2(u_\varepsilon(x,s)) + \alpha_1 \int_0^s \int_\Omega |\nabla G_k(u_\varepsilon)|^p \leq \int_0^s \int_\Omega f G_k(u_\varepsilon) + \frac{1}{2} \int_\Omega G_k^2(u_0).$$

Passing to the supremum in $s \in (0,T)$, we get

$$\begin{aligned}&\|G_k(u_\varepsilon)\|^2_{L^\infty(0,T;L^2(\Omega))} + \alpha_1 \|G_k(u_\varepsilon)\|^p_{L^p(0,T;W_0^{1,p}(\Omega))} \\ &\leq \int_0^T \int_\Omega f G_k(u_\varepsilon) + \frac{1}{2} \int_\Omega G_k^2(u_0).\end{aligned} \tag{2.43}$$

From now on, we can follow the standard technique used for non-singular in the case in [4] there exists a constant $C_\infty > 0$ (independent of ε) such that

$$\|u_\varepsilon\|_{L^\infty(Q)} \leq C_\infty \text{ in } Q. \tag{2.44}$$

We choose u_ε as a test function in problem (2.1). We have

$$\frac{1}{2} \int_\Omega u_\varepsilon^2(x,T) + \int_Q |\nabla u_\varepsilon|^p + \int_Q d(x,t) \frac{u_\varepsilon^2 |\nabla u_\varepsilon|^p}{(u_\varepsilon + \varepsilon)^{\gamma+1}} = \int_Q f_\varepsilon u_\varepsilon + \frac{1}{2} \int_\Omega u_0^2.$$

Since $0 < \beta_1 \leq d(x,t)$, we can drop the first and third non-negative terms. Then we get

$$\int_Q |\nabla u_\varepsilon|^p \leq \int_Q f_\varepsilon u_\varepsilon + \frac{1}{2} \int_\Omega u_0^2.$$

Given that $u_0 \in L^\infty(\Omega)$, Hölder's inequality is applied twice, and from (2.2), (2.44), it follows that

$$\begin{aligned}\int_Q |\nabla u_\varepsilon|^p &\le \int_Q f_\varepsilon u_\varepsilon + \frac{1}{2}\|u_0\|^2_{L^2(\Omega)} \\ &\le \|u_\varepsilon\|_{L^\infty(Q)}\|f\|_{L^m(Q)}|Q|^{\frac{1}{m'}} + \frac{1}{2}\|u_0\|^2_{L^2(\Omega)} \le C.\end{aligned} \tag{2.45}$$

As a consequence of estimates (2.44) and (2.45), u_ε is uniformly bounded in $L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q)$. From Hölder's inequality combined with (2.44) and (2.45), we have

$$\int_Q |u_\varepsilon^q |\nabla u_\varepsilon|^{p-1}|^{p'} = \int_Q u_\varepsilon^{qp'} |\nabla u_\varepsilon|^p \le C_\infty^{qp'} \int_Q |\nabla u_\varepsilon|^p \le C.$$

Hence, the boundedness of the sequence $u_\varepsilon^q |\nabla u_\varepsilon|^{p-1}$ in the space $L^{p'}(Q)$ is proved. □

3. PROOF OF THEOREMS 1.3, 1.5, 1.6 AND 1.8

Since the proofs of Theorems 1.5, 1.6, and 1.8 are similar to those of Theorem 1.3, here we only give in details the proof of Theorem 1.3.

Proof of Theorem 1.3. In view of Lemma 2.5 we have two cases.

(a) If $\frac{p(N+1+\delta)}{p(N+1+\delta)-N\delta} \le m < \frac{N}{p} + 1$, then u_ε is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$.
(b) If $1 < m < \frac{p(N+1+\delta)}{p(N+1+\delta)-N\delta}$, then u_ε is bounded in $L^s(0,T;W_0^{1,s}(\Omega))$.

Therefore,

$$u_\varepsilon \rightharpoonup u \ \text{ weakly in } L^\delta(0,T;W_0^{1,\delta}(\Omega)),\ \forall\, \delta \le s < p \text{ and a.e. in } Q.$$

By Remark 2.3, $f_\varepsilon - d(x,t)\frac{u_\varepsilon|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\gamma+1}} \in L^1(Q)$ and from Lemma 2.5 we have that $(a(x,t) + u_\varepsilon^q)|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon$ is bounded in $L^\rho(Q)$, for all $1 \le \rho < \frac{s}{p-1} < p$. Then $\operatorname{div}((a(x,t) + u_\varepsilon^q)|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon)$ is bounded in the space $L^{\rho'}(Q) \subset L^{p'}(Q) \subset L^{p'}(0,T;W^{-1,p'}(\Omega))$, and then $\frac{\partial u_\varepsilon}{\partial t}$ is bounded in the space $L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(Q)$. Using the compactness results in [41], we obtain

$$u_\varepsilon \to u \text{ strongly in } L^1(Q) \text{ and a.e. in } Q. \tag{3.1}$$

Since $f_\varepsilon - d(x,t)\frac{u_\varepsilon|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\gamma+1}} \in L^1(Q)$, we can use the same proof as in [1]. Then we obtain

$$T_k(u_\varepsilon) \to T_k(u) \quad \text{strongly in } L^p(0,T;W_0^{1,p}(\Omega)), \tag{3.2}$$

and also we have

$$\nabla u_\varepsilon \to \nabla u \text{ a.e. in } Q. \tag{3.3}$$

On the other hand, recalling (1.2), (3.1), (3.3), Lemma 2.5 and the dominated convergence theorem implies that the sequence $(a(x,t)+u_\varepsilon^q)|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon$ converges weakly to $(a(x,t)+u^q)|\nabla u|^{p-2}\nabla u$ in $L^\rho(Q)$ for every $1 \le \rho < \frac{s}{p-1}$. Therefore, for every $\varphi \in C_c^1(\Omega\times[0,T))$,

$$\lim_{\varepsilon\to 0}\int\limits_Q (a(x,t)+u_\varepsilon^q)|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon\nabla\varphi = \int\limits_Q (a(x,t)+u^q)|\nabla u|^{p-2}\nabla u\nabla\varphi. \tag{3.4}$$

Now we prove that

$$d(x,t)\frac{u_\varepsilon|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\gamma+1}} \longrightarrow d(x,t)\frac{|\nabla u|^p}{u^\gamma},\ \text{strongly locally in}\ L^1(Q).$$

For any measurable compact subset E of Q, we have

$$\begin{aligned}\int\limits_E \frac{d(x,t)u_\varepsilon|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\gamma+1}} &= \int\limits_{E\cap\{u_\varepsilon\le k\}} \frac{d(x,t)u_\varepsilon|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\gamma+1}} + \int\limits_{E\cap\{u_\varepsilon>k\}} \frac{d(x,t)u_\varepsilon|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\gamma+1}}\\ &\le \int\limits_{E\cap\{u_\varepsilon\le k\}} d(x,t)\frac{|\nabla u_\varepsilon|^p}{u_\varepsilon^\gamma} + \int\limits_{E\cap\{u_\varepsilon>k\}} d(x,t)\frac{u_\varepsilon|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\gamma+1}}.\end{aligned}$$

By Lemma 2.1, we get

$$\int\limits_E d(x,t)\frac{u_\varepsilon|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\gamma+1}} \le \frac{1}{c_\omega^\gamma}\int\limits_E d(x,t)|\nabla T_k(u_\varepsilon)|^p + \int\limits_{E\cap\{u_\varepsilon>k\}} d(x,t)\frac{u_\varepsilon|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\gamma+1}}.$$

Let $\nu>0$ be fixed. For $k>1$, we use $T_1(u_\varepsilon - T_{k-1}(u_\varepsilon))$ as a test function in (2.1), yielding

$$\begin{aligned}&\int\limits_0^T\int\limits_\Omega \frac{\partial u_\varepsilon}{\partial t}T_1(u_\varepsilon - T_{k-1}(u_\varepsilon))\\ &+\int\limits_Q (a(x,t)+u_\varepsilon^q)|\nabla u_n|^{p-2}\nabla u_n\nabla T_1(u_\varepsilon - T_{k-1}(u_\varepsilon))\\ &+\int\limits_Q d(x,t)\frac{u_\varepsilon|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\gamma+1}}T_1(u_\varepsilon - T_{k-1}(u_\varepsilon)) = \int\limits_Q f_\varepsilon T_1(u_\varepsilon - T_{k-1}(u_\varepsilon)).\end{aligned}$$

Recalling (1.2) and the fact $u_\varepsilon \ge 0$, we can write

$$\begin{aligned}&\int\limits_\Omega S_1(u_\varepsilon(T)) + \alpha_1\int\limits_{\{k-1\le u_\varepsilon\le k\}} |\nabla u_\varepsilon|^p + \int\limits_Q d(x,t)\frac{u_\varepsilon|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\gamma+1}}T_1(u_\varepsilon - T_{k-1}(u_\varepsilon))\\ &\le \int\limits_Q f_\varepsilon T_1(u_\varepsilon - T_{k-1}(u_\varepsilon)) + \int\limits_\Omega S_1(u_0),\end{aligned}$$

where

$$S_1(u_\varepsilon(T)) = \int\limits_0^{u_\varepsilon(T)} T_1(s - T_{k-1}(s))\, ds.$$

It is easy to see that $S_1(u_\varepsilon(T)) \geq 0$ a.e. in Ω. After the first and second non-negative terms of the previous inequality are removed, we arrive at

$$\begin{aligned}
&\int\limits_Q d(x,t)\frac{u_\varepsilon|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\gamma+1}} T_1(u_\varepsilon - T_{k-1}(u_\varepsilon)) \\
&\leq \int\limits_Q f_\varepsilon T_1(u_\varepsilon - T_{k-1}(u_\varepsilon)) + \int\limits_\Omega S_1(u_0) \\
&= \int\limits_Q f_\varepsilon T_1(u_\varepsilon - T_{k-1}(u_\varepsilon)) + \int\limits_\Omega \int\limits_0^{u_0} T_1(s - T_{k-1}(s))ds.
\end{aligned} \tag{3.5}$$

Since $T_1(u_\varepsilon - T_{k-1}(u_\varepsilon)) \geq 0$,

$$T_1(u_\varepsilon - T_{k-1}(u_\varepsilon)) = 0 \quad \text{if} \quad u_\varepsilon \leq k-1$$

and

$$T_1(u_\varepsilon - T_{k-1}(u_\varepsilon)) = 1 \quad \text{if} \quad u_\varepsilon > k,$$

recalling the condition (1.3) and the fact that $u_\varepsilon \geq 0$, we have

$$\begin{aligned}
&\int\limits_Q d(x,t)\frac{u_\varepsilon|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\gamma+1}} T_1(u_\varepsilon - T_{k-1}(u_\varepsilon)) \\
&= \int\limits_{Q\cap\{u_\varepsilon>k\}} d(x,t)\frac{u_\varepsilon|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{+1}} T_1(u_\varepsilon - T_{k-1}(u_\varepsilon)) \\
&\quad + \int\limits_{Q\cap\{u_\varepsilon\leq k\}} d(x,t)\frac{u_\varepsilon|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\gamma+1}} T_1(u_\varepsilon - T_{k-1}(u_\varepsilon)) \\
&= \int\limits_{Q\cap\{u_\varepsilon>k\}} d(x,t)\frac{u_\varepsilon|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\gamma+1}} + \int\limits_{Q\cap\{u_\varepsilon\leq k\}} d(x,t)\frac{u_\varepsilon|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\gamma+1}} T_1(u_\varepsilon - T_{k-1}(u_\varepsilon)) \\
&\geq \int\limits_{E\cap\{u_\varepsilon>k\}} d(x,t)\frac{u_\varepsilon|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\gamma+1}}.
\end{aligned}$$

and

$$
\begin{aligned}
&\int\limits_{Q} f_\varepsilon T_1(u_\varepsilon - T_{k-1}(u_\varepsilon))\\
&= \int\limits_{Q\cap\{u_\varepsilon\le k-1\}} f_\varepsilon T_1(u_\varepsilon - T_{k-1}(u_\varepsilon)) + \int\limits_{Q\cap\{k-1<u_\varepsilon\le k\}} f_\varepsilon T_1(u_\varepsilon - T_{k-1}(u_\varepsilon))\\
&\quad + \int\limits_{Q\cap\{u_\varepsilon>k\}} f_\varepsilon T_1(u_\varepsilon - T_{k-1}(u_\varepsilon))\\
&= \int\limits_{Q\cap\{k-1<u_\varepsilon\le k\}} f_\varepsilon T_1(u_\varepsilon - (k-1)) + \int\limits_{Q\cap\{u_\varepsilon>k\}} f_\varepsilon\\
&\le \int\limits_{Q\cap\{k-1<u_\varepsilon\le k\}} f_+ \int\limits_{Q\cap\{u_\varepsilon>k\}} f = \int\limits_{Q\cap\{u_\varepsilon\ge k-1\}} f,
\end{aligned}
$$

also we have

$$
\int\limits_{\Omega} S_1(u_0) = \int\limits_{\Omega}\int\limits_{0}^{u_0} T_1(s - T_{k-1}(s))ds = \int\limits_{\Omega}\int\limits_{[0,u_0]\cap\{s\ge k-1\}} T_1(s - T_{k-1}(s))ds.
$$

Therefore, from (3.5) combined with the two later inequalities and the above equality, we obtain

$$
\int\limits_{E\cap\{u_\varepsilon>k\}} d(x,t)\frac{u_\varepsilon|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\gamma+1}} \le \int\limits_{Q\cap\{u_\varepsilon\ge k-1\}} f + \int\limits_{E}\int\limits_{[0,u_0]\cap\{s\ge k-1\}} T_1(s - T_{k-1}(s))ds.
$$

It follows from $f \in L^m(Q)$ and $T_1(s - T_{k-1}(s)) \in L^1(\Omega)$ that

$$
\int\limits_{E\cap\{u_\varepsilon>k\}} d(x,t)\frac{u_\varepsilon|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\gamma+1}} \longrightarrow 0 \ \text{ as } k \longrightarrow \infty.
$$

Then, there exists $k_0 > 1$ such that

$$
\int\limits_{E\cap\{u_\varepsilon>k\}} d(x,t)\frac{u_\varepsilon|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\gamma+1}} \le \frac{\nu}{2}, \quad \forall k \ge k_0, \forall\, \varepsilon \in (0,T). \tag{3.6}
$$

Since from (3.2) ($T_k(u_\varepsilon) \to T_k(u)$ strongly in $L^p(0,T;W_0^{1,p}(\Omega))$), then there exits $\varepsilon_\nu, \theta_\nu$ such that $|E| \le \theta_\nu$, and we have

$$
\frac{1}{c_\omega^\gamma}\int\limits_{E} d(x,t)|\nabla T_k(u_\varepsilon)|^p \le \frac{\nu}{2}, \quad \forall\, \varepsilon \le \varepsilon_\nu. \tag{3.7}
$$

The estimates (3.6) and (3.7) imply that $d(x,t)\frac{u_\varepsilon|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\gamma+1}}$ is equi-integrable. This fact, together with the a.e. convergence of this term to $d(x,t)\frac{|\nabla u|^p}{u^\gamma}$, implies by the Vitali Theorem that

$$d(x,t)\frac{u_\varepsilon|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\gamma+1}} \longrightarrow d(x,t)\frac{|\nabla u|^p}{u^\gamma}, \text{ locally strongly in } L^1(Q). \tag{3.8}$$

Let $\varphi \in C_c^1(\Omega\times[0,T))$, taking φ test function in problem (2.1), by (2.2), (3.1), (3.4) and (3.8), we can let $\varepsilon \to 0$ yielding

$$\begin{aligned}&-\int_Q u\frac{\partial\varphi}{\partial t} + \int_Q (a(x,t)+u^q)|\nabla u|^{p-2}\nabla u\cdot\nabla\varphi + \int_Q d(x,t)\frac{|\nabla u|^p}{u^\gamma}\varphi\\ &= \int_Q f\varphi + \int_\Omega u_0(x)\varphi(x,0).\end{aligned}$$

Thus, Theorem 1.3 is proved. □

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