Dedicated with great pleasure to our friend Petru Cojuhari
on the occasion of his 65th birthday

EIGENVALUE ESTIMATES FOR OPERATORS WITH FINITELY MANY NEGATIVE SQUARES

Jussi Behrndt, Roland Möws, and Carsten Trunk

Communicated by A. Shkalikov

Abstract. Let A and B be selfadjoint operators in a Krein space. Assume that the resolvent difference of A and B is of rank one and that the spectrum of A consists in some interval $I \subset \mathbb{R}$ of isolated eigenvalues only. In the case that A is an operator with finitely many negative squares we prove sharp estimates on the number of eigenvalues of B in the interval I. The general results are applied to singular indefinite Sturm-Liouville problems.

Keywords: selfadjoint operator, Krein space, finitely many negative squares, eigenvalue estimate, indefinite Sturm-Liouville operator.

Mathematics Subject Classification: 47A55, 47B50.

1. INTRODUCTION

Let A and B be bounded or unbounded selfadjoint operators in a Krein space $(\mathcal{K}, [\cdot, \cdot])$ such that $\rho(A) \cap \rho(B) \neq \emptyset$ and assume that the resolvent difference of A and B is a rank one operator,

$$\dim \left(\operatorname{ran} \left((A - \lambda_0)^{-1} - (B - \lambda_0)^{-1} \right) \right) = 1$$

for some, and hence for all, $\lambda_0 \in \rho(A) \cap \rho(B)$. The main objective of this note is to provide sharp bounds on the number of distinct isolated eigenvalues of B in an open interval $I \subset \mathbb{R}$ in terms of the number of distinct isolated eigenvalues of A in I. In our considerations also eigenvalues of infinite geometric or algebraic multiplicity are allowed and hence multiplicities are not counted.

It is necessary to impose additional assumptions on the operators A and B in order to obtain meaningful bounds and estimates in the Krein space setting. In particular, in general it may happen that an open interval $I \subset \mathbb{R}$ is contained in $\rho(A)$ but the set $\sigma_p(B) \cap I$ is infinite, or I may even be in the interior of $\sigma(B)$, see, e.g.,

[25, §5, Remark 2]. In the recent work [3] the special case of a selfadjoint operator A which is nonnegative with respect to the indefinite inner product $[\cdot,\cdot]$ was considered; we refer the reader there for more details and further references.

In the present paper we go beyond the nonnegative case and consider a selfadjoint operator A with finitely many negative squares κ_A, that is, the form $[A\cdot,\cdot]$ has negative values on a κ_A-dimensional subspace of dom A, but there is no $\kappa_A + 1$-dimensional subspace with this property. We then employ typical techniques from perturbation theory to study interlacing properties of eigenvalues of A and B in a gap of the essential spectrum. Of particular interest are those eigenvalues of A and B which may destroy natural interlacing, and it is essential for our estimates that these special eigenvalues can be controlled in terms of a local quantity related to the number of negative squares of A and B, respectively. In summary this analysis leads to upper and lower bounds on the number of distinct eigenvalues of B in a gap of the essential spectrum of A in our first main result Theorem 3.1. It is remarkable that all these bounds are sharp; this is our second main result formulated as Theorem 4.1.

Finally, in the last section, the abstract results are illustrated for a typical Sturm-Liouville eigenvalue problem with an indefinite weight function which gives rise to a selfadjoint operator with finitely many negative squares in a weighted L^2-Krein space.

2. OPERATORS WITH FINITELY MANY NEGATIVE SQUARES AND RANK ONE PERTURBATIONS

A complex linear space $\mathcal{K}$ with a nondegenerate hermitian sesquilinear form $[\cdot,\cdot]$ is called a *Krein space* if there exists a decomposition

$$\mathcal{K} = \mathcal{K}_+ \dot{+} \mathcal{K}_-$$

such that the subspaces $(\mathcal{K}_\pm, \pm[\cdot,\cdot])$ are Hilbert spaces and orthogonal to each other with respect to $[\cdot,\cdot]$. An element x in the Krein space $(\mathcal{K},[\cdot,\cdot])$ is *positive* (*negative, neutral*) if $[x,x] > 0$ ($[x,x] < 0$, $[x,x] = 0$, respectively). For further information on Krein spaces we refer to the monographs [1] and [11].

For a densely defined linear operator A in the Krein space $(\mathcal{K},[\cdot,\cdot])$ the adjoint with respect to the indefinite inner product $[\cdot,\cdot]$ is denoted by A^+. The operator A is called *selfadjoint* if $A = A^+$ and *symmetric* if $A \subset A^+$. We denote the point spectrum by $\sigma_p(A)$, the spectrum by $\sigma(A)$ and the resolvent set by $\rho(A)$. The *root subspace* $\mathcal{L}_\lambda(A)$ *of A at λ* is the collection of all Jordan chains, $\mathcal{L}_\lambda(A) := \left\{ x \in \ker(A-\lambda)^j \ : \ j \in \mathbb{N} \right\}$. A real isolated eigenvalue λ of A is called of *positive* (*negative*) *type* if all its corresponding eigenvectors are positive (negative, respectively). In this case we write $\lambda \in \sigma_{++}(A)$ ($\lambda \in \sigma_{--}(A)$, respectively). An isolated eigenvalue λ of A which is not of positive neither of negative type is called a *critical point* of A and we write $\lambda \in c(A)$. Observe that for an isolated eigenvalue of positive or negative type there is no Jordan chain of length greater than one, that is, $\mathcal{L}_\lambda(A) = \ker(A-\lambda)$.

A selfadjoint operator A in the Krein space $(\mathcal{K}, [\cdot, \cdot])$ with $\rho(A) \neq \emptyset$ has κ_A *negative squares* if for some $\kappa_A \in \mathbb{N}_0$ the hermitian form $\langle \cdot, \cdot \rangle$ on dom A, defined by

$$\langle f, g \rangle := [Af, g], \qquad f, g \in \text{dom } A,$$

has κ_A negative squares, that is, there exists a κ_A-dimensional subspace $\mathcal{M}$ in dom A such that $\langle v, v \rangle < 0$ if $v \in \mathcal{M}$, $v \neq 0$, but there exists no $\kappa_A + 1$ dimensional subspace with this property. Selfadjoint operators with finitely many negative squares belong to the class of definitizable operators introduced and comprehensively studied by H. Langer in [23, 24]. We recall some well-known spectral properties of operators with finitely many negative squares. The statements in Theorem 2.1 below can be found in [23, 24], see also [6, Theorem 3.1].

Theorem 2.1. *Let A be a selfadjoint operator in the Krein space $(\mathcal{K}, [\cdot, \cdot])$ and assume that A has κ_A negative squares. Then the nonreal spectrum of A consists of at most κ_A pairs $\{\mu_i, \bar{\mu}_i\}$, $\text{Im } \mu_i > 0$, of eigenvalues with finite-dimensional root subspaces. If $\lambda \in \sigma_p(A)$ and $\{\kappa_-(\lambda), \kappa_0(\lambda), \kappa_+(\lambda)\}$ denotes the signature of the root subspace $\mathcal{L}_\lambda(A)$ then*

$$\sum_{\lambda \in \sigma_p(A) \cap (0, \infty)} \big(\kappa_-(\lambda) + \kappa_0(\lambda)\big) + \sum_{\lambda \in \sigma_p(A) \cap (-\infty, 0)} \big(\kappa_+(\lambda) + \kappa_0(\lambda)\big) + \sum_i \kappa_0(\mu_i) \leq \kappa_A$$

and if $0 \notin \sigma_p(A)$ then equality holds. Moreover, there are at most κ_A different real nonzero eigenvalues of A with corresponding Jordan chains of length greater than one.

Let A and B be selfadjoint operators in the Krein space $(\mathcal{K}, [\cdot, \cdot])$, let $\rho(A) \cap \rho(B) \neq \emptyset$ and assume that

$$\dim \text{ran} \big((A - \lambda_0)^{-1} - (B - \lambda_0)^{-1}\big) = 1 \tag{2.1}$$

holds for some, and hence for all, $\lambda_0 \in \rho(A) \cap \rho(B)$. It is not difficult to see that if A and B satisfy (2.1) and A has κ_A negative squares then it follows that also the operator B has $\kappa_B \geq 0$ negative squares, where

$$|\kappa_A - \kappa_B| \leq 1. \tag{2.2}$$

We recall a well known factorization of the resolvent difference of A and B with the help of scalar functions which can be viewed as Weyl functions or Q-functions corresponding to A and B, respectively; cf. [14, 26]. Proposition 2.2 below is taken from [3], where its proof was omitted. For the convenience of the reader a short proof of the resolvent formula with the help of boundary triples and their γ-fields and Weyl functions (see [13–15]) is given.

Proposition 2.2. *Let A and B be selfadjoint operators in the Krein space $(\mathcal{K}, [\cdot, \cdot])$ which satisfy (2.1). Then there exist holomorphic functions $M_A : \rho(A) \to \mathbb{C}$, $M_B : \rho(B) \to \mathbb{C}$ symmetric with respect to the real line and vectors φ_A, φ_B in $\mathcal{K}$ such that the following holds.*

(i) *For $\gamma_A(\lambda) := (1 + (\lambda - \lambda_0)(A - \lambda)^{-1})\varphi_A$, $\lambda \in \rho(A)$, we have*

$$M_A(\lambda) - M_A(\overline{\omega}) = (\lambda - \overline{\omega})[\gamma_A(\lambda), \gamma_A(\omega)], \quad \lambda, \omega \in \rho(A).$$

(ii) *For $\gamma_B(\lambda) := (1 + (\lambda - \lambda_0)(B - \lambda)^{-1})\varphi_B$, $\lambda \in \rho(B)$, we have*

$$M_B(\lambda) - M_B(\overline{\omega}) = (\lambda - \overline{\omega})[\gamma_B(\lambda), \gamma_B(\omega)], \quad \lambda, \omega \in \rho(B).$$

(iii) *For $\lambda \in \rho(A) \cap \rho(B)$ we have $M_B(\lambda) = -\frac{1}{M_A(\lambda)}$ and*

$$(A - \lambda)^{-1} - (B - \lambda)^{-1} = \frac{1}{M_A(\lambda)}[\cdot, \gamma_A(\overline{\lambda})]\gamma_A(\lambda) = -\frac{1}{M_B(\lambda)}[\cdot, \gamma_B(\overline{\lambda})]\gamma_B(\lambda).$$

Proof. Consider $S = A \cap B$, which is a (possibly nondensely defined) closed symmetric operator in $(\mathcal{K}, [\cdot, \cdot])$ of defect one. As in [4, Corollary 2.5] it follows that there exists a boundary triple $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ for the adjoint S^+ such that $A = S^+ \upharpoonright \ker \Gamma_0$ and $B = S^+ \upharpoonright \ker \Gamma_1$. Let γ and M be the corresponding γ-field and Weyl function and define $\varphi_A := \gamma(\lambda_0)$. It follows from the property $\gamma(\lambda) = (1 + (\lambda - \lambda_0)(A - \lambda)^{-1})\gamma(\lambda_0)$, $\lambda \in \rho(A)$, that $\gamma_A = \gamma$ holds. Moreover, $M_A := M$ satisfies the formula in (i). Observe that $\{\mathbb{C}, \Gamma_1, -\Gamma_0\}$ is also a boundary triple for S^+. Let $\widetilde{\gamma}$ and $\widetilde{M}$ be the corresponding γ-field and Weyl function and define $\varphi_B := \widetilde{\gamma}(\lambda_0)$. As above it follows that $\gamma_B = \widetilde{\gamma}$ and $M_B := \widetilde{M}$ satisfies the formula in (ii). It follows from the definition of the Weyl function corresponding to a boundary triple that $\widetilde{M}(\lambda) = -M(\lambda)^{-1}$, and hence $M_B(\lambda) = -M_A(\lambda)^{-1}$, $\lambda \in \rho(A) \cap \rho(B)$, as stated in (iii). The formula in (iii) is a special case of [13, Theorem 2.1] (see also [15, Theorem 3.1]). □

From now on we will suppose that the following assumption is satisfied.

Assumption (I). Let A and B be selfadjoint operators in the Krein space $(\mathcal{K}, [\cdot, \cdot])$ such that condition (2.1) on the resolvent difference of A and B holds for some, and hence for all, $\lambda_0 \in \rho(A) \cap \rho(B)$. Let $I \subset \mathbb{R}$ be an open interval and assume that $\rho(B) \cap I \neq \emptyset$ and that $\sigma(A) \cap I$ consists only of isolated eigenvalues which are poles of the resolvent of A.

From Assumption (I) and general perturbation results (see e.g. [16, 18]) it follows that the set $\sigma(B) \cap I$ consists only of eigenvalues which may only accumulate to the eigenvalues of infinite algebraic multiplicity of A or to the boundary of I. Note that any eigenvalue of A with an infinite dimensional root subspace is also an eigenvalue of B with an infinite dimensional root subspace. Furthermore, if $\mu \in \rho(A) \cap I$ then either $\mu \in \rho(B)$ or $\mu \in \sigma_p(B)$ with $\dim \ker(B - \mu) = 1$.

The next proposition is essentially a consequence of Proposition 2.2. For a proof we refer the reader to [3, Corollary 2.2 and Lemma 2.4].

Proposition 2.3. *Let A, B be as in Assumption* (I). *Then for all $\lambda \in I \cap \rho(A)$ we have*

(i) $\lambda \in \sigma_p(B)$ *if and only if $M_A(\lambda) = 0$;*
(ii) $\lambda \in \sigma_{++}(B)$ *if and only if $M_A(\lambda) = 0$ and $M_A'(\lambda) > 0$;*
(iii) $\lambda \in \sigma_{--}(B)$ *if and only if $M_A(\lambda) = 0$ and $M_A'(\lambda) < 0$.*

Let A and B be as in Assumption (I). According to [3, Lemma 2.5] the following holds: If $\mu \in I \cap \sigma_{++}(A)$ $(\mu \in I \cap \sigma_{--}(A))$ with $\mu \in \rho(B)$ then the function M_A has a pole at μ of order one with

$$\lim_{\lambda \nearrow \mu} M_A(\lambda) = +\infty, \quad \lim_{\lambda \searrow \mu} M_A(\lambda) = -\infty$$

$$\left(\lim_{\lambda \nearrow \mu} M_A(\lambda) = -\infty, \quad \lim_{\lambda \searrow \mu} M_A(\lambda) = +\infty, \text{ respectively} \right).$$

This observation immediately leads to the following interlacing property.

Proposition 2.4. *Let A, B be as in Assumption* (I). *Let $\mu_1, \mu_2 \in \rho(B) \cap I$ be such that the interval (μ_1, μ_2) satisfies $(\mu_1, \mu_2) \subset \rho(A)$.*

(i) *If $\mu_1, \mu_2 \in \sigma_{++}(A)$ then there exists $\mu \in (\mu_1, \mu_2)$ with $\mu \in \sigma_p(B) \setminus \sigma_{--}(B)$.*
(ii) *If $\mu_1, \mu_2 \in \sigma_{--}(A)$ then there exists $\mu \in (\mu_1, \mu_2)$ with $\mu \in \sigma_p(B) \setminus \sigma_{++}(B)$.*

3. EIGENVALUE ESTIMATES

Let A and B be as in Assumption (I) and assume that A has finitely many negative squares κ_A. The next theorem provides estimates from below and above on the number of distinct eigenvalues of B in terms of the number of distinct eigenvalues of A. This is the first main result of this paper. In order to formulate it, we denote the numbers of distinct eigenvalues of A and B in I by $n_A(I)$ and $n_B(I)$, respectively,

$$n_A(I) = \sharp\{\lambda : \lambda \in I \cap \sigma_p(A)\} \quad \text{and} \quad n_B(I) = \sharp\{\lambda : \lambda \in I \cap \sigma_p(B)\},$$

and we denote the number of common eigenvalues of A and B in I by $n_{A,B}(I)$,

$$n_{A,B}(I) = \sharp\{\lambda : \lambda \in I \cap \sigma_p(A) \cap \sigma_p(B)\};$$

here the symbol $\sharp$ stands for the number of elements in a given set. We emphasize that the multiplicities of the eigenvalues are not counted here. For a subset Δ of I the numbers $\mathfrak{K}_A(\Delta)$ and $\mathfrak{K}_B(\Delta)$ are defined as

$$\mathfrak{K}_A(\Delta) = \sharp\{\lambda \in \sigma_p(A) \cap \Delta : \lambda \in (\mathbb{R}^+ \setminus \sigma_{++}(A)) \cup (\mathbb{R}^- \setminus \sigma_{--}(A))\} \tag{3.1}$$

and

$$\mathfrak{K}_B(\Delta) = \sharp\{\lambda \in \sigma_p(B) \cap \Delta : \lambda \in (\mathbb{R}^+ \setminus \sigma_{++}(B)) \cup (\mathbb{R}^- \setminus \sigma_{--}(B))\},$$

respectively; where $\mathbb{R}^+ = (0, \infty)$ and $\mathbb{R}^- = (-\infty, 0)$. By Theorem 2.1 both numbers $\mathfrak{K}_A(\Delta)$ and $\mathfrak{K}_B(\Delta)$ are finite and satisfy the estimates

$$\mathfrak{K}_A(\Delta) \leq \kappa_A \quad \text{and} \quad \mathfrak{K}_B(\Delta) \leq \kappa_B. \tag{3.2}$$

We are ready to formulate our first main result. We mention that all estimates in the next theorem will turn out to be sharp, see Theorem 4.1 below.

Theorem 3.1. *Let A, B be as in Assumption* (I) *and assume that A has κ_A negative squares. Then the following estimates hold.*

(i) *If $n_A(I) < \infty$ and $0 \notin I$ then*

$$n_A(I) - n_{A,B}(I) - 2\mathfrak{K}_A(I \cap \rho(B)) - 1 \leq n_B(I) \leq n_A(I) + n_{A,B}(I) + 2\mathfrak{K}_B(I \cap \rho(A)) + 1.$$

(ii) *If $n_A(I) < \infty$ and $0 \in I$ then*

$$n_B(I) \geq n_A(I) - n_{A,B}(I) - 2\mathfrak{K}_A(I \cap \rho(B)) - \begin{cases} 3 & \text{if } 0 \in \rho(B) \cap c(A), \\ 2 & \text{if } 0 \in \rho(B) \setminus c(A), \\ 1 & \text{if } 0 \in \sigma(B), \end{cases}$$

and

$$n_B(I) \leq n_A(I) + n_{A,B}(I) + 2\mathfrak{K}_B(I \cap \rho(A)) + \begin{cases} 3 & \text{if } 0 \in \rho(A) \cap c(B), \\ 2 & \text{if } 0 \in \rho(A) \setminus c(B), \\ 1 & \text{if } 0 \in \sigma(A). \end{cases}$$

(iii) *We have $n_A(I) = \infty$ if and only if $n_B(I) = \infty$.*

Proof. (i) Let $n_A(I) < \infty$ and $0 \notin I$. It is no restriction to assume $I \subset \mathbb{R}^+$. Denote by $n \geq 0$ the number of points in the open interval I which are in $\sigma(A) \cap \sigma(B)$ or in $\sigma(A) \setminus \sigma_{++}(A)$. Obviously,

$$n = n_{A,B}(I) + \mathfrak{K}_A(I \cap \rho(B)). \tag{3.3}$$

The open interval I without these n points decomposes into $n + 1$ open intervals I_j, $j = 1, \ldots, n+1$. Each I_j satisfies $I_j \cap \sigma(A) \subset \sigma_{++}(A) \cap \rho(B)$. In view of Proposition 2.4 we have for $j = 1, \ldots, n + 1$

$$n_A(I_j) - 1 \leq n_B(I_j).$$

Summing up, we obtain

$$\left(\sum_{j=1}^{n+1} n_A(I_j) \right) - n - 1 \leq \sum_{j=1}^{n+1} n_B(I_j). \tag{3.4}$$

With (3.4) and with the identity $n_A(I) = n + \sum_{j=1}^{n+1} n_A(I_j)$ we obtain

$$n_B(I) = \left(\sum_{j=1}^{n+1} n_B(I_j) \right) + n_{A,B}(I) \geq \left(\sum_{j=1}^{n+1} n_A(I_j) \right) - n - 1 + n_{A,B}(I)$$

$$= n_A(I) - 2n - 1 + n_{A,B}(I).$$

Then (3.3) implies the lower estimate in (i). We recall from [5, Theorem 4.3 and Remark 4.4] that the assumption $n_A(I) < \infty$ is equivalent to $n_B(I) < \infty$, and hence by interchanging the roles of A and B, also the upper estimate in (i) is shown.

(ii) Let $n_A(I) < \infty$ and $0 \in I$. We show the lower estimate in (ii); the upper estimate follows by interchanging the roles of A and B. We apply (i) to the intervals $I \cap \mathbb{R}^+$ and $I \cap \mathbb{R}^-$ and obtain

$$n_B(I \cap \mathbb{R}^\pm) \geq n_A(I \cap \mathbb{R}^\pm) - n_{A,B}(I \cap \mathbb{R}^\pm) - 2\mathfrak{K}_A(I \cap \mathbb{R}^\pm \cap \rho(B)) - 1.$$

By adding the inequality for the interval $I \cap \mathbb{R}^-$ and the inequality for $I \cap \mathbb{R}^+$ we derive

$$n_B(I \setminus \{0\}) \geq n_A(I \setminus \{0\}) - n_{A,B}(I \setminus \{0\}) - 2\mathfrak{K}_A\left((I \setminus \{0\}) \cap \rho(B)\right) - 2. \qquad (3.5)$$

Obviously, if zero is in $\rho(A)$, then $n_A(I) = n_A(I \setminus \{0\})$. Otherwise $n_A(I) = n_A(I \setminus \{0\}) + 1$ and the same identities are valid for A replaced by B. Similarly, if zero is in $\sigma(A) \cap \sigma(B)$, then $n_{A,B}(I) = n_{A,B}(I \setminus \{0\}) + 1$. Otherwise $n_{A,B}(I) = n_{A,B}(I \setminus \{0\})$. From the definition of the number $\mathfrak{K}_A$ in (3.1) it follows $\mathfrak{K}_A(\Delta) = \mathfrak{K}_A(\Delta \setminus \{0\})$ for every subset Δ of I and hence we have with (3.5)

$$n_B(I) \geq n_A(I) - n_{A,B}(I) - 2\mathfrak{K}_A(I \cap \rho(B)) - \begin{cases} 3 & \text{if } 0 \in \rho(B) \cap \sigma(A), \\ 2 & \text{if } 0 \in \rho(B) \cap \rho(A), \\ 1 & \text{if } 0 \in \sigma(B). \end{cases} \qquad (3.6)$$

In order to show the lower bound in (ii) it remains to consider the case $0 \in (\rho(B) \cap \sigma(A)) \setminus c(A)$ and to show

$$n_B(I) \geq n_A(I) - n_{A,B}(I) - 2\mathfrak{K}_A(I \cap \rho(B)) - 2. \qquad (3.7)$$

For a spectral point λ of A which is in $I \setminus c(A)$ we have $\lambda \in \sigma_{++}(A) \cup \sigma_{--}(A)$. Hence, if $0 \in (\rho(B) \cap \sigma(A)) \setminus c(A)$, then

$$0 \in \rho(B) \cap (\sigma_{++}(A) \cup \sigma_{--}(A))$$

and it is sufficient to show (3.7) for $0 \in \rho(B) \cap \sigma_{++}(A)$. The case $0 \in \rho(B) \cap \sigma_{--}(A)$ is treated similarly. Therefore, we assume in the following

$$0 \in \rho(B) \cap \sigma_{++}(A). \qquad (3.8)$$

Then there exists $\epsilon > 0$ such that

$$[-\epsilon, 0) \subset \rho(A) \cap \rho(B)$$

and we set

$$A_\epsilon := A + \epsilon, \quad B_\epsilon := B + \epsilon, \quad \text{and} \quad I_\epsilon := \{\lambda + \epsilon : \lambda \in I\}.$$

It is not difficult to see that Theorem 2.1 and (3.8) yield that A_ϵ (B_ϵ) is also an operator with κ_A (resp. κ_B) negative squares and that Assumption (I) holds for A and B replaced by A_ϵ and B_ϵ. Furthermore, the spectrum of A_ϵ in I_ϵ consists of isolated eigenvalues only. Then (3.6) is also valid for A_ϵ, B_ϵ, I_ϵ and, as $0 \in \rho(A_\epsilon) \cap \rho(B_\epsilon)$, we have

$$n_{B_\epsilon}(I_\epsilon) \geq n_{A_\epsilon}(I_\epsilon) - n_{A_\epsilon, B_\epsilon}(I_\epsilon) - 2\mathfrak{K}_{A_\epsilon}(I_\epsilon \cap \rho(B_\epsilon)) - 2. \qquad (3.9)$$

Obviously, we have

$$n_A(I) = n_{A_\epsilon}(I_\epsilon), \quad n_B(I) = n_{B_\epsilon}(I_\epsilon), \quad \text{and} \quad n_{A,B}(I) = n_{A_\epsilon, B_\epsilon}(I_\epsilon). \qquad (3.10)$$

In particular, (3.8) implies

$$\mathfrak{K}_A(I \cap \rho(B)) = \mathfrak{K}_{A_\epsilon}(I_\epsilon \cap \rho(B_\epsilon)) \qquad (3.11)$$

and, therefore, (3.7) follows from (3.9), (3.10), and (3.11).

(iii) is a special case of [5, Theorem 4.3 and Remark 4.4]. $\qquad \square$

In view of (2.2) and (3.2) the operator B has $\kappa_B \leq \kappa_A + 1$ negatives squares and

$$\mathfrak{K}_A(I \cap \rho(B)) \leq \mathfrak{K}_A(I) \leq \kappa_A \quad \text{and} \quad \mathfrak{K}_B(I \cap \rho(A)) \leq \mathfrak{K}_B(I) \leq \kappa_B \leq \kappa_A + 1.$$

This implies the following corollary.

Corollary 3.2. *If $n_A(I) < \infty$ and $0 \notin I$ then*

$$n_A(I) - n_{A,B}(I) - 2\kappa_A - 1 \leq n_B(I) \leq n_A(I) + n_{A,B}(I) + 2\kappa_A + 3.$$

If $n_A(I) < \infty$ and $0 \in I$ then

$$n_A(I) - n_{A,B}(I) - 2\kappa_A - 3 \leq n_B(I) \leq n_A(I) + n_{A,B}(I) + 2\kappa_A + 5.$$

4. SHARPNESS

In this section we show that all the estimates in Theorem 3.1 are sharp in the following sense: There exists an interval I and two operators A, B satisfying Assumption (I) for which we obtain equality in the formulas in (i) and (ii) in Theorem 3.1. In particular, for given natural numbers k, n which stand for $\mathfrak{K}_B(I \cap \rho(A))$ or $\mathfrak{K}_A(I \cap \rho(B))$ and $n_A(I)$, respectively, and every number p ($= n_{A,B}(I)$) smaller than n we find for each of the inequalities in (i) and (ii) of Theorem 3.1 A, B and I such that equality holds. In this sense, the inequalities in Theorem 3.1 are optimal and the next theorem complements the estimates in Theorem 3.1. The following proof relies on minimal realizations of rational generalized Nevanlinna functions in finite dimensional Pontryagin spaces, and makes use of boundary triple techniques in a similar way as in the proof of Proposition 2.2. A similar method was used in [3] to show sharpness of related eigenvalue estimates for nonnegative operators in Krein spaces.

Theorem 4.1. *The following assertions* (i)–(iv) *hold.*

(i) *Let $k, n \in \mathbb{N}$ and $p \in \{0, 1, \ldots, n\}$. Then there exists an open interval $I \subset \mathbb{R}^+$, a finite dimensional Krein space $\mathcal{K}$ and selfadjoint matrices A and B in $\mathcal{K}$ which satisfy Assumption* (I) *such that*

$$n_B(I) = n_A(I) + n_{A,B}(I) + 2\mathfrak{K}_B(I \cap \rho(A)) + 1$$

holds with $n_A(I) = n$, $\mathfrak{K}_B(I \cap \rho(A)) = k$ and $n_{A,B}(I) = p$.

(ii) *Let $k, n \in \mathbb{N}$ and $p \in \{0, 1, \ldots, n\}$ be numbers with $n - p - 2k \geq 1$. Then there exists an open interval $I \subset \mathbb{R}^+$, a finite dimensional Krein space $\mathcal{K}$ and selfadjoint matrices A and B in $\mathcal{K}$ which satisfy Assumption (I) such that*

$$n_B(I) = n_A(I) - n_{A,B}(I) - 2\mathfrak{K}_A(I \cap \rho(B)) - 1$$

holds with $n_A(I) = n$, $\mathfrak{K}_A(I \cap \rho(B)) = k$ and $n_{A,B}(I) = p$.

(iii) *Let $k, n \in \mathbb{N}$ and $p \in \{0, 1, \ldots, n\}$. Then there exists an open interval $I \subset \mathbb{R}$ with $0 \in I$, finite dimensional Krein spaces $\mathcal{K}_j$ and selfadjoint matrices A_j, B_j in $\mathcal{K}_j$, $j = 1, 2, 3$, which satisfy Assumption (I) such that*

(a) $0 \in \sigma(A_1)$ *and* $n_{B_1}(I) = n_{A_1}(I) + n_{A_1, B_1}(I) + 2\mathfrak{K}_{B_1}(I \cap \rho(A_1)) + 1$,

(b) $0 \in \rho(A_2) \backslash c(B_2)$ *and* $n_{B_2}(I) = n_{A_2}(I) + n_{A_2, B_2}(I) + 2\mathfrak{K}_{B_2}(I \cap \rho(A_2)) + 2$,

(c) $0 \in \rho(A_3) \cap c(B_3)$ *and* $n_{B_3}(I) = n_{A_3}(I) + n_{A_3, B_3}(I) + 2\mathfrak{K}_{B_3}(I \cap \rho(A_3)) + 3$,

holds with $n_{A_j}(I) = n$, $\mathfrak{K}_{B_j}(I \cap \rho(A_j)) = k$ and $n_{A_j, B_j}(I) = p$ for $j = 1, 2, 3$.

(iv) *Let $k, n \in \mathbb{N}$ and $p \in \{0, 1, \ldots, n\}$. Then there exists an open interval $I \subset \mathbb{R}$ with $0 \in I$, finite dimensional Krein spaces $\mathcal{K}_j$ and selfadjoint matrices A_j, B_j in $\mathcal{K}_j$, $j = 1, 2, 3$, which satisfy Assumption (I) such that*

(a) $0 \in \sigma(B_1)$, $n - p - 2k \geq 1$, *and*

$$n_{B_1}(I) = n_{A_1}(I) - n_{A_1, B_1}(I) - 2\mathfrak{K}_{A_1}(I \cap \rho(B_1)) - 1,$$

(b) $0 \in \rho(B_2) \backslash c(A_2)$, $n - p - 2k \geq 2$, *and*

$$n_{B_2}(I) = n_{A_2}(I) - n_{A_2, B_2}(I) - 2\mathfrak{K}_{A_2}(I \cap \rho(B_2)) - 2,$$

(c) $0 \in \rho(B_3) \cap c(A_3)$, $n - p - 2k \geq 3$, *and*

$$n_{B_3}(I) = n_{A_3}(I) - n_{A_3, B_3}(I) - 2\mathfrak{K}_{A_3}(I \cap \rho(B_3)) - 3,$$

holds with $n_{A_j}(I) = n$, $\mathfrak{K}_{A_j}(I \cap \rho(B_j)) = k$ and $n_{A_j, B_j}(I) = p$ for $j = 1, 2, 3$.

Proof. (i) Let $k, n \in \mathbb{N}$ and fix positive numbers $\lambda_0, \ldots, \lambda_{n+1}$, $\mu_1, \ldots, \mu_{2k+1}$, and $\nu_1, \ldots, \nu_n$ such that

$$0 < \lambda_0 < \mu_1 < \cdots < \mu_{2k+1} < \lambda_1 < \nu_1 < \lambda_2 < \cdots < \nu_n < \lambda_{n+1}$$

and set $I := (\lambda_0, \lambda_{n+1})$. The rational function

$$M(\lambda) := -\frac{(\lambda - \mu_1) \ldots (\lambda - \mu_{2k+1})(\lambda - \nu_1) \ldots (\lambda - \nu_n)}{(\lambda - \lambda_0) \ldots (\lambda - \lambda_{n+1})(\lambda + 1)^{2k-1}}$$

has the following obvious properties:

(a) M is symmetric with respect to the real axis, $M(\bar{\lambda}) = \overline{M(\lambda)}$,

(b) M has n simple poles $\lambda_1, \ldots, \lambda_n$ in I,

(c) M has $2k + 1$ simple zeros $\mu_1, \ldots, \mu_{2k+1}$ in $(\lambda_0, \lambda_1) \subset I$ such that

$$M'(\mu_{2j+1}) > 0, \quad j = 0, \ldots, k, \quad \text{and} \quad M'(\mu_{2j}) < 0, \quad j = 1, \ldots, k,$$

and M has n simple zeros $\nu_1, \ldots, \nu_n \in I$ such that $M'(\nu_l) > 0$, $l = 1, \ldots, n$,

(d) $\lim_{\lambda \to \pm\infty} M(\lambda) = -1$.

Next we argue in the same way as in the proof of [3, Theorem 3.5] making use of a minimal realization of the function M as a Weyl function of some boundary triple; cf. [2, 17, 22]. More precisely, since M is a rational generalized Nevanlinna function there exists a Pontryagin space $(\mathcal{K}, [\cdot, \cdot])$, a (possibly nondensely defined) symmetric operator S of defect one and a boundary triple $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ for the adjoint S^+ such that the corresponding Weyl function coincides with M, see [2, Corollary 3.5]. The model can be chosen minimal, in which case $\mathcal{K}$ is a finite dimensional space and $A := S^+ \upharpoonright \ker \Gamma_0$ is a selfadjoint matrix with eigenvalues located at the poles of M. In particular, A has no multivalued part as M has no pole at $\pm\infty$. Note that $\sigma(A) \cap I$ consists of the n distinct eigenvalues $\lambda_1, \ldots, \lambda_n$, and hence

$$n_A(I) = n. \tag{4.1}$$

Next we use that $\{\mathbb{C}, \Gamma_1, -\Gamma_0\}$ is a boundary triple for S^+ with Weyl function $-M^{-1}$ and that $B := S^+ \upharpoonright \ker \Gamma_1$ is a selfadjoint matrix in $\mathcal{K}$. Note that B has no multivalued part as $-M^{-1}$ has no pole at $\pm\infty$. Since both A and B are selfadjoint extensions of the symmetric (nondensely defined) operator S with defect one the difference of A and B is a rank one operator and, hence, the difference of their resolvents is a rank one operator. Therefore Assumption (I) is satisfied. Moreover, the zeros of M in I coincide with $\sigma(B) \cap I$. Hence B has $2k + 1$ eigenvalues in the interval (λ_0, λ_1), where

$$\sigma_{++}(B) \cap (\lambda_0, \lambda_1) = \{\mu_1, \mu_3, \ldots, \mu_{2k+1}\}, \quad \sigma_{--}(B) \cap (\lambda_0, \lambda_1) = \{\mu_2, \mu_4, \ldots, \mu_{2k}\},$$

and one eigenvalue in each of the $n = n_A(I)$ intervals $(\lambda_1, \lambda_2), \ldots, (\lambda_n, \lambda_{n+1})$; cf. Proposition 2.3 and (4.1). In particular, we have

$$n_B(I) = 2k + 1 + n_A(I) \quad \text{and} \quad \mathfrak{K}_B(I \cap \rho(A)) = \mathfrak{K}_B((\lambda_0, \lambda_1)) = k,$$

and hence assertion (i) in the case $p = n_{A,B}(I) = 0$ follows. In order to obtain the assertion in the remaining case $1 \le p \le n$ add orthogonally to A and B a matrix C with $n_C(I) = p$ distinct eigenvalues such that $\sigma_p(C) \subset \sigma_p(A)$. Then,

$$\begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix} \tag{4.2}$$

have $n_C(I)$ common eigenvalues in the interval I, and their resolvents differ by a rank one matrix. We have shown assertion (i). Observe that assertion (ii) follows by interchanging the roles of A and B.

(iii) Let $k, n \in \mathbb{N}$, fix $\lambda_0, \ldots, \lambda_{n+1}$, $\mu_0, \ldots, \mu_{2k+1}$, and $\nu_1, \ldots, \nu_n$ such that

$$\lambda_0 < \mu_0 < 0 < \mu_1 < \cdots < \mu_{2k+1} < \lambda_1 < \nu_1 < \lambda_2 < \cdots < \nu_n < \lambda_{n+1}$$

and set $I := (\lambda_0, \lambda_{n+1})$. The rational function

$$N(\lambda) := -\frac{(\lambda - \mu_0)\lambda^2(\lambda - \mu_1)\ldots(\lambda - \mu_{2k+1})(\lambda - \nu_1)\ldots(\lambda - \nu_n)}{(\lambda - \lambda_0)\ldots(\lambda - \lambda_{n+1})(\lambda - (\lambda_0 - 1))^{2k+2}}$$

has the following obvious properties:

(a) N is symmetric with respect to the real axis, $N(\bar{\lambda}) = \overline{N(\lambda)}$,
(b) N has n simple poles $\lambda_1, \ldots, \lambda_n$ in I,
(c) N has a zero of multiplicity 2 at 0 and $2k+2$ simple zeros $\mu_0, \ldots, \mu_{2k+1}$ in the interval $(\lambda_0, \lambda_1) \subset I$ such that

$$N'(\mu_{2j+1}) > 0, \quad j = 0, \ldots, k, \quad \text{and} \quad N'(\mu_{2j}) < 0, \quad j = 0, \ldots, k, \qquad (4.3)$$

and N has n simple zeros $\nu_1, \ldots, \nu_n \in I$ such that $N'(\nu_l) > 0$, $l = 1, \ldots, n$,
(d) $\lim_{\lambda \to \pm\infty} N(\lambda) = -1$.

Now use a minimal realization of N and show the existence of $\mathcal{K}$ and A and B in exactly the same way as in the proof of assertion (i). Then $\sigma(A) \cap I$ consists of the poles of N in I, $\sigma(A) \cap I = \{\lambda_1, \ldots, \lambda_n\}$. The zeros of N in I coincide with $\sigma(B) \cap I$. From (4.3) and Proposition 2.2 (ii) and (iii) we conclude

$$\mu_{2j+1} \in \sigma_{++}(B), \quad j = 0, \ldots, k, \quad \text{and} \quad \mu_{2j} \in \sigma_{--}(B), \quad j = 0, \ldots, k,$$

and we see, as $\mu_0 < 0$ and $0 < \mu_1 < \ldots < \mu_{2k+1}$,

$$\mathfrak{K}_B(I \cap \rho(A)) = k.$$

Hence

$$n_B(I) = 2k + 3 + n = n_A(I) + 2\mathfrak{K}_B(I \cap \rho(A)) + 3.$$

Again one treats the case $1 \le p \le n$ as in (4.2). The sharpness in the remaining two cases in (iii) can be shown analogously with the help of a minimal model for the generalized Nevanlinna function

$$P(\lambda) := -\frac{(\lambda - \mu_0)(\lambda - \mu_1)\ldots(\lambda - \mu_{2k+1})(\lambda - \nu_1)\ldots(\lambda - \nu_n)}{(\lambda - \lambda_0)\ldots(\lambda - \lambda_{n+1})(\lambda - (\lambda_0 - 1))^{2k}},$$

which has the properties

(a) P is symmetric with respect to the real axis, $P(\bar{\lambda}) = \overline{P(\lambda)}$,
(b) P has n simple poles $\lambda_1, \ldots, \lambda_n$ in $I = (\lambda_0, \lambda_{n+1})$,
(c) P has $2k+2$ simple zeros $\mu_0, \ldots, \mu_{2k+1}$ in the interval $(\lambda_0, \lambda_1) \subset I$ such that

$$P'(\mu_{2j+1}) > 0, \quad j = 0, \ldots, k, \quad \text{and} \quad P'(\mu_{2j}) < 0, \quad j = 0, \ldots, k,$$

and P has n simple zeros $\nu_1, \ldots, \nu_n \in I$ such that $P'(\nu_l) > 0$, $l = 1, \ldots, n$,
(d) $\lim_{\lambda \to \pm\infty} P(\lambda) = -1$,

as well as with a minimal model for the generalized Nevanlinna function

$$Q(\lambda) := -\frac{(\lambda - \mu_0)(\lambda - \mu_1)\ldots(\lambda - \mu_{2k+1})(\lambda - \nu_1)\ldots(\lambda - \nu_n)}{(\lambda - \lambda_0)\lambda^2(\lambda - \lambda_1)\ldots(\lambda - \lambda_{n+1})(\lambda - (\lambda_0 - 1))^{2k-2}},$$

which has the properties

(a) Q is symmetric with respect to the real axis, $Q(\bar{\lambda}) = \overline{Q(\lambda)}$,
(b) Q has a pole of order 2 at 0 and n simple poles $\lambda_1, \ldots, \lambda_n$ in $I = (\lambda_0, \lambda_{n+1})$,

(c) Q has $2k + 2$ simple zeros $\mu_0, \ldots, \mu_{2k+1}$ in the interval $(\lambda_0, \lambda_1) \subset I$ such that

$$Q'(\mu_{2j+1}) > 0, \quad j = 0, \ldots, k, \quad \text{and} \quad Q'(\mu_{2j}) < 0, \quad j = 0, \ldots, k,$$

and Q has n simple zeros $\nu_1, \ldots, \nu_n \in I$ such that $Q'(\nu_l) > 0, l = 1, \ldots, n$,
(d) $\lim_{\lambda \to \pm\infty} Q(\lambda) = -1$.

This completes the proof of assertion (iii). It is clear that (iv) follows by interchanging A and B. $\square$

5. AN EXAMPLE: SINGULAR INDEFINITE STURM-LIOUVILLE PROBLEMS

In this section the general eigenvalue estimates are illustrated in a typical application from the theory of singular Sturm-Liouville problems with indefinite weight functions. We go beyond the so-called left-definite case, which was studied thoroughly from different points of view, see, e.g. [7–9, 12, 19–21, 27].

Let $r, p^{-1}, q \in L^1_{\text{loc}}(\mathbb{R})$ be real valued, $p > 0$ and $r \neq 0$ a.e., and consider the differential expression τ on $\mathbb{R}$,

$$\tau = \frac{1}{r} \left(-\frac{d}{dx} p \frac{d}{dx} + q \right).$$

We assume that τ is in the limit point case at $\pm\infty$ and that the weight function r has one sign change at some point $c \in \mathbb{R}$ such that $r_+ = r \upharpoonright (c, \infty) > 0$ and $r_- = r \upharpoonright (-\infty, c) < 0$ a.e.

The *indefinite* Sturm-Liouville operator B corresponding to τ is defined by

$$Bf = \tau(f) = \frac{1}{r}\big((-pf')' + qf\big), \quad f \in \text{dom } B,$$

where dom B consists of all locally absolutely continuous functions $f \in L^2(\mathbb{R}, |r|)$ such that pf' is locally absolutely continuous and $\tau(f) \in L^2(\mathbb{R}, |r|)$; here $L^2(\mathbb{R}, |r|)$ denotes the space of all equivalence classes of complex valued measurable functions f on $\mathbb{R}$ such that

$$(f, f) = \int_{\mathbb{R}} |f(x)|^2 |r(x)| \, dx < \infty. \tag{5.1}$$

Note that B is selfadjoint in the Krein space $(L^2(\mathbb{R}, |r|), [\cdot, \cdot])$, where $[\cdot, \cdot]$ is given by

$$[f, g] = \int_{\mathbb{R}} f(x)\overline{g(x)} \, r(x) \, dx, \qquad f, g \in L^2(\mathbb{R}, |r|). \tag{5.2}$$

In the following we will also make use of the selfadjoint realizations T_+ and T_- of τ restricted to (c, ∞) and $(-\infty, c)$, respectively, with Dirichlet boundary conditions at c in the Hilbert spaces $L^2((c, \infty), |r_+|)$ and $L^2((-\infty, c), |r_-|)$, respectively. Here $L^2((c, \infty), |r_+|)$ and $L^2((-\infty, c), |r_-|)$ stand for the spaces of all equivalence

classes of square integrable complex valued functions f with $\int_{(c,\infty)} |f|^2 |r_+| < \infty$ (resp. $\int_{(-\infty,c)} |f|^2 |r_-| < \infty$). The domain of T_+ (T_-) consists of all locally absolutely continuous functions f in $L^2((c,\infty), |r_+|)$ $(L^2((-\infty,c), |r_-|)$, respectively) which are zero in c such that pf' is locally absolutely continuous and the restrictions of $\tau(f)$ are in $L^2((c,\infty), |r_+|)$ $(L^2((-\infty,c), |r_-|)$, respectively).

Obviously, the direct sum $T_+ \oplus T_-$ is closely related to the operator B, see also Proposition 5.1 below. As we are interested in operators with finitely many squares we will impose assumptions on T_+ and T_- which imply that $T_+ \oplus T_-$ and the indefinite Sturm-Liouville operator B have finitely many negative squares.

Proposition 5.1. *Assume that the essential spectra of T_+ and T_- satisfy*

$$\eta_+ := \min \sigma_{\mathrm{ess}}(T_+) > 0 \quad and \quad \eta_- := \max \sigma_{\mathrm{ess}}(T_-) < 0, \tag{5.3}$$

and that T_+ has only finitely many eigenvalues in $(-\infty, \eta_+)$ and T_- has only finitely many eigenvalues in (η_-, ∞). Then the following holds.

(i) *The orthogonal sum $T_+ \oplus T_-$ is a selfadjoint operator in the Hilbert space $L^2(\mathbb{R}, |r|)$.*

(ii) *The indefinite Sturm-Liouville operator B has κ_B negative squares in the Krein space $(L^2(\mathbb{R}, |r|), [\cdot, \cdot])$ with $\rho(B) \neq \emptyset$ and*

$$\kappa_B \leq n_{T_+}((-\infty, 0)) + n_{T_-}((0, \infty)) + 1.$$

(iii) *We have*

$$\dim \mathrm{ran}\left((B - \lambda)^{-1} - ((T_+ \oplus T_-) - \lambda)^{-1}\right) = 1, \qquad \lambda \in \mathbb{C} \setminus \mathbb{R}, \tag{5.4}$$

and the essential spectrum of $T_+ \oplus T_-$ coincides with the essential spectrum of B. In particular,

$$(\eta_-, \eta_+) \cap \sigma_{\mathrm{ess}}(B) = \emptyset.$$

Proof. The selfadjointness of T_+ and T_- implies (i). Furthermore, it is not difficult to see that the operator $T_+ \oplus T_-$ is also selfadjoint in the Krein space $(L^2(\mathbb{R}, |r|), [\cdot, \cdot])$. Then [12, Remark 1.3] applied to $T_+ \oplus T_-$ and the symmetric operator $B \cap (T_+ \oplus T_-)$ gives $\rho(B) \neq \emptyset$. In order to show the assertion on κ_B in (ii), we consider the (definite) differential expression

$$\ell = \frac{1}{|r|}\left(-\frac{d}{dx} p \frac{d}{dx} + q\right)$$

on $\mathbb{R}$. Then ℓ gives rise to a selfadjoint operator

$$Tf = \ell(f) = \frac{1}{|r|}\left((-pf')' + qf\right), \quad f \in \mathrm{dom}\, T, \tag{5.5}$$

in the Hilbert space $L^2(\mathbb{R}, |r|)$, where the domain $\mathrm{dom}\, T$ coincides with the domain of B. In view of the definition of the inner products in (5.2) and (5.1), we easily see

$$[Bf, f] = (Tf, f) \qquad \text{for } f \in \mathrm{dom}\, B = \mathrm{dom}\, T.$$

Due to the limit point behaviour of τ at $\pm\infty$ all eigenvalues of T are simple. Hence the number of negative squares of the form $[B\cdot,\cdot]$ equals the number of negative eigenvalues of the selfadjoint operator T, $\kappa_B = n_T((-\infty,0))$. The orthogonal sum of T_+ and $-T_-$ differs from T only in the boundary condition at c. Therefore

$$\dim\left(\operatorname{ran}(T-\lambda)^{-1} - ((T_+ \oplus (-T_-)) - \lambda)^{-1}\right) = 1, \qquad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

This together with well-known perturbation results for selfadjoint operators in Hilbert spaces (see, e.g., [10, §9.3, Theorem 3]) gives

$$\kappa_B = n_T((-\infty,0)) \leq n_{T_+ \oplus (-T_-)}((-\infty,0)) + 1 = n_{T_+}((-\infty,0)) + n_{T_-}((0,\infty)) + 1.$$

This shows (ii). The orthogonal sum of the operators T_+ with T_- differs from B only in the boundary condition at c and (5.4) follows. The remaining assertions in (iii) are obvious. □

In Theorem 5.2 below we express the number of eigenvalues of B in a gap around zero in terms of the number of eigenvalues of the operators T_- and T_+. Observe that B is a selfadjoint operator in a Krein space and B, in general, is not left-definite. For such operators there are no results for the number of eigenvalues of B in terms of the coefficients r, p, q available. Contrary, as T_- and T_+ are selfadjoint operators in Hilbert spaces and correspond to definite Sturm-Liouville expressions on the intervals $(-\infty, c)$ and (c, ∞), respectively, their point spectrum has been studied in various situations. The estimates in the next theorem and Remark 5.4 below allow to translate results from the well-studied definite case to the indefinite case.

Theorem 5.2. *Assume* (5.3) *holds and that* T_+ *has only finitely many eigenvalues in* $(-\infty, \eta_+)$ *and* T_- *has only finitely many eigenvalues in* (η_-, ∞). *Moreover, we assume*

$$\sigma_p(T_+) \cap \sigma_p(T_-) \cap (\eta_-, \eta_+) = \emptyset. \tag{5.6}$$

Then the following holds.

(i) $n_B((\eta_-, \eta_+))$ *is finite.*

(ii) *If* $0 \in \rho(B) \cap \rho(T_+) \cap \rho(T_-)$ *then*

$$n_B((\eta_-, \eta_+)) \geq n_{T_+}(0, \eta_+) + n_{T_-}(\eta_-, 0) - n_{T_+}(\eta_-, 0) - n_{T_-}(0, \eta_+) - 2.$$

(iii) *If* $0 \notin \rho(B) \cap \rho(T_+) \cap \rho(T_-)$ *then*

$$n_B((\eta_-, \eta_+)) \geq n_{T_+}(0, \eta_+) + n_{T_-}(\eta_-, 0) - n_{T_+}(\eta_-, 0) - n_{T_-}(0, \eta_+) - 1.$$

Proof. As ℓ is in the limit point case at $\pm\infty$, all eigenvalues of T_+ and of T_- are simple. Assume $\lambda \in \sigma_p(T_+) \cap \sigma_p(B)$. Then the corresponding eigenfunction of the operator B is defined on $\mathbb{R}$ and coincides on (c, ∞) with the eigenfunction of T_+ corresponding to λ. Hence, it is zero at c. But this implies $\lambda \in \sigma_p(T_-)$, a contradiction to (5.6). The preceding argumentation remains valid if T_+ interchanges with T_- and, hence, we obtain

$$n_{T_+ \oplus T_-, B}((\eta_-, \eta_+)) = 0. \tag{5.7}$$

The operator $T_+ \oplus T_-$ is selfadjoint in the Krein space $(L^2(\mathbb{R}, |r|), [\cdot, \cdot])$ and the number $\mathfrak{K}_{T_+ \oplus T_-}((\eta_-, \eta_+) \cap \rho(B))$ defined in (3.1) can be calculated explicitly in terms of the eigenvalues of T_+ and of T_-,

$$\mathfrak{K}_{T_+ \oplus T_-}((\eta_-, \eta_+) \cap \rho(B)) = n_{T_+}(\eta_-, 0) + n_{T_-}(0, \eta_+). \tag{5.8}$$

By Proposition 5.1 the operators $T_+ \oplus T_-$ and B satisfy Assumption (I) and, hence, Theorem 3.1 (iii) and $n_{T_+ \oplus T_-}((\eta_-, \eta_+)) < \infty$ yield (i). Furthermore, Theorem 3.1 (ii), (5.7), and (5.8) give

$n_B((\eta_-, \eta_+)) \geq$

$$n_{T_+ \oplus T_-}((\eta_-, \eta_+)) - 2n_{T_+}(\eta_-, 0) - 2n_{T_-}(0, \eta_+) - \begin{cases} 3 & \text{if } 0 \in \rho(B) \cap c(T_+ \oplus T_-), \\ 2 & \text{if } 0 \in \rho(B) \setminus c(T_+ \oplus T_-), \\ 1 & \text{if } 0 \in \sigma(B). \end{cases}$$

Note that $0 \in c(T_+ \oplus T_-)$ if and only if $0 \in \sigma_p(T_+) \cap \sigma_p(T_-)$. By (5.6) the intersection of the set $\sigma_p(T_+) \cap \sigma_p(T_-)$ with (η_-, η_+) is empty and hence the first case in the above inequality is not present. Moreover,

$$n_{T_+ \oplus T_-}((\eta_-, \eta_+)) = n_{T_+}((\eta_-, \eta_+)) + n_{T_-}((\eta_-, \eta_+))$$
$$= n_{T_+}((0, \eta_+)) + n_{T_+}(\{0\}) + n_{T_+}((\eta_-, 0)) + n_{T_-}((0, \eta_+)) + n_{T_-}(\{0\}) + n_{T_-}((\eta_-, 0))$$

and we obtain the estimate

$$n_B((\eta_-, \eta_+)) \geq n_{T_+}((0, \eta_+)) - n_{T_+}((\eta_-, 0)) - n_{T_-}((0, \eta_+)) + n_{T_-}((\eta_-, 0))$$
$$+ n_{T_+}(\{0\}) + n_{T_-}(\{0\}) - \begin{cases} 2 & \text{if } 0 \in \rho(B), \\ 1 & \text{if } 0 \in \sigma(B). \end{cases}$$

If $0 \in \rho(B) \cap \rho(T_+) \cap \rho(T_-)$ then $n_{T_+}(\{0\}) = n_{T_-}(\{0\}) = 0$ and (ii) follows. If $0 \in \sigma_p(B)$ then (5.7) gives $0 \in \rho(T_+) \cap \rho(T_-)$ and $n_{T_+}(\{0\}) = n_{T_-}(\{0\}) = 0$ implies (iii). If $0 \in \rho(B)$ and $0 \in \sigma_p(T_+)$, then (5.6) implies $0 \in \rho(T_-)$. Hence $n_{T_+}(\{0\}) = 1$, $n_{T_-}(\{0\}) = 0$ and (iii) follows. The case $0 \in \rho(B)$ and $0 \in \sigma_p(T_-)$ is shown analogously. $\square$

Corollary 5.3. *Assume in addition to the assumption in Theorem 5.2 that T_+ is a nonnegative operator in the Hilbert space $L^2((c, \infty), |r_+|)$ and T_- is a nonpositive operator in the Hilbert space $L^2((-\infty, c), |r_-|)$. Then the following holds.*

(i) *If $0 \in \rho(B) \cap \rho(T_+) \cap \rho(T_-)$ then*

$$n_B((\eta_-, \eta_+)) \geq n_{T_+}(0, \eta_+) + n_{T_-}(\eta_-, 0) - 2.$$

(ii) *If $0 \notin \rho(B) \cap \rho(T_+) \cap \rho(T_-)$ then*

$$n_B((\eta_-, \eta_+)) \geq n_{T_+}(0, \eta_+) + n_{T_-}(\eta_-, 0) - 1.$$

In the situation of Corollary 5.3 we refer to [3] for a related estimate. In [3] the number of eigenvalues in a gap of the essential spectrum of B is estimated with the help of the number of eigenvalues of the definite Sturm-Liouville operator T in (5.5).

Remark 5.4. In Theorem 5.2 there are only estimates for the number $n_B((\eta_-, \eta_+))$ from below. For the corresponding estimates from above in Theorem 3.1, applied to the operators $T_+ \oplus T_-$ and B, the quantity $\mathfrak{K}_B((\eta_-, \eta_+) \cap \rho(B))$ appears. In general it is difficult to find an estimate for this quantity in terms of the number of eigenvalues of T_+ and T_- in the interval (η_-, η_+). However one can use the general (and rough) estimate from Corollary 3.2 together with (5.7) and Proposition 5.1 (ii) and concludes

$$n_B((\eta_-, \eta_+)) \leq n_{T_+ \oplus T_-}((\eta_-, \eta_+)) + 2n_{T_+}((-\infty, 0)) + 2n_{T_-}((0, \infty)) + 5.$$

REFERENCES

[1] T.Ya. Azizov, I.S. Iokhvidov, *Linear Operators in Spaces with an Indefinite Metric*, John Wiley and Sons, Chichester, New York, 1989.

[2] J. Behrndt, P. Jonas, *Boundary value problems with local generalized Nevanlinna functions in the boundary condition*, Integral Equations Operator Theory **55** (2006), 453–475.

[3] J. Behrndt, L. Leben, F. Martínez Pería, R. Möws, C. Trunk, *Sharp eigenvalue estimates for rank one perturbations of nonnegative operators in Krein spaces*, J. Math. Anal. Appl. **439** (2016), 864–895.

[4] J. Behrndt, M. Malamud, H. Neidhardt, *Finite rank perturbations, scattering matrices and inverse problems*, Oper. Theory Adv. Appl. **198** (2009), 61–85.

[5] J. Behrndt, R. Möws, C. Trunk, *On finite rank perturbations of selfadjoint operators in Krein spaces and eigenvalues in spectral gaps*, Complex Anal. Oper. Theory **8** (2014), 925–936.

[6] J. Behrndt, C. Trunk, *On the negative squares of indefinite Sturm-Liouville operators*, J. Differential Equations **238** (2007), 491–519.

[7] C. Bennewitz, B.M. Brown, R. Weikard, *Scattering and inverse scattering for a left-definite Sturm Liouville problem*, J. Differential Equations **253** (2012), 2380–2419.

[8] P. Binding, P. Browne, *Left definite Sturm-Liouville problems with eigenparameter dependent boundary conditions*, Differential Integral Equations **12** (1999), 167–182.

[9] P. Binding, P. Browne, B. Watson, *Inverse spectral problems for left-definite Sturm-Liouville equations with indefinite weight*, J. Math. Anal. Appl. **271** (2002), 383–408.

[10] M.Sh. Birman, M.Z. Solomjak, *Spectral Theory of Selfadjoint Operators in Hilbert Space*, Kluwer, Dordrecht, 1987.

[11] J. Bognar, *Indefinite Inner Product Spaces*, Springer, 1974.

[12] B. Ćurgus, H. Langer, *A Krein space approach to symmetric ordinary differential operators with an indefinite weight function*, J. Differential Equations **79** (1989), 31–61.

[13] V.A. Derkach, *On Weyl function and generalized resolvents of a Hermitian operator in a Krein space*, Integral Equations Operator Theory **23** (1995), 387–415.

[14] V.A. Derkach, M.M. Malamud, *Generalized resolvents and the boundary value problems for Hermitian operators with gaps*, J. Funct. Anal. **95** (1991), 1–95.

[15] V.A. Derkach, M.M. Malamud, *The extension theory of Hermitian operators and the moment problem*, J. Math. Sci. (N.Y.) **73** (1995), 141–242.

[16] I. Gohberg, M.G. Krein, *Introduction to the Theory of Linear Nonselfadjoint Operators*, Translations of Mathematical Monographs, vol. 18, American Mathematical Society, Providence, R.I. 1969.

[17] S. Hassi, H.S.V. de Snoo, H. Woracek, *Some interpolation problems of Nevanlinna-Pick type. The Krein-Langer method*, Oper. Theory Adv. Appl. **106** (1998), 201–216.

[18] T. Kato, *Perturbation Theory for Linear Operators*, 2nd ed., Springer, 1976.

[19] Q. Kong, M. Möller, H. Wu, A. Zettl, *Indefinite Sturm-Liouville problems*, Proc. Roy. Soc. Edinburgh Sect. A **133** (2003), 639–652.

[20] Q. Kong, H. Wu, A. Zettl, *Left-definite Sturm-Liouville problems*, J. Differential Equations **177** (2001), 1–26.

[21] Q. Kong, H. Wu, A. Zettl, *Singular left-definite Sturm-Liouville problems*, J. Differential Equations **206** (2004), 1–29.

[22] M.G. Krein, H. Langer, *Über einige Fortsetzungsprobleme, die eng mit der Theorie hermitescher Operatoren im Raum Π_κ zusammenhängen, I. Einige Funktionenklassen und ihre Darstellungen*, Math. Nachr. **77** (1977), 187–236.

[23] H. Langer, *Spektraltheorie linearer Operatoren in J-Räumen und einige Anwendungen auf die Schar $L(\lambda) = \lambda^2 + \lambda B + C$*, Habilitationsschrift, Technische Universität Dresden, 1965.

[24] H. Langer, *Spectral functions of definitizable operators in Krein spaces*, [in:] *Functional Analysis Proceedings of a Conference held at Dubrovnik, Yugoslavia, November* 2–14, 1981, Lecture Notes in Mathematics, vol. 948, Springer (1982), 1–46.

[25] H. Langer, A. Markus, V. Matsaev, *Locally definite operators in indefinite inner product spaces*, Math. Ann. **308** (1997), 405–424.

[26] H. Langer, B. Textorius, *On generalized resolvents and Q-functions of symmetric linear relations (subspaces) in Hilbert space*, Pacific J. Math. **72** (1977), 135–165.

[27] A. Zettl, *Sturm-Liouville Theory*, American Mathematical Society, Providence, RI, 2005.

Jussi Behrndt
behrndt@tugraz.at

Technische Universität Graz
Institut für Numerische Mathematik
Steyrergasse 30, 8010 Graz, Austria

Roland Möws
acad@rolandmoews.de

Krossener Str. 17, D-10245 Berlin, Germany

Carsten Trunk (corresponding author)
carsten.trunk@tu-ilmenau.de

Technische Universität Ilmenau
Institut für Mathematik
Postfach 100565, D-98684 Ilmenau, Germany