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A PARTIAL REFINING OF THE ERDŐS-KELLY REGULATION

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Abstract. The aim of this note is to advance the refining of the Erdős-Kelly result on graphical inducing regularization. The operation of inducing regulation (on graphs or multigraphs) with prescribed maximum vertex degree is originated by D. König in 1916. As is shown by Chartrand and Lesniak in their textbook Graphs & Digraphs (1996), an iterated construction for graphs can result in a regularization with many new vertices. Erdős and Kelly have presented (1963, 1967) a simple and elegant numerical method of determining for any simple n-vertex graph G with maximum vertex degree Δ, the exact minimum number, say $\theta = \theta(G)$, of new vertices in a Δ-regular graph H which includes G as an induced subgraph. The number $\theta(G)$, which we call the cost of regulation of G, has been upper-bounded by the order of G, the bound being attained for each $n \geq 4$, e.g. then the edge-deleted complete graph $K_n - e$ has $\theta = n$. For $n \geq 4$, we present all factors of K_n with $\theta = n$ and next $\theta = n - 1$. Therein in case $\theta = n - 1$ and n odd only, we show that a specific extra structure, non-matching, is required.

Keywords: inducing Δ-regulation, cost of regulation.

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1. ON THE ERDŐS-KELLY RESULT

Let G be a simple n-vertex non-regular graph with maximum and minimum vertex degree Δ and δ, resp. Hence $\delta < \Delta$.

Definition 1.1. Let H be a Δ-regular graph which contains an induced subgraph isomorphic to G and has the minimal order possible. Then we call H to be an *intrinsic regularization* of the graph G (since any such H seems to be the most natural inducing regularization).

If v is a vertex of G, the difference $\Delta - \deg_G v$ is called the *deficiency* of v in G. Hence $\Delta - \delta$ is the maximum deficiency among vertices. Let $s = \sum_v (\Delta - \deg_G v)$ be the *sum of all deficiencies.*

An improved version of the original theorem (as in [13] and next in [9,14]) follows.

Theorem 1.2 (Erdős and Kelly [13,14]). *The necessary and sufficient condition that $n+t$ be the order for an intrinsic regularization H of G is that t is the least positive integer such that*

(1) $t\Delta \geq s$,
(2) $t^2 - (\Delta+1)t + s \geq 0$,
(3) $t \geq \Delta - \delta$,
(4) $(t+n)\Delta$ *is an even integer.*

Moreover, $t \leq n$, and for each $n \geq 4$ there exists a graph G such that $t = n$, e.g. $G = K_n - e$, where e is an edge of the complete graph K_n.

Notes on Theorem 1.2.

1. Since the minimal t depends on G, the original phrase in [13] "maximum value of t is n" is replaced above by the correct inequality $t \leq n$ as in [9,14].
2. Only $n \geq 4$ is considered above because otherwise trivially $n = 3$, and then either $G = P_3$ and $H = C_4$ or $G = \overline{P_3}$ and $H = 2K_2$. Hence, if $n = 3$ then $t(G) = 1$ only.
3. Each of conditions (1)–(4) is proved to be both necessary and independent from remaining ones.
4. Sufficiency is proved by presenting the operation of regulation $G \mapsto H$, i.e. by the construction of a supergraph H of G.

Definition 1.3. The minimum value of t in question, denoted by $\theta = \theta(G)$, will be called the *cost of the intrinsic regulation* $G \mapsto H$ of G.

2. A REFINING OF THE ERDŐS-KELLY RESULT

In the first part we list n-vertex graphs G for which $n > \Delta > \delta$ and the intrinsic regulation cost $\theta = \theta(G) = n = |G|$. Actually we characterize Gs as factors of the complete graph K_n, $G = K_n - E_k$ obtained from K_n by removal of a subset of k edges under the requirement that, depending on n and k, Δ is $n-1$ or $n-2$.

We now show that neither condition (1) nor (3) can contribute to making $\theta = n$.

Lemma 2.1. *Both conditions* (1) *and* (3) *hold for* $t = n-2$.

Proof. The sum s of deficiencies is the largest possible if one vertex (of degree Δ) has deficiency 0, each of its Δ neighbors has degree 1, deficiency $\Delta - 1$, and the remaining vertices are isolated. Consequently, $s \leq \Delta(\Delta-1) + (n-1-\Delta)\Delta = \Delta(n-2)$, i.e. (1) holds for $t = n-2$. Since clearly $\Delta - \delta \leq n-2$ if either $\Delta = n-1$ or $\Delta \leq n-2$, condition (3) also holds for $t = n-2$. □

Consequently, in proofs which follow we disregard both conditions (1) and (3) and refer only to the quadratic inequality (2) and the parity requirement (4). The graph of the left-hand side of (2) is a convex parabola whose vertex has t-coordinate $t_v = (\Delta+1)/2$. Moreover, the equation $t = t_v$ represents the axis of symmetry. Condition (2) is seen true for symmetric values $t = 0$, $\Delta + 1$. Thus we get the following.

Corollary 2.2. *Condition* (2) *is true for all* $t \geq \Delta + 1$.

Corollary 2.3. $\theta(G)$ *can equal* n *for even* n *if* $\Delta(G) = n - 1$ *only, for odd* n *if* $\Delta(G) \geq n - 2$ *only.*

Theorem 2.4. *Three lists* $A.j$ *of graphs* G *with* $\theta(G) = n$ *follow together with relevant proofs,* $j = 1, 2, 3$.

A.1. *For any even* $n \geq 4$, *let* $1 \leq k \leq n - 3$ *and let* $G = K_n - E_k$ *such that* $\Delta = n - 1$.
A.2. *For any odd* $n \geq 5$, *let* $1 \leq k \leq (n - 3)/2$ *and let* $G = K_n - E_k$.
A.3. *For any odd* $n \geq 5$, *let* $(n + 1)/2 \leq k \leq n - 2$ *and let* $G = K_n - E_k$, *where* E_k *covers all* n *vertices so that* $\Delta < n - 1$.

Proof. Ad A.1. Then Δ is odd whence, due to (4), the cost θ must be even. Moreover, each of k removed edges contributes 2 to the sum s in G whence $s = 2k$. Therefore, condition (2) reads as $t(t - n) + 2k \geq 0$ whence $t_v = n/2$ and the condition is seen false for symmetric even $t = 2$, $n - 2$ and true for $t = n$, as required.

Ad A.2. Then G has three or more vertices of degree $\Delta = n - 1$ (which is even). Hence each of k removed edges contributes 2 to the sum s in G, whence $s = 2k$. Therefore condition (2) reads as $t(t - n) + 2k \geq 0$ whence $t_v = n/2$ and the condition is seen false for symmetric $t = 1$, $n - 1$ and true for $t = n$, as required.

Ad A.3. Since $k < n$, $\Delta = n - 2$, which is odd. Hence, due to (4), the cost θ must be odd, too. Moreover, the degree sum of the induced subgraph $\langle E_k \rangle$ is
$2k =: n + s \in \{n+1, n+2, \ldots, n+(n-4)\}$. Therefore the parameter s in the resulting graphs G's is among numbers 1, 2, $\ldots$, $n - 4$ whence $s < \Delta = n - 2$.

Condition (2) reads as $t(t - n + 1) + s \geq 0$ whence $t_v = (n - 1)/2$ and the condition is seen false for symmetric odd $t = 1, n - 2$ and is true for odd $t = n$, whence odd cost $\theta = n$, as required. □

Theorem 2.5. *All* n*-vertex graphs* G, $G = K_n - E_k$, *with largest possible* (*intrinsic*) Δ*-regulation cost* $\theta(G) = n$ *are listed in Theorem 2.4* (*in items* $A.j$, $j = 1, 2, 3$).

Proof. Let N_n^k denote the number of nonisomorphic graphs $K_n - E_k$ with $\theta = n$. Then if $k = 1$, $N_n^1 = 1$ for each n in question, $n \geq 4$. Let $k \geq 2$. Then $n \geq 5$. By inspection of the graph diagrams in Harary's book [18] (wherein $n \leq 6$), on referring to Theorem 1.2 and Corollary 2.3, we get $N_5^3 = 1$ and $N_6^k = 2, 4$ for $k = 2, 3$, resp. In fact, we find all the corresponding induced graphs $\langle E_k \rangle$:
$n = 5$: $k = 3$, and $\langle E_3 \rangle = P_3 \cup K_2$;
$n = 6$: if $k = 2$ then $\langle E_2 \rangle = P_3,\ 2K_2$; if $k = 3$ then $\langle E_3 \rangle = P_3 \cup K_2,\ P_4,\ C_3,\ K_{1,3}$.
By inspection, none of the corresponding graphs G is exceptional, all of them are listed in Theorem 2.4.

It remains $n \geq 7$. Due to Corollary 2.3, the relevant remaining graphs G split into three classes, say $B.j$, complementing the above $A.j$. In order to complete the proof, we simply show that conditions (2) and (4) in Theorem 1.2 are satisfied for some $t < n$.

B.1. Let $n \geq 8$ be even, $k \geq n-2$ and still $\Delta = n-1$ (which is odd as in case A.1 above). Hence each of k removed edges contributes 2 to s in G, whence $s = 2k \geq 2n-4$. Therefore (2) reads as $t^2 - nt + 2k \geq 0$ which holds for even and symmetric $t = 2, n-2$.

B.2. Let $n \geq 7$ be odd, $k \geq (n-1)/2$ and $\Delta = n-1$ which is even. Hence condition (4) is true and $s = 2k \geq n-1$. Therefore, (2) is true for $t = n-1$.

B.3. Let $n \geq 7$ be odd, $k \geq n-1$ and $\Delta = n-2$ which is odd. Hence $2k = n+s \geq 2n-2$ whence $s \geq n-2$. Therefore, condition (2), namely $t(t-n+1)) + s \geq 0$, and condition (4) are both true for odd $t = n-2$. □

We now pass on to the second part of refining wherein $\theta(G) = n-1$.

Theorem 2.6. *All n-vertex graphs $G = K_n - E_k$ with Δ-regulation cost $\theta(G) = n-1$ are the following. For odd $n \geq 5$, $(n-1)/2 \leq k \leq n-3$, E_k is not a matching (if $k = (n-1)/2$), and $\Delta = n-1$. For even $n \geq 6$, $(n+2)/2 \leq k \leq n-2$ and $\Delta = n-2$.*

Proof. The proof (similar to what is above) is left to the reader. Note: It would be $\theta = 1$ for odd n if E_k in Theorem 2.6 were a (maximum) matching. □

By inspection of graph diagrams in Harary [18] we find all induced graphs $\langle E_k \rangle$ for edge sets E_k which are listed in Theorem 2.6 if $n = 5, 6, 7$.
$n = 5$: $k = 2$, and $\langle E_k \rangle = P_3$;
$n = 6$: $k = 4$, and $\langle E_4 \rangle = P_4 \cup K_2,\ K_{1,3} \cup K_2,\ 2P_3$;
$n = 7$: if $k = 3$ then $\langle E_3 \rangle = P_4,\ C_3,\ P_3 \cup K_2,\ K_{1,3}$; if $k = 4$ then $\langle E_4 \rangle$ is as in case $n = 6$, or $\langle E_4 \rangle = C_3 \cup K_2,\ C_4,\ K_{1,4},\ P_5$, or else is C_3 with a hanging edge or $K_{1,3}$ with a subdivided edge.

Hence, it follows that the smallest graphs G_n with $\theta = n-1$ (and $5 \leq n \leq 6$) are $G_5' := C_5 \cup P_4$, where the path P_4 comprises three chords of the cycle C_5 and next are the three 6-vertex graphs $G_6 = K_6 - E_4$ with $\Delta = 4$. Moreover, there are exactly 13 such graphs of order $n = 7$ with $\Delta = 6$.

Remark 2.7. Only numerical requirements are imposed on graphs G in Theorems 1.2, 2.4 and 2.5 above. A structural requirement (non-matching) is imposed in an odd part of Theorem 2.6 only.

Problems 2.8.

1. Characterize n-vertex graphs G_n with a smaller θ, e.g. $\theta = n-2$.
2. Estimate the number of graphs G_n with a fixed intrinsic regulation cost θ. (The set of complements of those graphs G_n with $\theta = n$ includes almost all forests on n-vertices.)
3. In general, study the statistics of the distribution of graphs G_n among classes comprising graphs with $\theta = 0, 1, \ldots, n$.

3. CONCLUDING REMARKS

Related results on inducing superstructures are presented in [3, 10, 12, 15–17, 20]. A generalization of the above Erdős-Kelly result in case of r-regulation ($r \geq \Delta$ as in König [21, 22]) is presented in the authors papers [15–17]. For non-inducing regulations (which are described briefly in [17]), see [1, 2, 4–8, 19].

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