

Synchronization of FitzHugh-Nagumo reaction-diffusion systems via one-dimensional linear control law

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The Fitzhugh-Nagumo model (FN model), which is successfully employed in modeling the function of the so-called membrane potential, exhibits various formations in neuronal networks and rich complex dynamics. This work deals with the problem of control and synchronization of the FN reaction-diffusion model. The proposed control law in this study is designed to be uni-dimensional and linear law for the purpose of reducing the cost of implementation. In order to analytically prove this assertion, Lyapunov's second method is utilized and illustrated numerically in one- and/or two-spatial dimensions.

Key words: FitzHugh-Nagumo; synchronization; uni-dimensional control; linear control; reaction-diffusion system; neuronal networks; Lyapunov's second method

1. Introduction

Due to the extreme complexity of the nervous system and its importance in the human body, many biologists, chemists, psychiatrists, computer scientists, physicists, and even mathematicians contributed to the study of this central part

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of the body. One of the most interesting questions in studying the nervous system is how neurons can synchronize. Synchronization is a process that makes two or more systems oscillate in a harmonious way and have the same behavior over time [13, 25]. In consequence of its efficient implementations in secure communications, laser technology, cryptography, combinatorial optimization and ecological systems [8], different methods were developed as well as various control techniques were implemented to accomplish synchronization in low- [1, 7, 12, 19, 20, 22, 23, 26], or high-dimension domains [10, 14–16, 21, 24, 29, 32, 33].

Like most natural processes, the activities of a neuron can be described by several equations analyzing the evolution of its characteristics over time. This description is usually called a mathematical model. In the literature, a few neuron models have been recently developed to explain neuronal dynamics. One of the simplest modification of the well-known Hodgkin-Huxley model [11] is the Fitzhugh-Nagumo model (FN model) [9, 17] that usefully describes such dynamics. Several significant researches were devoted to analyze the synchronization of the uni-dimensional FN model through describing its dynamics via some appropriate Ordinary Differential Equations (ODEs). For instance, H_∞ universe fuzzy approach [28], single- and two-input control technique [5], feedback control scheme [18], nonlinear feedback and adaptive controls approaches [27], and the internal model technique for spatially homogeneous FN model [31], are some examples of those researches. Although, in real neural modeling, the effect of the spatial component can not be avoided and the FN model must be presented by Partial Differential Equations (PDEs), the synchronization in the spatio-temporal domain of such model remains limited, just some results can be found in [2–4]. However, this work addresses some control techniques and some synchronization's analysis for the FN reaction-diffusion model that was expressed in [30] as follows:

$$\begin{cases} \frac{\partial u_1(x, t)}{\partial t} = d_1 \Delta u_1 - u_2 + f(u_1) + I, \\ \frac{\partial u_2(x, t)}{\partial t} = d_2 \Delta u_2 + \varepsilon u_1 - \varepsilon \gamma u_2, \end{cases} \quad x \in \Omega, \quad t > 0. \quad (1)$$

where Ω is a bounded domain in \mathbb{R}^n ($n \geq 1$), and $f(u)$ is a nonlinear function given by:

$$f(u) = -u^3 + (1 + \alpha)u^2 - \alpha u. \quad (2)$$

In this spatially extended system, the state u_1 corresponds to the membrane potential, while u_2 represents a combination of potassium activation and sodium inactivation at point $(x, t) \in \Omega \times (0, \infty)$. The parameters α , ε and γ are positive constants in which $0 < \alpha < 1$ and $\varepsilon \ll 1$. The parameter I corresponds to the external injected current. At the boundary of Ω , we assume that system (1)

satisfies the following homogeneous Neumann conditions:

$$\frac{\partial u_1}{\partial \eta} = \frac{\partial u_2}{\partial \eta} = 0, \quad x \in \partial\Omega, \tag{3}$$

where η is the unit vector normal to $\partial\Omega$.

Due to significant benefit of this model, a suitable control scheme will be next designed in a viable format to achieve synchronization between two neurons. The resultant findings will be analytically proved using some properties of the solution of system (1) together with the Lyapunov’s second method. These findings will be then displayed numerically in one- and two-spatial dimensions. However, the structure of this paper is arranged as follow: In section 2, the problem of drive-response is formulated and the error associated with synchronization is defined mathematically. Section 3 presents the main resultant findings of this work. Section 4 illustrates how such findings can be applicable through using several numerical simulations. At the end section, the final conclusion and some concluding remarks are reported.

2. Problem formulation

To assess the possibility of synchronizing couple of FN models, the drive-response method is implemented where these two models can be coupled using certain functions called controllers. The role of these functions is to force the response system’s output follows the drive system’s output over time. This mechanism is called the complete synchronization and, consequently, the two considered systems are said to be completely synchronized. In particular, the drive system (1) can be coupled with the following response system:

$$\begin{cases} \frac{\partial v_1(x, t)}{\partial t} = d_1 \Delta v_1 - v_2 + f(v_1) + I + C, \\ \frac{\partial v_2(x, t)}{\partial t} = d_2 \Delta v_2 + \varepsilon v_1 - \varepsilon \gamma v_2, \\ \frac{\partial v_1}{\partial \eta} = \frac{\partial v_2}{\partial \eta} = 0, \quad x \in \partial\Omega, \end{cases} \quad x \in \Omega, \quad t > 0. \tag{4}$$

We assume here that the parameters and the nonlinear function are the same as in system (1), but the associate initial conditions are arbitrary. The main aim of this study is to design a suitable controller C in a simplest form, making the control scheme cheaper in the application framework and easier to implement. To express the aforementioned details mathematically, we define the synchronization error

between the two systems, system (1) and system (4), as follows:

$$e(x, t) = \begin{pmatrix} e_1 = v_1 - u_1 \\ e_2 = v_2 - u_2 \end{pmatrix}. \quad (5)$$

Afterwards, we intend to prove that this error converges to zero as t goes to infinity. For purpose of clarity, we state the following definition:

Definition 1 *The drive and response systems given, respectively, in (1) and (4) are called completely synchronized if*

$$\lim_{t \rightarrow \infty} \|e(x, t)\|_{L^2} = 0. \quad (6)$$

Before we present the main result of this study, it is necessary to note that model (1) is well-posed and its associated solution is uniformly bounded. In particular, the following Lemma confirms and summarised all these affirmations.

Lemma 1 [6] *Model (1) admits a global unique solution (u_1, u_2) and $\exists K \in \mathbb{R}^+$ such that:*

$$u_1(x, t), u_2(x, t) \leq K$$

for all (x, t) in $\Omega \times [0, \infty)$.

3. Main results

In a logically equivalent manner to definition 1, we can prove the complete synchronization between system (1) and system (4) by showing that the zero solution of the following synchronization error system:

$$\begin{cases} \frac{\partial e_1(x, t)}{\partial t} = d_1 \Delta e_1 - e_2 + f(v_1) - f(u_1) + C, \\ \frac{\partial e_2(x, t)}{\partial t} = d_2 \Delta e_2 + \varepsilon e_1 - \varepsilon \gamma e_2, \end{cases} \quad x \in \Omega, \quad t > 0 \quad (7)$$

is globally asymptotically stable. At the boundary, it is clear that the above system satisfies the Neumann conditions:

$$\frac{\partial e_1}{\partial \eta} = \frac{\partial e_2}{\partial \eta} = 0, \quad x \in \partial\Omega. \quad (8)$$

Theorem 1 *The drive and the response systems given, respectively, in (1) and (4) are completely synchronized in accordance with the following one-dimensional linear control law:*

$$C = - \left(3K^2 + 2(1 + \alpha)K \right) e_1 + (1 - \varepsilon)e_2, \quad (9)$$

where K is positive constant given in Lemma 1.

Proof. Substituting controller (9) in the error system given in (7) yields:

$$\begin{cases} \frac{\partial e_1(x, t)}{\partial t} = d_1 \Delta e_1 - \varepsilon e_2 - (3K^2 + 2(1 + \alpha)K) e_1 + f(v_1) - f(u_1), \\ \frac{\partial e_2(x, t)}{\partial t} = d_2 \Delta e_2 + \varepsilon e_1 - \varepsilon \gamma e_2, \quad x \in \Omega, \quad t > 0 \end{cases}$$

To prove the global stability of the zero solution of system (7), we use the Lyapunov's direct (second) method together with the following positive definite Lyapunov functional:

$$V = \frac{1}{2} \int (e^T e) \, dx.$$

This leads us to prove that the derivative of this functional with respect to t is negative definite. Actually, this can be carried out as follows:

$$\begin{aligned} \frac{\partial V}{\partial t} &= \int_{\Omega} \left(e_1 \frac{\partial e_1}{\partial t} + e_2 \frac{\partial e_2}{\partial t} \right) dx \\ &= \int_{\Omega} e_1 \left(d_1 \Delta e_1 - \varepsilon e_2 - (3K^2 + 2(1 + \alpha)K) e_1 + f(v_1) - f(u_1) \right) dx \\ &\quad + \int_{\Omega} e_2 \left(d_2 \Delta e_2 + \varepsilon e_1 - \varepsilon \gamma e_2 \right) dx = \int_{\Omega} (d_1 e_1 \Delta e_1 + d_2 e_2 \Delta e_2) dx \\ &\quad - \int_{\Omega} \left((3K^2 + 2(1 + \alpha)K) e_1^2 - \varepsilon \gamma e_2^2 \right) dx + \int_{\Omega} e_1 (f(v_1) - f(u_1)) dx. \end{aligned}$$

In view of Eq. (2) and Lemma 1, we can attain the following assertion:

$$\begin{aligned} \frac{\partial V}{\partial t} &= \int_{\Omega} (d_1 e_1 \Delta e_1 + d_2 e_2 \Delta e_2) dx - \int_{\Omega} \left((3K^2 + 2(1 + \alpha)K) e_1^2 - \varepsilon \gamma e_2^2 \right) dx \\ &\quad + \int_{\Omega} \left(|v_1|^2 + |v_1| |u_1| + |u_1|^2 + (1 + \alpha) (|u_1| + |v_1|) - \alpha \right) e_1^2 dx, \\ &\leq \int_{\Omega} (d_1 e_1 \Delta e_1 + d_2 e_2 \Delta e_2) dx - \int_{\Omega} \left((3K^2 + 2(1 + \alpha)K) e_1^2 - \varepsilon \gamma e_2^2 \right) dx \\ &\quad + \int_{\Omega} \left(3K^2 + 2(1 + \alpha)K - \alpha \right) e_1^2 dx, \\ &\leq \int_{\Omega} (d_1 e_1 \Delta e_1 + d_2 e_2 \Delta e_2) dx - \int_{\Omega} \left(\alpha e_1^2 + \varepsilon \gamma e_2^2 \right) dx. \end{aligned}$$

By using Green's formula, one can show that the derivative of the Lyapunov functional V satisfies the following estimate:

$$\begin{aligned} \frac{\partial V}{\partial t} \leq & -d_1 \int_{\Omega} |\nabla e_1|^2 dx + d_1 \int_{\partial\Omega} e_1 \frac{\partial e_1}{\partial \eta} d\sigma - d_2 \int_{\Omega} |\nabla e_2|^2 dx \\ & + d_2 \int_{\partial\Omega} e_2 \frac{\partial e_2}{\partial \eta} d\sigma - \alpha \int_{\Omega} e_1^2 dx - \varepsilon\gamma \int_{\Omega} e_1^2 dx. \end{aligned}$$

Accordingly, using the homogeneous Neumann conditions given in (8) leads one to obtain:

$$\frac{\partial V}{\partial t} \leq - \left(d_1 \int_{\Omega} |\nabla e_1|^2 dx + d_2 \int_{\Omega} |\nabla e_2|^2 dx + \alpha \int_{\Omega} e_1^2 dx + \varepsilon\gamma \int_{\Omega} e_1^2 dx \right) < 0,$$

which finishes the proof. \square

4. Numerical simulations

This part intends to demonstrate some numerical illustrations in one- and two-spatial dimensions to explain the suitability of the synchronization scheme described in this article. These simulations have been carried out using the finite difference approach and MATLAB software. In such simulations, we let $\Omega = [0, 50]$, $t \leq 100$ and

$$(d_1, d_2, \alpha, \varepsilon, \gamma, I) = (0.5, 0.8, 0.139, 0.008, 2.54, 2). \quad (10)$$

On the other hand, we select the initial condition associated with the drive system to be as follows:

$$(u_1(x, 0), u_2(x, 0)) = \left(0.5 + 0.1 \sin\left(\frac{\pi x}{5}\right), 0.8 + 0.2 \cos\left(\frac{\pi x}{5}\right) \right). \quad (11)$$

The dynamics of the spatio-temporal solutions of system (1) are exhibited in Fig. 1. For more illustration, Fig. 2 depicts these solutions in 2-dimensional space (2D-space). For comparison reasons, we plot the uncontrolled response system (i.e., system (4) with $C = 0$) in one- and two-spatial dimensions (see Fig. 3 and Fig. 4), where the initial condition associated with system (4) is given as follows:

$$(v_1(x, 0), v_2(x, 0)) = (1.5 + 0.2 \sin(x), 0.28 + 0.21 \cos(x)). \quad (12)$$

Comparing with the dynamics of the drive system (Fig. 1 and 2), one might notice that both figures, Fig. 3 and Fig. 4, have confirmed that the uncontrolled response

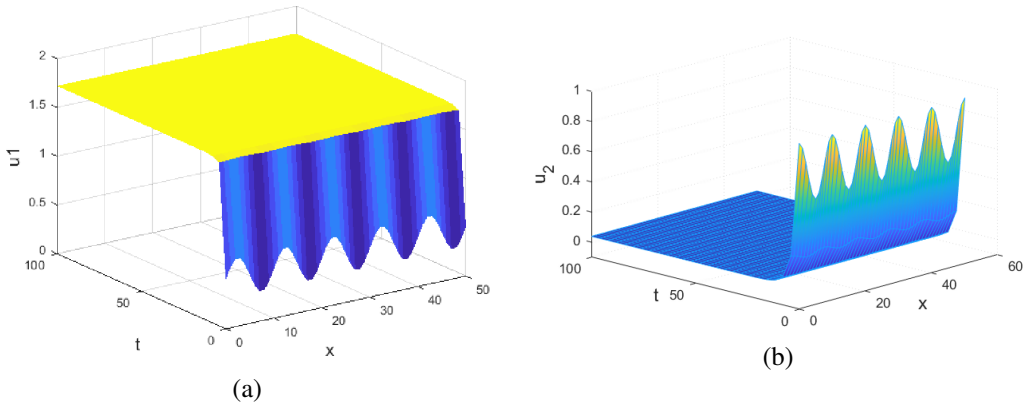


Figure 1: Dynamic behavior of the drive system (1) with $d_1 = 0.5$, $d_2 = 0.8$, $\alpha = 0.139$, $\varepsilon = 0.008$, $\gamma = 2.54$, $I = 2$ according to the initial conditions given in (11)

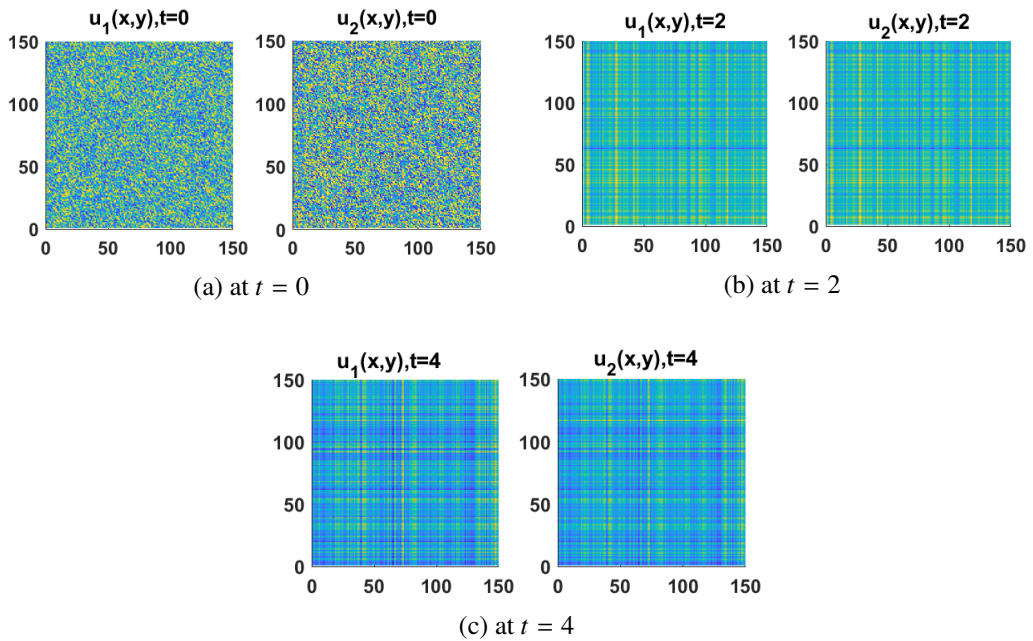


Figure 2: Solution of the drive system (1) in 2D-space

system (4) does not synchronize with the drive system (1). In the same perspective and based on Theorem 1, if one selects $K = 0.2$, then the uni-dimensional linear controller will be designed as follows:

$$C = 3.259e_1 + 0.992e_2$$

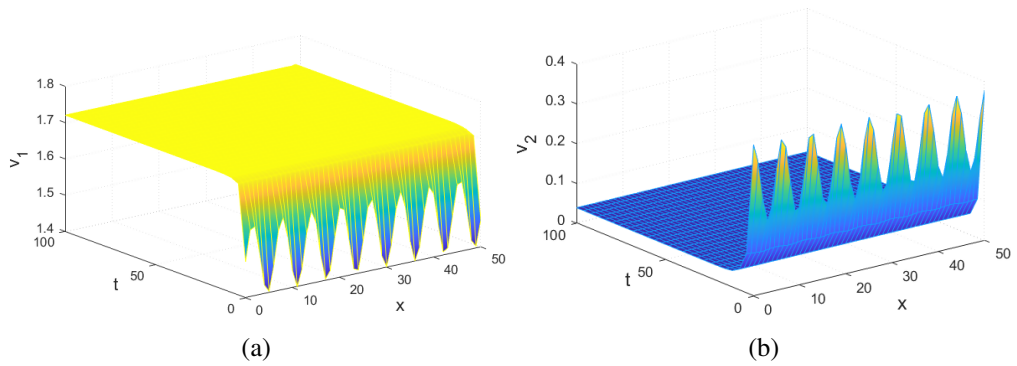


Figure 3: Dynamic behavior of the response system (4) with $d_1 = 0.5$, $d_2 = 0.8$, $\alpha = 0.139$, $\varepsilon = 0.008$, $\gamma = 2.54$, $I = 2$ according to the initial conditions given in (12)

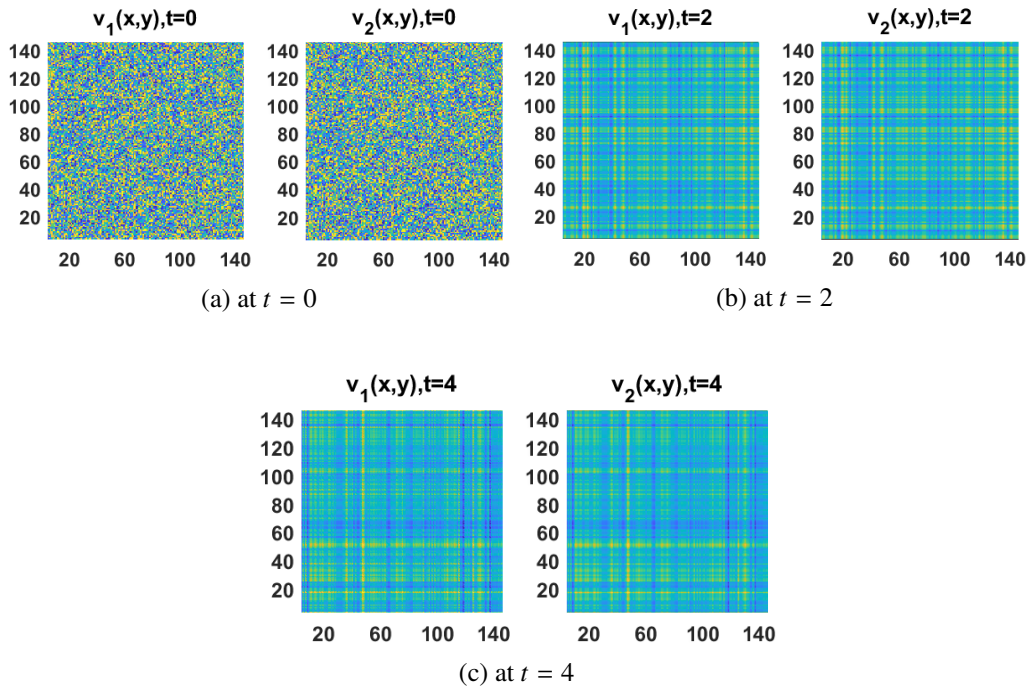


Figure 4: Solution of the response system (4) in 2D-space

with noting that the drive system (1) and the response system (4) will be also globally synchronized. To illustrate this numerically, the spatiotemporal solutions of the error synchronization system (7) are provided in Fig. 5 and Fig. 6 in one- and two-dimensional space. This evolution indicates, consequently, that all errors go to 0 as t goes to $+\infty$.

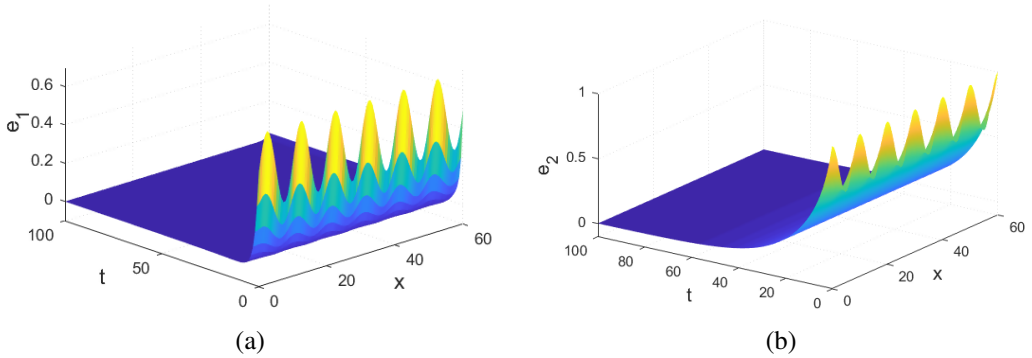


Figure 5: Dynamic behavior of the solutions of the spatiotemporal synchronization error system (5) with $d_1 = 0.5$, $d_2 = 0.8$ and $K = 0.2$

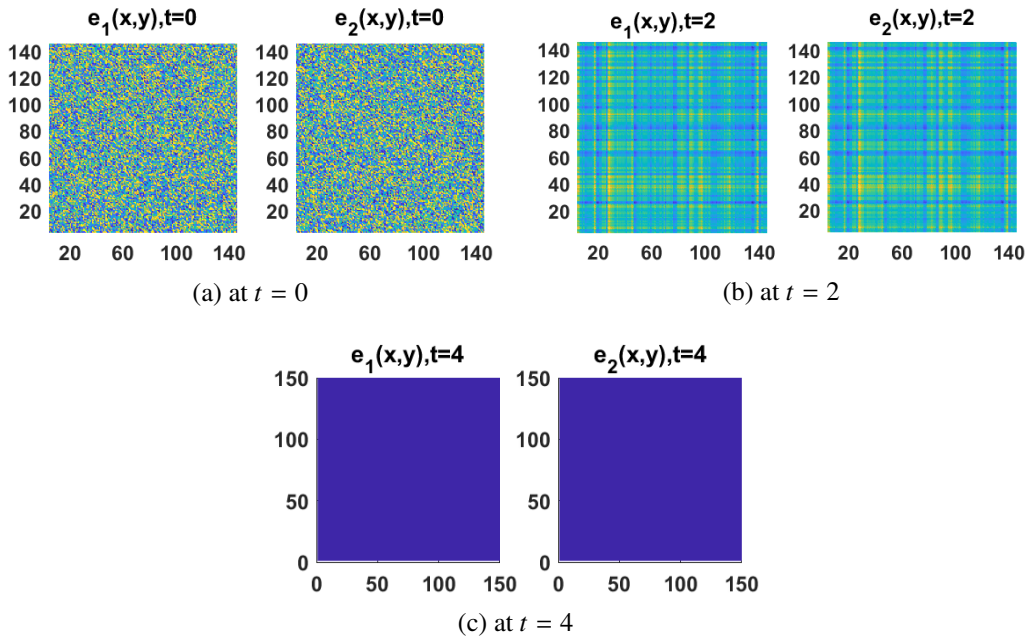


Figure 6: Solution of the spatiotemporal synchronization error system (5) in 2D-space

5. Conclusion

Over the past few years, several researchers were focused on studying the nervous system, especially in how neurons synchronize with each others. In this paper, we have developed a novel control scheme to achieve synchronization between two neurons in spatially extended domain. This contribution has been proved rigorously with the use of the uniform boundedness of the unique solution of the Fitzhugh-Nagumo model and the Lyapunov's second method. In order

to reduce the implementation cost, this controller has been designed in uni-dimensional and linear form, and then it has been illustrated numerically in one- and two-spatial dimensions. There is no question that the findings of this study will motivate us to discuss this subject further. For this reason, we are planning to analyze synchronization in many types of spatially extended systems, including lattice maps, stochastic and fractional-order models.

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