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INVARIANT UNIVERSALITY FOR PROJECTIVE PLANES

A b s t r a c t. We continue the work of [1, 2, 3] by analyzing the equivalence relation of bi-embeddability on various classes of countable planes, most notably the class of countable non-Desarguesian projective planes. We use constructions of the author from [13] to show that these equivalence relations are invariantly universal, in the sense of [3], and thus in particular complete analytic. We also introduce a new kind of Borel reducibility relation for standard Borel G -spaces, which requires the preservation of stabilizers, and explain its connection with the notion of full embeddings commonly considered in category theory.

1. Introduction

Definition 1.1. A plane is a system of points and lines satisfying:

(A) every pair of distinct points determines a unique line;

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- (B) every pair of distinct lines intersects in at most one point;
- (C) every line contains at least two points;
- (D) there exist at least three non-collinear points.

A plane is *projective* if in addition:

- (B') every pair of lines intersects in exactly one point.

A plane is *simple*¹ if except for a finite number of points every point is incident with at most two non-trivial lines (i.e. lines containing more than two points).

The class of simple planes and the class of (non-Desarguesian) projective planes are first-order classes, and so we can regard them as standard Borel spaces, and use invariant descriptive set theory to analyze the complexity of analytic equivalence relations defined on them. We recall that a binary relation R defined on a standard Borel space X is called *analytic* (or Σ_1^1), if it is an analytic subset of the product space $X \times X$, i.e., it is the projection of a Borel set $B \subseteq Y \times X \times X$, for some Polish space Y .

The main tool used to compare equivalence relations is *Borel reducibility*. If E and F are two equivalence relations on the standard Borel spaces X and Y , we say that E *Borel reduces* to F (and write $E \leq_B F$) if there is a Borel map $f: X \rightarrow Y$ witnessing that $x E y \iff f(x) F f(y)$, for every $x, y \in X$. We can take the statement “ E Borel reduces to F ” as a formal way of saying that E is not more complicated than F , as any set of complete invariants for F includes a set of complete invariants for E . When $E \leq_B F$ and $F \leq_B E$, the complexity of E and F is considered the same, and we say that E and F are *Borel bi-reducible* (in symbols, $E \sim_B F$).

In [9] the authors proved that the bi-embeddability relation \equiv_{Gr} on countable graphs is a *complete analytic equivalence relation*. That is, \equiv_{Gr} is a \leq_B -maximum among all analytic equivalence relations. It follows that \equiv_{Gr} is strictly more complicated than any isomorphism relation between countable structures, and so it can be argued that the problem of classifying countable graphs up to bi-embeddability is highly intractable.

In [4] the authors proved that the bi-embeddability relation on countable graphs is analytic complete in a very strong sense: every analytic equivalence relation is Borel bi-reducible with the restriction of \equiv_{Gr} to some $\mathfrak{L}_{\omega_1\omega}$ -subclass of the standard Borel space of countable graphs. Such property reappeared thereafter in [3], where it was considered in a more general framework and called *invariant universality* — the definition given in [3] is stated for all analytic equivalence relations (not only for those defined on spaces of countable structures).

Next, the work of [3] was continued by Calderoni et al., who proved invariant universality for the bi-embeddability relation on several $\mathfrak{L}_{\omega_1\omega}$ -classes, which include countable

¹The notion of simple plane is not standard and we use it for lack of a better name.

groups (cf. [2, Theorem 3.5]), and countable fields of fixed characteristic $p \neq 2$ (cf. [1, Theorem 5.12]). The main technique used in [1] and [2] requires to have a Borel reduction from the bi-embeddability relation between graphs to the bi-embeddability relation on the class under consideration, and the possibility to explicitly describe the automorphism group of each structure in the image of the reduction – this technique is taken from [3].

In [13] the author proved the Borel completeness of both the class of simple planes and the class of non-Desarguesian projective planes. That is, the isomorphism relation on both of those classes of planes is a \leq_B -maximum for all orbit equivalence relations arising from a Borel action of S_∞ , the Polish group of permutations on \mathbb{N} . In each case he defined a Borel reduction from the isomorphism relation between countable graphs to the isomorphism relation between the class under consideration. Furthermore, the constructions have the remarkable additional property of preserving automorphism groups. As we point out in the last section, this feature is common to many categorical construction which give a full embedding between two $\mathfrak{L}_{\omega_1\omega}$ -class, and can be adapted to define a Borel reduction between the isomorphism relations defined on the corresponding standard Borel spaces.

Our aim in this paper is twofold:

1. To study the bi-embeddability relation on the classes of countable planes considered in [13], with the stipulation that the bi-embeddability relation between planes coincide with the bi-embeddability relation between the corresponding geometric lattices.
2. To develop some generalities on the kind of stabilizer preserving Borel reduction (or SPB reduction for short) mentioned above.

Concerning the first aim, we use the main constructions of [13] to prove:

Theorem 1.2. *The bi-embeddability relation \equiv_{pl} between countable simple planes is invariantly universal.*

Theorem 1.3. *The bi-embeddability relation \equiv_{ppl} between countable non-Desarguesian projective planes is invariantly universal.*

Corollary 1.4. *Every Σ_1^1 equivalence relation is Borel bi-reducible with the bi-embeddability relation restricted to some $\mathfrak{L}_{\omega_1\omega}$ -subclass of countable simple planes.*

Corollary 1.5. *Every Σ_1^1 equivalence relation is Borel bi-reducible with the bi-embeddability relation restricted to some $\mathfrak{L}_{\omega_1\omega}$ -subclass of countable non-Desarguesian projective planes.*

Consequently, the bi-embeddability relation in the class of countable non-Desarguesian projective planes is strictly more complicated than isomorphism. In fact, we get that \equiv_{ppl} and \equiv_{pl} are complete analytic equivalence relations in the sense of [9, Def. 1.2]. It follows

that we cannot classify the class of countable non-Desarguesian projective planes up to bi-embeddability in any reasonable way: neither in terms of Ulm-type invariants, nor in terms of orbits of Polish group actions.

Concerning the second aim, we point out how in some cases SPB reductions can be obtained from the existing literature in category theory and list a couple of open questions.

2. Invariant Universality

Following [9] we consider Borel reducibility between quasi-orders, i.e., reflexive and transitive binary relations.

Definition 2.1. Let Q and R be quasi-orders on the standard Borel spaces X and Y .

- Q *Borel reduces* to R (in symbols, $Q \leq_B R$) if there exists a Borel map $f: X \rightarrow Y$ such that for all $x, y \in X$,

$$x Q y \iff f(x) R f(y).$$

In this case we say that f is a Borel reduction from Q to R (in symbols, $f: Q \leq_B R$).

- Q is *Borel bi-reducible* with R (in symbols, $Q \sim_B R$) if $Q \leq_B R$ and $R \leq_B Q$.

In particular, when Q and R are equivalence relations, one obtains the usual notion of Borel reducibility mentioned in the introduction. When Q is an analytic quasi-order on X and A is a Borel subset of X , we can regard A as a standard Borel space with its relative standard Borel structure and the quasi-order on A obtained by the restriction of Q . We shall denote by $Q \upharpoonright A$ the restriction of Q to A .

We now recall the main definitions from [3, Definition 1.1].

Definition 2.2. Let Q be a Σ_1^1 quasi-order on some standard Borel space X and let E be a Σ_1^1 equivalence subrelation of Q (i.e. $E \subseteq E_Q$, where E_Q is the equivalence relation induced by Q). We say that (Q, E) is *invariantly universal* if for every Σ_1^1 quasi-order P there is a Borel subset $A \subseteq X$ which is E -invariant and such that $P \sim_B Q \upharpoonright A$.

Definition 2.3. Let F be a Σ_1^1 equivalence relation on some standard Borel space X and let E be a Σ_1^1 equivalence subrelation of F . We say that (F, E) is *invariantly universal* if for every Σ_1^1 equivalence relation D there is a Borel subset $A \subseteq X$ which is E -invariant and such that $D \sim_B F \upharpoonright A$.

Notice that if (F, E) is invariantly universal, then F is in particular a complete analytic equivalence relation in the sense of [9, Definition 1.2]. Moreover, our interest for quasi-orders is easily explained: if (Q, E) is an invariantly universal quasi-order and E_Q is the equivalence relation generated by Q , then (E_Q, E) is an invariantly universal equivalence relation.

Throughout this paper we will make use of the following notation.

Notation 2.4. Let X be a standard Borel space of countable structures.

- (i) We denote by \sqsubseteq_X (or simply \sqsubseteq) the embeddability relation on X .
- (ii) We denote by \cong_X (or simply \cong) the isomorphism relation on X .
- (iii) We denote by \equiv_X (or simply \equiv) the bi-embeddability relation on X .
- (iv) We say that the quasi-order Q on X is invariantly universal if (Q, \cong_X) is (cf. Definition 2.2).
- (v) We say that the equivalence relation E on X is invariantly universal if (E, \cong_X) is (cf. Definition 2.3).

The following fact is an immediate consequence of the López-Escobar Theorem (cf. [8, Theorem 16.8]) and gives a further insight of the phenomenon of invariantly universality on spaces of countable structures.

Fact 2.5. *If X is a standard Borel space of countable structures, and F is a Σ_1^1 equivalence relation on X , then F is invariantly universal if and only if every Σ_1^1 equivalence relation is Borel bi-reducible with the restriction of F to some $\mathfrak{L}_{\omega_1\omega}$ subclass of X .*

We now present a sufficient condition for invariant universality. Let X_{Gr} be the standard Borel space of countable graphs. First we abstract the following fact from [3, Section 3].

Fact 2.6. *There is a Borel subset $\mathbb{X} \subseteq X_{\text{Gr}}$ such that the following hold:*

- (i) $=_{\mathbb{X}}$ and $\cong_{\mathbb{X}}$ coincide;
- (ii) each graph in \mathbb{X} is rigid; that is, it has no non-trivial automorphism;
- (iii) for every Σ_1^1 quasi-order P on $2^{\mathbb{N}}$, there exists an injective Borel reduction $\alpha \mapsto T_\alpha$ from P to $\sqsubseteq_{\mathbb{X}}$.

Convention 2.7. Throughout the paper we will denote by \mathbb{X} the set from Fact 2.6.

Notation 2.8. We denote by S_∞ the Polish group of permutations on \mathbb{N} , and by $\text{Subg}(S_\infty)$ the standard Borel space of closed subgroup of S_∞ , endowed with the Effros-Borel structure (see [8, Section 12.C]).

Now we recall the following fact, which is a particular case of [3, Theorem 4.2].

Fact 2.9. *Let X be a standard Borel space of countable structures. Then the relation \sqsubseteq_X is an invariantly universal quasi-order provided that the following conditions hold:*

- (I) *there is a Borel map $f: \mathbb{X} \rightarrow X$ such that:*

$$(i) f: \sqsubseteq_{\mathbb{X}} \leq_B \sqsubseteq_X;$$

$$(ii) f: \cong_{\mathbb{X}} \leq_B \cong_X;$$

(II) the map $f(\mathbb{X}) \rightarrow \text{Subg}(S_\infty) : f(T) \mapsto \text{Aut}(f(T))$ is Borel.

We stress:

Remark 2.10. Since every graph in \mathbb{X} is rigid, whenever the reduction f witnessing (I) of Fact 2.9 further preserves the automorphisms groups (i.e. $\text{Aut}(T) \cong \text{Aut}(f(T))$), then clearly condition (II) from Fact 2.9 is automatically satisfied.

3. Planes

Definition 3.1. Let P_1 and P_2 be planes (cf. Definition 1.1).

- (1) We say that P_1 is a *subplane* of P_2 if (the domain of) P_1 is a subset of P_2 , points of P_1 are points of P_2 , lines of P_1 are lines of P_2 , and the point p is on the line ℓ in P_1 if and only if the point p is on the line ℓ in P_2 .
- (2) We say that P_1 is a *complete subplane* of P_2 if P_1 is a subplane of P_2 and any point of intersection of lines of P_1 which lies in P_2 also lies in P_1 , and every line joining two points of P_1 which lies in P_2 also lies in P_1 .

Definition 3.2 (Cf. [10, Theorem 11.4]). Given a plane P we define by recursion on $n < \omega$ a chain of planes $(P_n : n < \omega)$ as follows:

$n = 0$. Let $P_n = P$.

$n = m + 1$. For every pair of parallel lines $\ell \neq \ell'$ in P_m add a new point $\ell \wedge \ell'$ to P_m incident only with ℓ and ℓ' . Let P_n be the resulting plane.

We define the *free projective extension* of P to be $F(P) := \bigcup_{n < \omega} P_n$.

Definition 3.3. Let P be a plane.

- (1) We say that a line from P is *trivial* if it contains exactly two points.
- (2) When P is finite, then we say that P is *confined* if every point of P is incident with at least three lines of P , and every line of P is non-trivial.
- (3) We say that P is confined if every point and every line of P is contained in a finite confined subplane of P .

Fact 3.4. Let P_1 and P_2 be confined planes and $f: P_1 \rightarrow P_2$ a complete embedding (i.e. $f(P_1)$ is a complete subplane of P_2). Then there exists a complete embedding $\hat{f}: F(P_1) \rightarrow F(P_2)$ such that $f \upharpoonright P_1 = f$.

Proof. This is essentially [7, Theorem 4.3]. \square

Fact 3.5. *Let P_1 and P_2 be confined planes (cf. Definition 3.3) and $f: F(P_1) \rightarrow F(P_2)$ an embedding. Then $f \upharpoonright P_1 \subseteq P_2$.*

Proof. In the terminology of [10, Chapter XI], the core of $F(P_\ell)$ equals the core of P_ℓ (cf. [10, Corollary at p. 224]), which in turn is the whole of P_ℓ , since by assumption P_ℓ is confined. It easily follows that $f \upharpoonright P_1 \subseteq P_2$, since otherwise the core of $F(P_2)$ is not equal to P_2 . \square

Definition 3.6. A plane is said to be Desarguesian if the following condition is satisfied: whenever p, q, r and p', q', r' are triples of non-collinear points such that lines pp' , qq' and rr' pass through the same point, then the points:

$$pq \wedge p'q', pr \wedge p'r' \text{ and } qr \wedge q'r'$$

are collinear (where $\ell \wedge \ell'$ is the unique point incident with both lines).

As well known, the class of planes corresponds canonically to the class of geometric lattices of length 3 (cf. [11, Section 2]). For our purposes the perspective of geometric lattices is preferable (see Convention 3.11 and the proofs of Theorems 4.2 and 4.3), and thus in the next two remarks we make explicit this correspondence. Before doing this, we recall the basic definitions needed.

Recall that a lattice is an order (L, \leq) such that any two elements a, b have a least upper bound and a greatest lower bound, denoted by $a \wedge b$ and $a \vee b$, respectively. A chain in a lattice (L, \leq) is a subset $X \subseteq L$ such that (X, \leq) is a linear order.

Assumption: In this paper all lattices have a maximum element 1 and a minimum element 0. Furthermore, any chain between any two elements is finite.

Given a lattice (L, \leq) (as in the assumption above) and $x \in L$, we let $h(x)$, the height of x , to be the length of the longest maximal chain between 0 and x . Furthermore, given $a, b \in L$, we say that a is *covered* by b , for short $a \lessdot b$, if $a < b$ and for every $a \leq c \leq b$ we have that either $a = c$ or $c = b$. Finally, we say that a is an *atom* if it covers 0.

Definition 3.7. Let (L, \leq) be a lattice.

1. We say that (L, \leq) is *semimodular* if for every $a, b \in L$ we have that

$$a \wedge b \lessdot a \Rightarrow b \lessdot a \vee b.$$

2. We say that (L, \leq) is a *point lattice* if every $a \in L$ is a supremum of atoms.
3. We say that (L, \leq) is *geometric* if (L, \leq) is a semimodular point lattice.

Remark 3.8. Let P be a plane as in Definition 1.1. First of all, add to P a largest element 1 and a smallest element 0, and let $P^+ = P \cup \{0, 1\}$. Then, for every pair of points a and b from P let $a \vee b$ denotes the unique line they determine. For every pair of lines ℓ_1 and ℓ_2 from P let $\ell_1 \wedge \ell_2$ be 0 if the two lines are parallel, and let it be the unique point in their intersection otherwise. Then $(P^+, 0, 1, \vee, \wedge)$ is a geometric lattice of length 3 (cf. Section 2 of the present paper).

Remark 3.9. Let $(P, 0, 1, \vee, \wedge)$ be a geometric lattice of length 3 (cf. [11, Section 2]), and $P^- = P - \{0, 1\}$. Let A be the set of atoms of P and let B be the set of co-atoms of P , and for $a \in A$ and $b \in B$, let $a E b$ if $a \vee b = b$. Then (P^-, A, B, E) is a plane, where A denotes the set of points of the plane, B denotes the set of lines of the plane and E is the incidence relation between points and lines.

Remark 3.10. Let P_1 and P_2 be planes, and P_1^+ and P_2^+ be the associated geometric lattices. Then P_1 is a complete subplane of P_2 iff P_1^+ is a sublattice of P_2^+ .

Convention 3.11. For the rest of the paper, formally, by a plane we will mean a geometric lattice of length 3 considered with respect to the signature $L = \{0, 1, \vee, \wedge\}$. In particular, an embedding of planes will mean an embedding of geometric lattices.

4. Proofs of Main Theorems

First of all we stress:

We invite the reader to keep in mind Convention 3.11.

- Notation 4.1.** (1) We denote by X_{Gr} the standard Borel space of countable graphs.
- (2) We denote by X_{pl} the standard Borel space of simple planes² (cf. Definition 1.1), and by \sqsubseteq_{pl} and \cong_{pl} the relation of embeddability and isomorphism on X_{pl} , respectively.
- (3) We denote by X_{ppl} the standard Borel space of countable non-Desarguesian projective planes³, and by \sqsubseteq_{ppl} and \cong_{ppl} the relation of embeddability and isomorphism on X_{ppl} , respectively.

Theorem 4.2. *For every $\Gamma \in X_{\text{Gr}}$ let P_Γ be defined as in [13, Section 3]. The map $X_{\text{Gr}} \rightarrow X_{\text{pl}}: \Gamma \mapsto P_\Gamma$ is Borel and:*

²Clearly the class of countable simple planes is a first-order class.

³The class of countable non-Desarguesian projective planes is first-order (see e.g. [15, Definition 5.1.1]).

- (1) $\Gamma_1 \cong \Gamma_2$ if and only if $P_{\Gamma_1} \cong P_{\Gamma_2}$;
- (2) $\text{Aut}(\Gamma) \cong \text{Aut}(P_\Gamma)$;
- (3) $\Gamma_1 \sqsubseteq \Gamma_2$ if and only if $P_{\Gamma_1} \sqsubseteq P_{\Gamma_2}$.

Proof. Items (1) and (2) are proved in [13]. Concerning (3), argue as in the end of the proof of [13, Theorem 12] (where it is proved that $\Gamma \mapsto P_\Gamma$ is isomorphism-invariant). Notice that the choice of signature (and thus of embedding) is crucial for the argument to go through, since we need that intersection of lines are preserved by the embedding in order to use the (\star_1) of the proof in the way we use it there. \square

Theorem 4.3. *For every $\Gamma \in X_{\text{Gr}}$ let P_Γ^* be defined as in [13, Section 4]. The map $X_{\text{Gr}} \rightarrow X_{\text{ppl}}: \Gamma \mapsto P_\Gamma^*$ is Borel and:*

- (1) $\Gamma_1 \cong \Gamma_2$ if and only if $P_{\Gamma_1} \cong P_{\Gamma_2}$;
- (2) $\text{Aut}(\Gamma) \cong \text{Aut}(P_\Gamma)$;
- (3) $\Gamma_1 \sqsubseteq \Gamma_2$ if and only if $P_{\Gamma_1} \sqsubseteq P_{\Gamma_2}$.

Proof. Items (1) and (2) are proved in [13]. Concerning (3), it follows from Remark 3.10, Facts 3.4 and 3.5, [13, (\star_1) of Proof of Theorem 3] and Theorem 4.2(3). Notice that also in this case the choice of signature (and thus of embedding) is crucial. \square

Proof of Theorem 1.2. Consider the restriction of the map $\Gamma \mapsto P_\Gamma$ on \mathbb{X} from Fact 2.6. By items (1) and (3) of Theorem 4.2 the map $\Gamma \mapsto P_\Gamma$ simultaneously reduces $\sqsubseteq_{\mathbb{X}}$ to \sqsubseteq_{pl} and $\cong_{\mathbb{X}}$ to \cong_{pl} . Condition (II) of Fact 2.9 follows by Theorem 4.2(2) and Remark 2.10. The statement now follows from Fact 2.9. \square

Proof of Corollary 1.4. The statement follows from Theorem 1.2 and Fact 2.5. \square

Proof of Theorem 1.3. Argue as in the proof of Theorem 1.2 using Theorem 4.3. \square

Proof of Corollary 1.5. The statement follows from Theorem 1.3 and Fact 2.5. \square

5. SPB reductions

In this section we will denote by G a Polish group and by X and Y two standard Borel spaces. If $a: G \times X \rightarrow X$ is a Borel action of G on X , we shall denote by E_a the orbit equivalence relation arising from a (i.e., $x E_a y \iff \exists g \in G (a(g, x) = y)$). The stabilizer of any point $x \in X$ is the subgroup $G_x := \{g \in G \mid a(g, x) = x\}$.

Definition 5.1. Let $a: G \times X \rightarrow X$, $b: G \times Y \rightarrow Y$ be two Borel actions so that X and Y are standard Borel G -spaces. We say that E_a *SPB reduces* to E_b (in symbols, $E_a \leq_{\text{SPB}} E_b$) if there is a Borel map $f: X \rightarrow Y$ witnessing that $E_a \leq_B E_b$ and such that

$$\forall x \in X (G_x \cong G_{f(x)}). \quad (\text{SP})$$

We stress the following:

Remark 5.2. When $G = S_\infty$, condition (SP) is equivalent to saying that

$$\forall x \in X (\text{Aut}(x) \cong \text{Aut}(f(x))).$$

Items (1)–(2) of both Theorem 4.2 and Theorem 4.3 can be briefly reformulated as follows.

Theorem 5.3 ([13]). *The following SPB reductions hold:*

- $\cong_{\text{Gr}} \leq_{\text{SPB}} \cong_{\text{pl}}$;
- $\cong_{\text{Gr}} \leq_{\text{SPB}} \cong_{\text{ppl}}$.

We highlight the following fact which follows directly from Fact 2.9 and Remark 2.10, and exhibits how Theorem 5.3 can be used to prove Theorem 1.2 and Theorem 1.3.

Fact 5.4. *Let X be a standard Borel space of countable structures. Then the relation \sqsubseteq_X is an invariantly universal quasi-order provided that there is a Borel map $f: \mathbb{X} \rightarrow X$ such that:*

- (i) $f: \sqsubseteq_{\mathbb{X}} \leq_B \sqsubseteq_X$;
- (ii) $f: \cong_{\mathbb{X}} \leq_{\text{SPB}} \cong_X$.

Some examples of SPB reductions directly follow from the existence of full embeddings between categories. In category theory there has been quite a lot of work concerning the complexity of different categories by means of (categorical) embeddings⁴. Several classical examples of categorical embedding typically concern categories whose objects are algebraic structures of a fixed type, and whose morphisms are the respective homomorphisms (or embedding) between those structures. A comprehensive reference for this kind of results is the book [14]. One of the strongest notion of (categorical) embedding that has been considered in the literature is the one of *full embedding*, an injective functor which further induces a bijection between the morphisms in the domain category and the morphisms in the target category.

⁴The categorical notion of embedding should not be confused with the one of embedding between structures.

Definition 5.5. If \mathcal{C} and \mathcal{D} are categories, a *full embedding* F from \mathcal{C} into \mathcal{D} is a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ such that

- F is injective on the objects of \mathcal{C} ;
- for every a, b the map $\text{Hom}_{\mathcal{C}}(a, b) \rightarrow \text{Hom}_{\mathcal{D}}(F(a), F(b)): f \mapsto F(f)$ is a bijection.

An example of full embedding is given by the constructions of [13], that we previously mentioned in the statements of Theorem 4.2 and Theorem 4.3. E.g., the map $\Gamma \mapsto P_{\Gamma}^*$ can be redefined for the category of all graphs, regardless of their cardinality (and in fact this is the setting of [13]), to prove the following:

Theorem 5.6 (essentially [13]). *There exists a full embedding from the category of graphs together with graph embeddings into the category of non-Desarguesian projective planes together with planes embeddings (recall Convention 3.11).*

Our interest in full embeddings is easily explained. First, notice that any $\mathfrak{L}_{\omega_1\omega}$ -class \mathcal{C} can be regarded as a category — the morphisms of \mathcal{C} are the usual embeddings between the structures of \mathcal{C} . The next proposition explains how certain full embedding induce a Borel reduction.

Proposition 5.7. *Let \mathcal{C} and \mathcal{D} be two $\mathfrak{L}_{\omega_1\omega}$ -classes so that we can consider the corresponding standard Borel spaces $X_{\mathcal{C}}$ and $X_{\mathcal{D}}$. Suppose that F is a full embedding from \mathcal{C} into \mathcal{D} such that*

- (i) F maps objects whose domain is ω to object whose domain is ω ;
- (ii) $F \upharpoonright X_{\mathcal{C}}$ can be realized as a Borel function from $X_{\mathcal{C}}$ to $X_{\mathcal{D}}$; i.e., there is a Borel function $f: X_{\mathcal{C}} \rightarrow X_{\mathcal{D}}$ such that for every $x \in X_{\mathcal{D}}$, $f(x) \cong F(x)$.

Then, the isomorphism relation $\cong_{\mathcal{C}}$ SPB reduces to $\cong_{\mathcal{D}}$.

Proof. Since F is full, for every x, y , the sets of isomorphisms between x and y and their images, respectively denoted by $\text{Iso}(x, y)$ and $\text{Iso}(F(x), F(y))$, are isomorphic via the map

$$\text{Iso}(x, y) \rightarrow \text{Iso}(F(x), F(y)): h \mapsto F(h).$$

In particular, for every $x \in X_{\mathcal{C}}$, the map:

$$\text{Aut}(x) \rightarrow \text{Aut}(F(x)): h \mapsto F(h).$$

is a bijection, indeed it is a group isomorphism.

Now let $f: X_{\mathcal{C}} \rightarrow X_{\mathcal{D}}$ be a Borel function as in (ii). Since every $F(x)$ is isomorphic to $f(x)$, we have that for every $x \in X_{\mathcal{C}}$,

$$\text{Aut}(x) \cong \text{Aut}(F(x)).$$

□

Following the approach of [12], we can regard the subcategories of \mathcal{C} and \mathcal{D} formed by $X_{\mathcal{C}}$ and $X_{\mathcal{D}}$, respectively, together with the isomorphism maps as analytic groupoids. The SPB reduction we get is in particular a functorial reduction (see [12, Definition 2.8.1]).

The following full embeddings between categories are well-known in the literature. When not specified, we consider categories with respect to homomorphism embeddings as morphisms.

Fact 5.8. *There is a full embedding from the category of graphs into any of the following categories.*

- the category of partial orders PO ([14, Chapter IV, 5.6]);
- the category of semigroups Smg ([14, Chapter V, 2.9]);
- the category of unital rings Rng_1 ([5, Section 3]).

One can check that for any of the aforementioned categorical embeddings, items (i)–(ii) of Proposition 5.7 are satisfied, thus we obtain the following.

Proposition 5.9. *The isomorphism relation between countable graphs \cong_{Gr} SPB reduces to any of the following isomorphism relation*

- the isomorphism relation between countable partial orders \cong_{PO} ;
- the isomorphism relation between countable semigroups \cong_{Smg} ;
- the isomorphism relation between countable unital rings \cong_{Rng_1} .

We conclude this section with a few more thoughts about SPB reductions. If the $\mathfrak{L}_{\omega_1\omega}$ -elementary classes X and Y are Borel complete, then the isomorphism relations \cong_X and \cong_Y are necessarily Borel bi-reducible, but they need not be SPB bi-reducible. E.g., the isomorphism relation between countable graphs \cong_{Gr} does not SPB reduce to isomorphism between countable groups \cong_{Gr} , because every infinite countable group has nontrivial automorphisms.

Let \cong_{Tr} be the isomorphism relation between countable trees (i.e., connected acyclic graphs) and \cong_{LO} be the isomorphism relation between countable linear orders. Although \cong_{Tr} and \cong_{LO} have been known to be Borel complete, they are not equivalent to \cong_{Gr} up to faithful Borel reducibility (cf. [6, Theorem 4.5]). It is then natural to ask the following questions.

Question 5.10. *Does $\cong_{\text{Gr}} \leq_{\text{SPB}} \cong_{\text{Tr}}$?*

Question 5.11. *Does $\cong_{\text{Gr}} \leq_{\text{SPB}} \cong_{\text{LO}}$?*

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