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Fuzzy dynamic programming: interpolative reasoning for an efficient derivation of optimal control policies*

by

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Abstract: We consider multistage optimal control of a fuzzy dynamic system under fuzzy constraints on controls and fuzzy goals on states in the setting of Bellman and Zadeh (1970) and Kacprzyk (1983, 1997). First, we present the solution by dynamic programming which is a standard techniques in the class of problems considered. We indicate its limitations, mainly related to its inherent curse of dimensionality. We propose to replace the source problem by its auxiliary counterpart with a small number of reference fuzzy states and reference fuzzy controls, solve it by dynamic programming to obtain optimal reference control policies relating optimal reference fuzzy controls to reference fuzzy states. Then, we show the use of an interpolative reasoning approach to derive optimal fuzzy controls, not necessarily reference ones, for current fuzzy states, not necessarily reference ones.

Keywords: multistage fuzzy control, fuzzy dynamic programming, fuzzy system under control, interpolative reasoning.

1. Introduction

Fuzzy (logic) control has been generally considered a flagship example of success of fuzzy technology. Basically, its essence may be viewed as that while considering a control problem we do not intend to build a model of the control process itself, as is customary in traditional control theory and engineering, because this may be too difficult or costly, or even such a model may be unknown. We just assume that we know how to control the process, in the sense of control laws that maybe do not guarantee optimality but yield good results in practice. Such

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control laws may be either derived from experienced process operators by some knowledge elicitation procedure, or can be derived by some (semi)automatic data mining or knowledge discovery process from data. Normally, linguistic IF–THEN rules are used to represent control laws, and also models of systems under control. This paradigm is clearly an example of a *descriptive* approach since we explicitly *describe* how to control the process considered.

Even if the above descriptive approach to fuzzy control has been so successful, it may be viewed as somehow counter-intuitive, contradicting the long tradition of control, the aim of which is to find (prescribe) a *best* course of action, assuming knowledge of the dynamics of the system to be controlled, and some *goals* to be attained and *constraints* under which to operate. Such a *prescriptive* approach to fuzzy control is even earlier, see Bellman and Zadeh (1970); see also Kacprzyk's (1983, 1997) books for a detailed description. This approach is of a fuzzy optimal control type, and notably dynamic programming is employed therein.

This paper is concerned with that prescriptive approach to fuzzy control, to be more specific: with some problems related to the use of dynamic programming. We assume a more difficult case of a fuzzy system under control, given as some known and prespecified fuzzy state transition equation. The multistage control proceeds over a finite, fixed and specified horizon, under fuzzy constraints imposed on the controls at the consecutive stages, and fuzzy goals imposed on the states (equated, for simplicity with outputs) attained. An optimal sequence of fuzzy controls (in fact, clearly, of control policies) is sought which maximizes the membership function of the fuzzy decision given as the intersection of the consecutive fuzzy constraints and goals. Basically, to obtain solution we employ a fuzzy dynamic programming scheme as proposed by Kacprzyk (1997). However, due to the inherent curse of dimensionality of dynamic programming, which is even more dangerous in the case considered, owing to an increase of dimensionality caused by the use of fuzzy constraints, goals and state transitions, we first use a simplification procedure for a reduction of problem dimensionality through the replacement of a huge number of possible fuzzy states, fuzzy controls, fuzzy state transitions, etc. by a small number of the so-called reference fuzzy controls, fuzzy states, fuzzy state transitions, etc. This has much to do with the problem of sparsity of rule bases in fuzzy control as first analyzed by Kacprzyk and Fedrizzi (1995), and then further developed by many, e.g. by Wu, Mizumoto and Shi (1996), Chang, Chen and Liau (2008), Chen and Ko (2008), or Hsiao, Chen and Lee (1998).

Obviously, through this simplification we obtain a slightly different problem so that the solution obtained should be adjusted. This is done by using an interpolation approach as initiated, and then mainly extended by Kacprzyk (1997), with an application to regional development planning shown in Kacprzyk, Romero and Gomide (1999). We will present that approach in much detail, but with a somehow broader perspective by taking into account some newer developments in a very relevant field of interpolation in (fuzzy) rule bases, notably by Kóczy and his collaborators (Baranyi, Koćzy and Gedeon, 2004; Koćzy, Hirota and Gedeon, 2000; Tikk and Baranyi, 2000; Yam, Wong, and Baranyi, 2006; Wong et al., 2005).

A broad view of the entire area of multistage fuzzy control, or – more specifically – fuzzy dynamic programming, which will be provided, should make it possible to better present such a general perspective.

First, we outline Bellman and Zadeh's (1970) seminal framework of (optimal) fuzzy decision making and control within which the problem will be discussed. Then, we present its solution by dynamic programming, which is a standard solution technique, and indicate its limitations, mainly related to its inherent curse of dimensionality. We also present some alternative techniques. We then present an extension of Kacprzyk's (1993a,b,c, 1997) proposal to replace the source problem by its auxiliary counterpart with a small number of reference fuzzy states and reference fuzzy controls, solve it by dynamic programming, and then "adjust" the solution obtained by using the concept of Kóczy and Hirota's (1993a,b, 1997) interpolative reasoning technique, with a reference to newest development in this field, exemplified by Baranyi, Kóczy and Gedeon (2004), Kóczy, Hirota and Gedeon (2000), Tikk and Baranyi (2000), Yam, Wong and Baranyi (2006) or Wong, Tikk, Gedeon and Kóczy (2005). We show illustrative examples.

2. Multistage fuzzy control: the case of a deterministic system under control

To start, we will outline the general Bellman and Zadeh's (1970) approach to decision making in a fuzzy environment, and show how it can be employed in multistage fuzzy control with a deterministic system. This will provide us with a point of departure.

2.1. Bellman and Zadeh's general approach to decision making and control under fuzziness

If $X = \{x\}$ is a set of possible *options* (alternatives, choices, ...), then:

- a *fuzzy goal* is defined as a fuzzy set G in X, characterized by its membership function $\mu_G : X \longrightarrow [0,1]$ such that $\mu_G(x) \in [0,1]$ specifies the grade of membership of a particular option $x \in X$ in the fuzzy goal G;
- a *fuzzy constraint* is similarly defined as a fuzzy set C in the set of options X, characterized by $\mu_C : X \longrightarrow [0,1]$.

The general problem formulation is "Attain G and satisfy C", which leads to the *fuzzy decision* D defined as a fuzzy set in X such that

$$\mu_D(x) = \mu_G(x) \wedge \mu_C(x), \qquad \text{for each } x \in X \tag{1}$$

where $a \wedge b = \min(a,b)$ may be replaced by another operation, e.g., a t-norm (see Kacprzyk, 1997), which is defined as a function $T : [0,1] \times [0,1] \longrightarrow [0,1]$, satisfying the following properties, $\forall a,b,c \in [0,1]$:

- Commutativity: $T(a,b) = T(b,a)$,

- Monotonicity: $T(a,b) \leq T(c,d)$ if $a \leq c$ and $b \leq d$,
- Associativity: $T(a,T(b,c)) = T(T(a,b),c)$,
- The identity element is 1, i.e. $T(a,1) = a$.

If we have $n > 1$ fuzzy goals, $G_1,\ldots,G_n$, and $m > 1$ fuzzy constraints, $C_1,\ldots,C_m$, all defined in X, then the *fuzzy decision* is

$$\mu_D(x) = \mu_{G_1}(x) \wedge \ldots \mu_{G_n}(x) \wedge \\ \wedge \mu_{C_1}(x) \wedge \ldots \wedge \mu_{C_m}(x), \qquad \text{for each } x \in X. \tag{2}$$

If, on the other hand, C is defined in $X = \{x\}$, G is defined in $Y = \{y\}$, and a function $f : X \longrightarrow Y$, $y = f(x)$, is known, then the *fuzzy decision* is clearly

$$\mu_D(x) = \mu_G[f(x)] \wedge \mu_C(x), \qquad \text{for each } x \in X \tag{3}$$

and, for $G_1,\ldots,G_n$ defined in Y, $C_1,\ldots,C_m$ defined in X, and $f : X \longrightarrow Y$, $y = f(x)$, we have

$$\mu_D(x) = \mu_{G_1}[f(x)] \wedge \cdots \wedge \mu_{G_n}[f(x)] \wedge \\ \wedge \mu_{C_1}(x) \wedge \cdots \wedge \mu_{C_n}(x), \qquad \text{for each } x \in X. \tag{4}$$

In all the above cases the so-called *maximizing decision*, which is the solution sought, is defined as an $x^* \in X$ such that

$$\mu_D(x^*) = \max_{x \in X} \mu_D(x). \tag{5}$$

It is easy to see that the above general decision making related setting can readily be extended to the multistage control context.

Now, the control space is $U = \{u\} = \{c_1,\ldots,c_m\}$, the state space is $X = \{x\} = \{s_1,\ldots,s_n\}$, and both are finite. The system under control is assumed to be deterministic, governed by a *state transition equation*

$$x_{t+1} = f(x_t, u_t), \qquad t = 0,1,\ldots \tag{6}$$

where $x_t, x_{t+1} \in X = \{s_1,\ldots,s_n\}$ are the states at t and $t+1$, respectively, and $u_t \in U = \{c_1,\ldots,c_m\}$ is the control at t.

At t, $t = 0,1,\ldots$, $u_t \in U$ is subject to a *fuzzy constraint* $\mu_{C^t}(u_t)$, and on $x_{t+1} \in X$ a *fuzzy goal*, $\mu_{G^{t+1}}(x_{t+1})$, is imposed.

The *initial state* is $x_0 \in X$ and is assumed to be known and given in advance. The *termination time* (planning, or control, horizon), $N \in \{1,2,\ldots\}$, is assumed to be finite, fixed and specified in advance (for other types of the termination time, see Kacprzyk's book, 1997).

The *performance function* of the multistage fuzzy control process is evaluated by the *fuzzy decision*

$$\mu_D(u_0,\ldots,u_{N-1} \mid x_0) = \\ = \mu_{C^0}(u_0) \wedge \mu_{G^1}(x_1) \wedge \ldots \wedge \mu_{C^{N-1}}(u_{N-1}) \wedge \mu_{G^N}(x_N) \tag{7}$$

and the problem is to find an optimal sequence of controls $u_0^*, \ldots, u_{N-1}^*$ such that

$$\mu_D(u_0^*, \ldots, u_{N-1}^* \mid x_0) = \max_{u_0, \ldots, u_{N-1} \in U} \mu_D(u_0, \ldots, u_{N-1} \mid x_0). \tag{8}$$

Problem (8) can be solved using the following two basic traditional techniques:

- dynamic programming (Bellman and Zadeh, 1970, Kacprzyk, 1983,1997), and
- branch-and-bound (Kacprzyk, 1978a, 1979),

and also using the following two new ones, based on:

- a neural network (Francelin, Gomide and Kacprzyk, 2001a,b), and
- a genetic algorithm (Kacprzyk, 1995a,b,d,1996).

For the purpose of this paper the use of dynamic programming is the most relevant.

First, we rewrite problem (8) as to: find $u_0^*, \ldots, u_{N-1}^*$ such that

$$\begin{aligned} \mu_D(u_0^*, \ldots, u_{N-1} \mid x_0) = \max_{u_0, \ldots, u_{N-1}} [\mu_{C^0}(u_0) \wedge \mu_{G^1}(x_1) \wedge \ldots \\ \ldots \wedge \mu_{C^{N-1}}(u_{N-1}) \wedge \mu_{G^N}(f(x_{N-1}, u_{N-1}))] \end{aligned} \tag{9}$$

and since the term $\mu_{C^{N-1}}(u_{N-1}) \wedge \mu_{G^N}(f(x_{N-1}, u_{N-1}))$ depends only on u_{N-1}, then the maximization over $u_0, \ldots, u_{N-1}$ in (9) can be split into:

- the maximization over $u_0, \ldots, u_{N-2}$, and
- the maximization over u_{N-1},

which may be continued for the maximization over $u_0, \ldots,$ u_{N-3} u_{N-2}, etc.

This backward iteration implies the set of fuzzy dynamic programming recurrence equations:

$$\begin{cases} \mu_{\overline{G}^{N-i}}(x_{N-i}) = \\ \qquad = \max_{u_{N-i}} [\mu_{C^{N-i}}(u_{N-i}) \wedge \mu_{G^{N-i}}(x_{N-i}) \wedge \mu_{\overline{G}^{N-i+1}}(x_{N-i+1})] \\ x_{N-i+1} = f(x_{N-i}, u_{N-i}); \qquad i = 0, 1, \ldots, N \end{cases} \tag{10}$$

where $\mu_{\overline{G}^{N-i}}(x_{N-i})$ is a fuzzy goal at $t = N - i$ induced by the fuzzy goal at $t = N - i + 1$, $i = 0, 1, \ldots, N$; $\mu_{\overline{G}^N}(x_N) = \mu_{G^N}(x_N)$.

The $u_0^*, \ldots, u_{N-1}^*$ sought is given by the successive maximizing values of u_{N-i}, $i = 1, \ldots, N$ in (10), which are obviously obtained as functions of x_{N-i}, i.e. as an *optimal policy.*

3. Multistage fuzzy control with a fuzzy system

For our purposes a relevant and natural extension of the multistage fuzzy control problem with a deterministic system under control, as outlined in Section 2.1, is by assuming a fuzzy system under control, the dynamics of which is given as a fuzzy state transition equation

$$X_{t+1} = F(X_t, U_t), \qquad t = 0, 1, \ldots \tag{11}$$

where $X_t, X_{t+1} \in \mathcal{X}$ are fuzzy states at control stage t and $t+1$, respectively, and $U_t \in \mathcal{U}$ is a fuzzy control at control stage t, $t = 0, 1, \ldots, N-1$; $\mathcal{U} = \{C_1, \ldots, C_l\}$ is the set of fuzzy controls, and $\mathcal{X} = \{S_1, \ldots, S_q\}$ is the set of fuzzy states; both are assumed finite, for simplicity and practical reasons. The finite termination time is fixed and specified in advance; for the problem with infinite termination time, see Kacprzyk and Staniewski (1983).

First, notice that in the previously discussed case of a deterministic system under control, the consecutive controls applied, $u_0, \ldots, u_{N-1} \in U$, and the states attained, $x_1, \ldots, x_N \in X$, were non-fuzzy, hence we could directly determine their grade of membership in the fuzzy constraints, $\mu_{C^0}(u_0), \ldots, \mu_{C^{N-1}}(u_{N-1})$, and in the fuzzy goals, $\mu_{G^1}(x_1), \ldots, \mu_{G^N}(x_N)$, respectively.

Unfortunately, this is not the case for a fuzzy system as the control applied and states attained are fuzzy. Thus, their grade of membership in $\mu_{C^0}(u_0), \ldots,$ $\mu_{C^{N-1}}(u_{N-1})$, and in $\mu_{G^1}(x_1), \ldots, \mu_{G^N}(X_N)$, respectively, cannot be directly determined, and some indirect mechanism ("trickery") is needed.

At each t, $U_t \in \mathcal{U}$ is subject to a fuzzy constraint $\mu_{C^t}(u_t)$, and on $X_{t+1} \in \mathcal{X}$ a fuzzy goal $\mu_{G^{t+1}}(x_{t+1})$ is imposed, $t = 0, 1, \ldots, N-1$. We basically need to redefine the fuzzy constraints and fuzzy goals, for instance, as follows:

$$\mu_{\overline{C}^t}(U_t) = 1 - \text{diss}(C^t, U_t), \qquad t = 0, 1, \ldots, N-1 \tag{12}$$

and

$$\mu_{\overline{G}^{t+1}}(X_{t+1}) = 1 - \text{diss}(G^{t+1}, X_{t+1}), \qquad t = 0, 1, \ldots, N-1 \tag{13}$$

where diss : $[0,1] \times [0,1] \longrightarrow [0,1]$ is some measure of dissemblance (dissimilarity, ...) which is traditionally assumed to be a normalized distance between two fuzzy sets, $d(.,.) \in [0,1]$.

The simplest and most widely used normalized distances are (given below for the fuzzy states and goals, and analogously for the fuzzy controls and constraints):

- the normalized linear (Hamming) distance

$$d_l(X_N, G^N) = \frac{1}{N} \sum_{i=1}^{N} \mid \mu_{X_N}(s_i) - \mu_{G^N}(s_i) \mid \tag{14}$$

- the normalized quadratic (Euclidean) distance

$$d_q(X_N, G^N) = \sqrt{\frac{1}{N} \sum_{i=1}^{N} [\mu_{X_N}(s_i) - \mu_{G^N}(s_i)]^2}. \tag{15}$$

As to other choices, the use of a degree of equality of two fuzzy sets proposed by Kacprzyk and Staniewski (1983) is also a plausible choice, i.e. $\mu_{\overline{G}^N}(X_N) = e(X_N, G^N)$ (see Kacprzyk, 1997). Moreover, one can also use one of numerous measures of similarity or dissimilarity, or mose generally, incompatibility, between two fuzzy sets as given, e.g., in Cross and Sudkamp's (2002) book.

Then, generally, the fuzzy decision is

$$\begin{aligned} &\mu_D(U_0, \ldots, U_{N-1} \mid X_0) = \\ &= \mu_{\overline{C}^0}(U_0) \wedge \mu_{\overline{G}^1}(X_1) \wedge \ldots \wedge \mu_{\overline{C}^{N-1}}(U_{N-1}) \wedge \mu_{\overline{G}^N}(X_N) \end{aligned} \tag{16}$$

and we seek an optimal sequence of fuzzy controls $U_0^*, \ldots, U_{N-1}^*$ such that

$$\mu_D(U_0^*, \ldots, U_{N-1}^* \mid X_0) = \max_{U_0, \ldots, U_{N-1}} \mu_D(U_0, \ldots, U_{N-1} \mid X_0). \tag{17}$$

Now, since we focus on dynamic programming here, we will show how problem (17) may be solved by dynamic programming. For the use of branch-and-bound (see Kacprzyk, 1979) and a genetic algorithm (see Kacprzyk, 1995a, b, d), see the book by Kacprzyk (1997).

3.1. Solution by dynamic programming

The application of a dynamic programming scheme to solving problem (17) is due to Baldwin and Pilsworth (1992).

The fuzzy system under control is described by a fuzzy state transition equation (11), i.e. $X_{t+1} = F(X_t, U_t)$, $t = 0, 1, \ldots$, where $X_t, X_{t+1} \in \mathcal{X}$ are fuzzy states at control stages t and $t+1$, respectively, and $U \in \mathcal{U}$ is a fuzzy control at control stage t; $\mathcal{X}$ is the set of fuzzy states and $\mathcal{U}$ is the set of fuzzy controls.

At each control stage t, U_t is subject to a fuzzy constraint $\mu_{C^t}(u_t)$, and on the resulting X_{t+1} a fuzzy goal $\mu_{G^{t+1}}(x_{t+1})$ is imposed, $t = 0, 1, \ldots, N-1$.

Both U_t and X_{t+1} are now fuzzy, and hence their grades of membership in C^t and G^{t+1} cannot be directly determined as the values of $\mu_{C^t}(u_t)$ and $\mu_{G^{t+1}}(x_{t+1})$, respectively. Therefore, for each t we construct a fuzzy relation R in $U \times X$ such that

$$\mu_{R^t}(u_t, x_t) = \mu_{C^t}(u_t) \wedge \mu_{G^{t+1}}(x_{t+1}), \quad \text{for each } u_t \in U, x_{t+1} \in X \tag{18}$$

which represents the degree, between 0 and 1, to which the fuzzy constraint C^t and fuzzy goal G^{t+1} are satisfied.

In turn, the degree to which a U_t and X_{t+1} satisfy C^t and G^{t+1}, respectively, is given as

$$\begin{aligned} T(U_t, R^t, X_{t+1}) &= \max_{x_{t+1} \in X} [\max_{u_t \in U} (\mu_{U_t}(u_t) \wedge \mu_{R^t}(u_t, x_t)) \wedge \mu_{X_{t+1}}(x_{t+1})] = \\ &= \max_{x_{t+1} \in X} [\max_{u_t} (\mu_{U_t}(u_t) \wedge \mu_{C^t}(u_t) \wedge \mu_{G^{t+1}}(x_{t+1}) \wedge \mu_{X_{t+1}}(x_{t+1}))] = \\ &= \max_{u_t \in U} [\mu_{U_t}(u_t) \wedge \mu_{C^t}(u_t)] \wedge \max_{x_{t+1} \in X} [\mu_{X_{t+1}}(x_{t+1}) \wedge \mu_{G^{t+1}}(x_{t+1})]. \end{aligned} \tag{19}$$

The fuzzy decision is now given as

$$\begin{aligned} &\mu_D(U_0, \ldots, U_{N-1} \mid X_0) = \\ &= T(U_0, R^0, X_1) \wedge \ldots \wedge T(U_{N-1}, R^{N-1}, X_N) \end{aligned} \tag{20}$$

which yields the degree to which a particular sequence of fuzzy controls $U_0, \ldots,$ U_{N-1}, and the resulting sequence of fuzzy states $X_1, \ldots, X_N$, satisfy the respective fuzzy constraints and fuzzy goals.

The problem is now to determine an optimal sequence of fuzzy controls $U_0^*, \ldots, U_{N-1}^*$ such that

$$\begin{aligned}
&\mu_D(U_0^*, \ldots, U_{N-1}^* \mid X_0) = \\
&= \max_{U_0,\ldots,U_{N-1}} \mu_D(U_0, \ldots, U_{N-1} \mid X_0) = \\
&= \max_{U_0,\ldots,U_{N-1}} [T(U_0, R^0, X_1) \wedge \ldots \wedge T(U_{N-1}, R^{N-1}, X_N)]. \qquad (21)
\end{aligned}$$

Notice that the operations assumed, the minimum ("$\wedge$") and maximum ("$\vee$") can be replaced by a suitable t-norm and s-norm, i.e. those which would make it possible to split the respective optimization problems into the consecutive optimization steps over the consecutive single controls; for instance, the minimum or algebraic product are such t-norms.

Assume now for simplicity that the fuzzy constraints are imposed at control stages $t = 0, 1, \ldots, N-1$, but the fuzzy goal is imposed only at control stage $t = N$. Problem (21) becomes, therefore, the one to find an optimal sequence of fuzzy controls $U_0^*, \ldots, U_{N-1}^*$ such that

$$\begin{aligned}
&\mu_D(u_0^*, \ldots, U_{N-1}^* \mid X_0) = \\
&= \max_{U_0,\ldots,U_{N-1}} \max_{u_0 \in U} [\mu_{U_0}(u_0) \wedge \mu_{C^0}(u_0) \wedge \ldots \wedge \max_{U_{N-1} \in U} (\mu_{U_{N-1}}(u_{N-1}) \wedge \\
&\quad \wedge \mu_{C^{N-1}}(u_{N-1}) \wedge \max_{x_N} (\mu_{X_N}(x_N) \wedge \mu_{G^N}(x_N)) \ldots] \qquad (22)
\end{aligned}$$

where the final state X_N is reached from the initial state X_0 through the sequence of fuzzy controls $U_0, \ldots, U_{N-1}$ by applying the fuzzy state transition equation (11).

It is now easy to see that the structure of (22) is essentially the same as that of (8), i.e. the two right-most terms depend only on the fuzzy control U_{N-1} and not on the other controls, the next right-most term depends only on U_{N-2}, etc.

This leads to the following set of dynamic programming recurrence equations:

$$\begin{cases}
\mu_{\overline{G}^N}(X_N) = \max_{x_N \in X} [\mu_{X_N}(x_N) \wedge \mu_{G^N}(x_N)] \\
\mu_{\overline{G}^{N-i}}(X_{N-i}) = \max_{U_{N-i} \in \mathcal{U}} [\max_{u_{N-i} \in U} (\mu_{U_{N-i}}(u_{N-i}) \wedge \\
\qquad \wedge \mu_{C^{N-i}}(u_{N-i})) \wedge \mu_{\overline{G}^{N-i+1}}(x_{N-i+1})] \\
\mu_{X_{N-i+1}}(x_{N-i+1}) = \max_{x_{N-i} \in X} [\max_{u_{N-i} \in U} (\mu_{U_{N-i}}(u_{N-i}) \wedge \\
\qquad \wedge \mu_{X_{N-i+1}}(x_{N-i+1} \mid x_{N-i}, u_{N-i})) \wedge \mu_{X_{N-i}}(x_{N-i})] \\
i = 1, \ldots N.
\end{cases} \qquad (23)$$

In principle, this set of dynamic programming recurrence equations can be solved. However, one can notice a serious potential difficulty. On the one hand, $\mu_{\overline{G}^{N-i}}(X_{N-i})$ is to be specified for each possible fuzzy state $X_{N-i} \in \mathcal{X}$. On the other hand, the maximization, $\max_{U_{N-i}}(.)$, is to proceed over all fuzzy controls $U_{N-i} \in \mathcal{U}$. Evidently, the number of all possible fuzzy controls U_{N-i} and fuzzy states X_{N-i} may be very high, which clearly makes the solution of (23) practically impossible.

An obvious first idea to overcome this difficulty is that the numbers of possible fuzzy controls and fuzzy states are to be considerably reduced to make (23) solvable in reasonable time. This is virtually the essence of Baldwin and Pilsworth's (1992) approach (as it is of Kacprzyk and Staniewski's, 1982, approach). This reduction is there called *fuzzy interpolation*, though interpolation will be meant later in this paper in a different, even broader sense.

Consider first $\mu_{\overline{G}^{N-i+1}}(X_{N-i+1})$, $i = 1, \ldots, N$. To overcome its possible troublesome dependence on a very high number of fuzzy states X_{N-i+1}, a relatively small number, say r, of the so-called *reference fuzzy states* $\overline{X}^1_{N-i+1}, \ldots, \overline{X}^r_{N-i+1}$ is introduced. Then, using the fuzzy conditional statements, $\mu_{G^{N-i+1}}(X_{N-i+1})$ is approximately specified as

$$\begin{cases} \text{IF } (X_{N-i+1} = \overline{X}^1_{N-i+1}) \text{ THEN } [\mu_{\overline{G}^{N-i+1}}(X_{N-i+1}) = \mu_{\overline{G}^{N-i+1}}(X^1_{N-i+1})] \\ \text{ELSE} \ldots \text{ELSE} \\ \text{IF } (X_{N-i+1} = \overline{X}^r_{N-i+1}) \text{ THEN } [\mu_{\overline{G}^{N-i+1}}(X_{N-i+1}) = \mu_{\overline{G}^{N-i+1}}(X^r_{N-i+1})] \end{cases} \tag{24}$$

which evidently corresponds to a fuzzy relation $\mu_{R^{N-i+1}}(x_{N-i+1}, w_{N-i+1})$ defined in $X \times [0,1]$, $x_{N-i+1} \in X$ and $w_{N-i+1} \in [0,1]$.

Now, if we have a fuzzy state X_{N-i+1}, not necessarily a reference one, then it yields via the above fuzzy relation R^{N-i+1} through the max–min composition of X_{N-i+1} and R^{N-i+1} the following induced fuzzy goal at control stage $N-i+1$:

$$\begin{aligned} &\mu_{\overline{G}^{N-i+1}}(X_{N-i+1}) = \\ &= \max_{x_{N-i+1} \in X} [\mu_{X_{N-i+1}}(x_{N-i+1}) \wedge \mu_{R^{N-i+1}}(x_{N-i+1}, w_{N-i+1})]. \end{aligned} \tag{25}$$

Thus, for given U_{N-i} and X_{N-i+1} the expression to be maximized with respect to U_{N-i} in (23) is

$$\begin{aligned} b(U_{N-i}, X_{N-i}) = &\max_{u_{N-i} \in U} [\mu_{U_{N-i}}(u_{N-i}) \wedge \\ &\wedge \max_{x_{N-i+1} \in X} (\mu_{X_{N-i+1}}(x_{N-i+1}) \wedge \mu_{R^{N-i+1}}(x_{N-i+1}, w_{N-i+1}))]. \end{aligned} \tag{26}$$

Since the number of possible U_{N-i}'s may also be very high, then, similarly as in the case of fuzzy states, we introduce a relatively small number, say p, of the so-called *reference fuzzy controls* $\overline{U}^1_{N-i}, \ldots, \overline{U}^p_{N-i}$, and, for a given X_{N-i}, the following fuzzy conditional statements are constructed:

$$\begin{cases} \text{IF } (U_{N-i} = \overline{U}^1_{N-i}) \text{ THEN } [b(X_{N-i}, U_{N-i}) = b(X_{N-i}, \overline{U}^1_{N-i})] \\ \text{ELSE} \ldots \text{ELSE} \\ \text{IF } (U_{N-i} = \overline{U}^p_{N-i}) \text{ THEN } [b(X_{N-i}, U_{N-i}) = b(X_{N-i}, \overline{U}^p_{N-i})] \end{cases} \tag{27}$$

which is equivalent to a fuzzy relation defined in $U \times [0,1]$, $\mu_{R'}(u_{N-i}, v_{N-i} \mid X_{N-i})$, $u_{N-i} \in U$, $v_{N-i} \in [0,1]$. For simplicity, this fuzzy relation will be denoted by $R'(X_{N-i})$.

Now we need to find a method to determine such U^*_{N-i} that maximizes $b(X_{N-i}, U_{N-i})$ with respect to U_{N-i}, for a given X_{N-i}.

We introduce first a fuzzy set labelled "k-large" defined as

$$\mu_{k-\mathrm{large}}(x) = \begin{cases} 0 & \text{for } x \le k \\ z > 0 & \text{for } x > k \end{cases} \tag{28}$$

such that if $x' > x'' > k$, then $\mu_{k-\mathrm{large}}(x') > \mu_{k-\mathrm{large}}(x'')$, and $\mu_{k-\mathrm{large}}(1) = 1$. The value of k should be carefully chosen so that $\mu_{k'-\mathrm{large}}(.)$ be preferred to $\mu_{k''-\mathrm{large}}(.)$ if $k' > k''$.

Using the $R'(X_{N-i})$, obtained via (27), we determine through the use of the max–min composition of "k-large" and $R'(X_{N-i})$ the following fuzzy sets induced by "k-large":

$$\mu_{U(k)}(u_{N-i}) = \max_{x_{N-i}\in X}[\mu_{R'}(u_{N-i}, x_{N-i} \mid X_{N-i}) \wedge \\ \wedge \mu_{k-\mathrm{large}}(x_{N-i})], \qquad \text{for each } u_{N-i} \in U. \tag{29}$$

As an optimal fuzzy control U^*_{N-i} we take such $U(k)$ which corresponds to the highest value of k, i.e.

$$\max_{u_{N-i}\in U} \mu_{U(k)}(u_{N-i}) \ge m \tag{30}$$

where m is some parameter to be chosen.

The above procedure is performed for each particular X_{N-i} so that the U^*_{N-i} determined using (30) is in fact a function of X_{N-i}, i.e. is a *control policy*. The above procedure is repeated for each reference fuzzy state $\overline{X}^1_{N-i}, \ldots, \overline{X}^r_{N-i}$ yielding $U^*_{N-i}(\overline{X}^1_{N-i}), \ldots, U^*_{N-i}(\overline{X}^r_{N-i})$, respectively, which may be represented as the following fuzzy conditional statement:

$$\begin{cases} \text{IF } (X_{N-i} = \overline{X}^1_{N-i}) \text{ THEN } [U^*_{N-i}(X_{N-i}) = U^*_{N-i}(\overline{X}^1_{N-i})] \\ \text{ELSE} \ldots \text{ELSE} \\ \text{IF } (X_{N-i} = \overline{X}^r_{N-i}) \text{ THEN } [U^*_{N-i}(X_{N-i}) = U^*_{N-i}(\overline{X}^1_{N-r})] \end{cases} \tag{31}$$

which is equivalent to a fuzzy relation defined in $X \times U$, $\mu_{R^{N-i}_{U^*}}(x_{N-i}, u_{N-i})$, that represents the *optimal fuzzy control policy.*

If we are therefore currently at control stage $t = N - i$ in a state X_{N-i}, then the *optimal fuzzy control* given by this optimal fuzzy control policy is

$$\mu_{U^*_{N-i}}(u_{N-i}) = \max_{x_{N-i}\in X}[\mu_{X_{N-i}}(x_{N-i}) \wedge \\ \wedge \mu_{R^{N-i}_{U^*}}(x_{N-i}, u_{N-i})], \qquad \text{for each } u_{N-i} \in U. \tag{32}$$

Example 1. Suppose that: $X = \{s_1, s_2, s_3\}$, $U = \{c_1, c_2\}$, $N = 3$, the fuzzy constraints are:

$$C^0 = 1/c_1 + 0.5/c_2 \qquad C^1 = 0.8/c_1 + 0.7/c_2 \qquad C^2 = 1/c_1 + 1/c_2$$

and the fuzzy goal is

$$G^3 = 1/s_1 + 0.4/s_2 + 0.1/s_3.$$

Let the fuzzy system under control be given by

$\mu_{X_{t+1}}(x_{t+1} \mid x_t, u_t) =$

		$x_{t+1} = s_1$	s_2	s_3
$u_t = c_1$	$x_t = s_1$	1	0.7	0.3
	s_2	0.7	1	0.7
	s_3	0.3	0.7	1

		$x_{t+1} = s_1$	s_2	s_3
$u_t = c_2$	$x_t = s_1$	1	0.7	0.3
	s_2	1	0.7	0.3
	s_3	1	0.7	0.3

.

First, we introduce the following three reference fuzzy states:

$$\begin{aligned}
\overline{X}_1 &= 1/s_1 + 0.4/s_2 + 0.1/s_3 \\
\overline{X}_2 &= 0.4/s_1 + 1/s_2 + 0.4/s_3 \\
\overline{X}_3 &= 0.1/s_1 + 0.4/s_2 + 1/s_3
\end{aligned}$$

and the following two reference fuzzy controls:

$$\begin{aligned}
\overline{U}_1 &= 1/c_1 + 0.2/c_2 \\
\overline{U}_2 &= 0.2/c_1 + 1/c_2.
\end{aligned}$$

Now, we present $\mu_{X_{t+1}}(x_{t+1} \mid x_t, u_t)$ in the form involving only the reference fuzzy states and controls, that is

$X_{t+1} =$

	$U_t = \overline{U}^1$	$\overline{U}^2$
$X_t = \overline{X}^1$	$1/s_1 + 0.7/s_2 + 0.4/s_3$	$1/s_1 + 0.7/s_2 + 0.3/s_3$
$\overline{X}^2$	$0.7/s_1 + 1/s_2 + 0.7/s_3$	$1/s_1 + 0.7/s_2 + 0.3/s_3$
$\overline{X}^3$	$0.4/s_1 + 0.7/s_2 + 1/s_3$	$1/s_1 + 0.7/s_2 + 0.3/s_3$

.

We solve (23) for $i = 0$, i.e. for $N - i = 3$, and obtain

$$\begin{aligned}
\mu_{\overline{G}^3}(\overline{X}_1) &= \max_{x_3 \in \{s_1, s_2, s_3\}} [\mu_{\overline{X}_1}(x_3) \wedge \mu_{G^3}(x_3)] = \\
&= \quad (1 \wedge 1) \vee (0.4 \wedge 0.4) \vee (0.1 \wedge 0.1) = 1
\end{aligned}$$

and analogously we obtain

$$\mu_{\overline{G}^3}(\overline{X}_2) = 0.4 \quad \mu_{\overline{G}^3}(\overline{X}_3) = 0.4.$$

The above is equivalent to

$$\begin{cases}
\text{IF } \overline{X}_1 \text{ THEN } \mu_{\overline{G}^3}(\overline{X}_1) = 1 \\
\text{ELSE} \\
\text{IF } \overline{X}_2 \text{ THEN } \mu_{\overline{G}^3}(\overline{X}_2) = 0.4 \\
\text{ELSE} \\
\text{IF } \overline{X}_3 \text{ THEN } \mu_{\overline{G}^3}(\overline{X}_3) = 0.4
\end{cases}$$

which, due to (24), corresponds to the following fuzzy relation R^3 in $X \times [0,1]$:

$$\mu_{R^3}(x_3, w_3) = \begin{array}{r|ccc} & x_3 = s_1 & s_2 & s_3 \\ \hline w_3 = 0.4 & 0.4 & 0.4 & 1 \\ 1 & 1 & 1 & 0.4 \end{array} .$$

Now, for $i = 2$, i.e. $N - i = 2$, we obtain via (25) the following fuzzy goal $\mu_{\overline{G}^3}(X_3)$ induced by X_3 and R^3:

$$\mu_{\overline{G}^3}(X_3) = \max_{x_3}[\mu_{X_3}(x_3) \wedge \mu_{R^3}(x_3, w_3)] =$$

$$= \begin{array}{r|cc} & U_2 = \overline{U}^1 & \overline{U}^2 \\ \hline X_2 = \overline{X}^1 & 0.4/0.7 + 1/1 & 0.4/0.7 + 1/1 \\ \overline{X}^2 & 0.4/0.7 + 1/0.7 & 0.4/0.7 + 1/1 \\ \overline{X}^3 & 1/0.4 + 0.4/1 & 0.4/0.7 + 1/1 \end{array}$$

and via (26) we obtain

$$b(X_2, U_2) =$$

$$= \max_{U_2 \in \{\overline{U}_1, \overline{U}_2\}} \begin{array}{r|cc} & U_2 = \overline{U}^1 & \overline{U}^2 \\ \hline X_2 = \overline{X}^1 & 1 \wedge \{0.4/0.7 + 1/1\} & 1 \wedge \{0.4/0.7 + 1/1\} \\ \overline{X}^2 & 1 \wedge \{0.4/0.7 + 1/0.7\} & 1 \wedge \{0.4/0.7 + 1/1\} \\ \overline{X}^3 & 1 \wedge \{1/0.4 + 0.4/1\} & 1 \wedge \{0.4/0.7 + 1/1\} \end{array}$$

which is to be meant as

- for $X_2 = \overline{X}^1$
  IF $\overline{U}^1$ THEN $b(\overline{X}^1, \overline{U}^1)$ ELSE IF $\overline{U}^2$ THEN $b(\overline{X}^1, \overline{U}^2)$
- for $X_2 = \overline{X}^2$
  IF $\overline{U}^1$ THEN $b(\overline{X}^2, \overline{U}^1)$ ELSE IF $\overline{U}^2$ THEN $b(\overline{X}^2, \overline{U}^2)$
- for $X_2 = \overline{X}^3$
  IF $\overline{U}^1$ THEN $b(\overline{X}^3, \overline{U}^1)$ ELSE IF $\overline{U}^2$ THEN $b(\overline{X}^3, \overline{U}^2)$

which in turn gives, via (27), the following fuzzy relation $R'_2(X_1)$ defined in $U \times [0,1]$:

$$\mu_{R'_2}(u_2, w_2) = \begin{array}{r|cc} & w_2 = 0.4 & 1 \\ \hline u_2 = c_1 & 1 & 0.7 \\ c_2 & 0.7 & 1 \end{array} .$$

Now, if we suppose the test function "k-large" $= 1/1$, then due to (30), for $m = 1$ we find the optimal control at $t = 2$ to be

$$U_2^* = 0.7/c_1 + 1/c_2$$

and, in a similar way, we can determine optimal controls for $X_2 = \overline{X}^2$ and $X_2 = \overline{X}^3$.

The same is repeated for the next control stages, i.e. for $t = 1$ and $t = 0$. The consecutive optimal fuzzy policies are obtained as:

$R^0_{u^*} =$

	$u_0 = c_1$	c_2
$x_0 = s_1$	1	0.2
s_2	1	0.2
s_3	1	0.2

$R^1_{u^*} =$

	$u_1 = c_1$	c_2
$x_1 = s_1$	1	0.2
s_2	1	0.2
s_3	1	0.2

.

$R^2_{u^*} =$

	$u_2 = c_1$	c_2
$x_2 = s_1$	1	1
s_2	0.7	1
s_3	0.4	1

If, for instance, the initial fuzzy state at control stage $t = 0$ is $X_0 = 0.2/s_2 + 1/s_3$, then – using consecutively the above optimal fuzzy policies – we obtain for the consecutive control stages $t = 1, 2, 3$ the following fuzzy states X_t [via (11)] and the optimal fuzzy controls U^*_t [via (32)]:

	X_t	U^*_t
$t = 0$	$0.2/s_2 + 1/s_3$	$1/c_1 + 0.2/c_2$
1	$0.3/s_1 + 0.7/s_2 + 1/s_3$	$1/c_1 + 0.2/c_2$
2	$0.7/s_1 + 0.7/s_2 + 1/s_3$	$0.7/c_1 + 1/c_2$
3	$1/s_1 + 0.7/s_2 + 0.7/s_3$	

.

This concludes our short exposition of Baldwin and Pilsworth's (1992) dynamic–programming–based approach. Notice, first, that it is very complicated, a couple of parameters have to be prespecified, and for solving non-trivial problems some conceptual simplifications are necessary, namely the use of reference fuzzy controls and reference fuzzy states, and then the formulation of the problems in terms of them. However, even then, to obtain meaningful results a high number of reference fuzzy states and reference fuzzy controls is needed because the use of the compositional rule of inference requires "overlapping" reference fuzzy states and reference fuzzy controls that should cover the entire space of states and controls. This may lead to too high numbers of them, and hence in the next section we will present a more effective and efficient approach that is based on interpolative reasoning in fuzzy rule based that was initiated by Kóczy and Hirota (1993a,b, 1997) (see also Dubois and Prade, 1992). The approach presented is based on early ideas by Kacprzyk (1993a, b,c, 1995d) but is extended and refocused here in view of many new results on fuzzy interpolation, exemplified notably by the works of Kóczy and his collaborators (Baranyi, Koćzy, and Gedeon, 2004; Koćzy, Hirota and Gedeon, 2000; Tikk and Baranyi, 2000; Yam, Wong and Baranyi, 2006; Wong et al., 2015).

4. Multistage control of a fuzzy system using dynamic programming with interpolative reasoning

The essence of a dynamic programing based approach with interpolative reasoning was initially proposed in Kacprzyk (1993a,b,c, 1995) and it boils down to the use of a very small number of non-overlapping reference fuzzy states and controls to formulate in their terms auxiliary fuzzy constraints, fuzzy goals, and a resulting auxiliary (much simpler!) control problem. Its solution yields an auxiliary optimal reference control policy relating optimal reference fuzzy controls to the particular reference fuzzy states. Such a policy is equated with a fuzzy relation, which is then used through the max–min composition to determine an auxiliary optimal control (not necessarily a reference one) for a particular current fuzzy state (not necessarily a reference one). Then, the (auxiliary) optimal control obtained should be in some way adjusted to become a "real" optimal fuzzy control.

First, a relatively small number, r, of reference fuzzy states $\overline{X}^1_{N-i+1}, \ldots, \overline{X}^r_{N-i+1}$, and a relatively small number, p, of reference fuzzy controls $\overline{U}^1_{N-i}, \ldots, \overline{U}^p_{N-i}$, $i = 1, \ldots, N-1$, are introduced.

The control problem is then expressed in terms of the reference fuzzy states and reference fuzzy controls only, and its solution, i.e. an (auxiliary!) optimal reference control policy, a^*_{N-i}, is expressed by the following fuzzy conditional statement [see (31)]:

$$\begin{cases} \text{IF } (X_{N-i} = \overline{X}^1_{N-i}) \text{ THEN } [U^*_{N-i}(X_{N-i}) = U^*_{N-i}(\overline{X}^1_{N-i})] \\ \text{ELSE} \ldots \text{ELSE} \\ \text{IF } (X_{N-i} = \overline{X}^r_{N-i}) \text{ THEN } [U^*_{N-i}(X_{N-i}) = U^*_{N-i}(\overline{X}^r_{N-i})] \end{cases} \tag{33}$$

which is equivalent to a fuzzy relation defined in $X \times U$, $\mu_{R^{N-i}_{U^*}}(x_{N-i}, u_{N-i})$.

This optimal reference control policy obtained should now be used to determine an optimal fuzzy control for a particular fuzzy state; and both need not be reference. At $t = N-i$ we are in a fuzzy state X_{N-i}. Then, the optimal fuzzy control given by this optimal reference fuzzy control policy is [see (32)]

$$\mu_{U^*_{N-i}}(u_{N-i}) = \max_{x_{N-i} \in X} [\mu_{X_{N-i}}(x_{N-i}) \wedge \wedge \mu_{R^{N-i}_{U^*}}(x_{N-i}, u_{N-i})], \qquad \text{for each } u_{N-i} \in U. \tag{34}$$

Let t denote the control stage instead of $N-i$, for simplicity, and X_t be a fuzzy number between the two reference fuzzy states $\overline{S}_i$ and $\overline{S}_{i+1}$. We seek its corresponding fuzzy control U^*_t. Clearly, it need not be a reference fuzzy optimal control.

For effectiveness and efficiency, we assume that the fuzzy states and controls are defined as triangular fuzzy numbers, i.e. as fuzzy sets in the real line. The determination of U^*_t boils down therefore to the determination of its mean value and width. The width is the length of interval given as the lowest and the highest values of the support of U_t^*, and the mean value is the middle of the support of the fuzzy set.

The procedure is as follows, for $t = 0, 1, \ldots, N-1$:

1. If X_t is the current (not necessarily reference) fuzzy state, with its mean value $\underline{X}_t$, find such two neighboring reference fuzzy states $\overline{S}_i, \overline{S}_{i+1} \in \mathcal{X}$ that their mean values $\underline{S}_i$ and $\underline{S}_{i+1}$, respectively, satisfy
$$\underline{S}_i \leq \underline{X}_t \leq \underline{S}_{i+1}. \tag{35}$$
2. Denote the supports of the above fuzzy states as:
$$\begin{aligned} &\text{supp } \overline{X}_t = [X_t^L, X_t^R] \\ &\text{supp } \overline{S}_i = [S_i^L, S_i^R] \\ &\text{supp } \overline{S}_{i+1} = [S_{i+1}^L, S_{i+1}^R] \end{aligned} \tag{36}$$
where the upper indices L and R denote the left-most and right-most values of the respective intervals.
3. For the two reference fuzzy states found as above, which satisfy (35), take their two corresponding optimal reference control policies of type (33), i.e.
$$\begin{cases} \text{IF } (X_t = \overline{S}_i) \text{ THEN } U_t^*(X_t) = \overline{C}_j \\ \text{IF } (X_t = \overline{S}_{i+1}) \text{ THEN } U_t^*(X_t) = \overline{C}_k \end{cases} \tag{37}$$
and denote the supports of $\overline{C}_j$ and $\overline{C}_k$ as
$$\begin{aligned} &\text{supp } \overline{C}_j = [C_j^L, C_j^R] \\ &\text{supp } \overline{C}_k = [C_k^L, C_k^R]. \end{aligned} \tag{38}$$
4. For the current fuzzy state X_t, whose support is supp $X_t = [X(t)^L, X(t)^R]$, find the optimal fuzzy control U_t^* (not necessarily reference!) the support of which is $[U_t^{*L}, U_t^{*R}]$, using Kóczy and Hirota's (1993a, b, 1997) idea of interpolative reasoning in fuzzy rule bases:
$$U_t^{*L} = C_j^L + \frac{X_t^L - S_i^L}{S_{i+1}^L - S_i^L}[C_j^L - C_k^L] \tag{39}$$
$$U_t^{*R} = C_k^L + \frac{X_t^R - S_i^R}{S_{i+1}^R - S_i^R}[C_j^R - C_k^R]. \tag{40}$$
5. Find the mean value of U_t^* as
$$\underline{U}_t^* = \frac{1}{2}(U_t^{*L} + U_t^{*R}). \tag{41}$$

Notice, first, that – in view of notable recent developments concerning interpolation in fuzzy rule bases, as mentioned in the beginning of this section, we have employed a slightly different form of the interpolative reasoning scheme than in the source papers by Kacprzyk (1993a,b,c, 1995).

The approach presented is conceptually convincing, effective, computationally efficient, and has proved to be useful in solving some planning problems in regional development – see Kacprzyk, Romero and Gomide (1999).

We will now present an illustrative example which will best illustrate the essence of our approach. In fact, this example is analogous to that given in Bellman and Zadeh (1970) for the deterministic system under control so that even more insight can be gained.

Example 2. Let the state space consist of three reference fuzzy states, $\overline{X} = \{\overline{S}_1, \overline{S}_2, \overline{S}_3\}$, and the control space consist of two reference fuzzy controls, $\overline{U} =$

$\{\overline{C}_1, \overline{C}_2\}$, all defined, for simplicity, as fuzzy numbers in $[0,1]$ with trapezoid (triangular) membership functions shown in Fig. 1. The mean value of the triangular fuzzy number is $\underline{S}_2$, while its counterparts for the trapezoid fuzzy numbers are meant to be $\underline{S}_1$, $\underline{S}_3$, $\underline{C}_1$ and $\underline{C}_2$. The respective spreads, assumed to be symmetrically distributed around the mean values, are $2\Delta(\underline{S}_i)$, $i = 1, 2, 3$, $2\Delta(\underline{C}_1)$ and $2\Delta(\underline{C}_2)$.

Suppose that the mean values and spreads are:

- for the reference fuzzy states:
  $\underline{S}_1 = 0.1 \qquad \underline{S}_2 = 0.5 \qquad \underline{S}_3 = 0.9$
  and
  $2\Delta(\underline{S}_1) = 0.2 \qquad 2\Delta(\underline{S}_2) = 0.2 \qquad 2\Delta(\underline{S}_3) = 0.2$
- for the reference fuzzy controls
  $\underline{C}_1 = 0.2 \qquad \underline{C}_2 = 0.8$
  and
  $2\Delta(\underline{C}_1) = 0.4 \qquad 2\Delta(\underline{C}_2) = 0.4.$

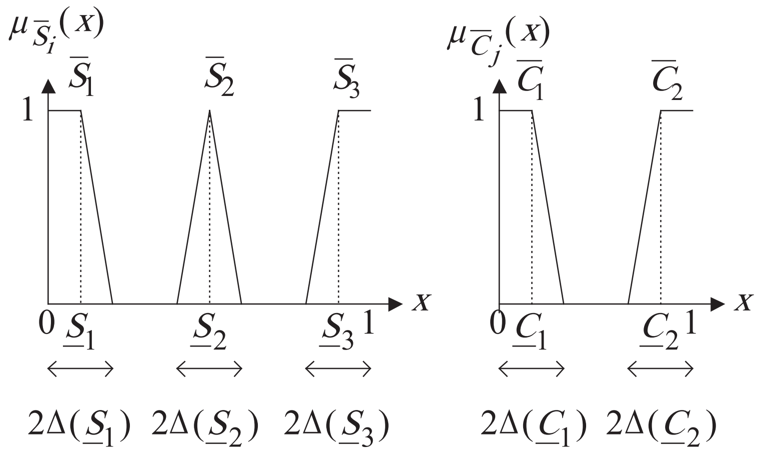

Figure 1. Reference fuzzy states $\overline{S}_i$, $i = 1, 2, 3$, and reference fuzzy controls $\overline{C}_1$ and $\overline{C}_2$, with their respective mean values and spreads for Example 2

Suppose that the termination time is $N = 2$, and the fuzzy constraints and fuzzy goal are defined as, respectively:

$$\begin{aligned} C^0 &= 0.7/\overline{C}_1 + 1/\overline{C}_2 \\ C^1 &= 1/\overline{C}_1 + 0.8/\overline{C}_2 \\ G^2 &= 0.3/\overline{S}_1 + 1/\overline{S}_2 + 0.8/\overline{S}_3. \end{aligned}$$

Dynamics of the fuzzy system under control is governed by the fuzzy state transition equation of type (11), which is given now in terms of reference fuzzy states and controls, as:

$$\overline{X}_{t+1} = \quad \begin{array}{r|ccc} & \overline{X}_t = \overline{S}_1 & \overline{S}_2 & \overline{S}_3 \\ \hline \overline{U}_t = \overline{C}_1 & \overline{S}_1 & \overline{S}_3 & \overline{S}_1 \\ \overline{C}_2 & \overline{S}_2 & \overline{S}_1 & \overline{S}_3 \end{array} \quad . \tag{42}$$

The fuzzy dynamic programming recurrence equations (10) are now employed, first for $i = 1$, and we obtain $G^1 = 0.6/\overline{S}_1 + 0.8/\overline{S}_2 + 0.6/\overline{S}_3$, and the corresponding optimal reference fuzzy control policy (expressed in terms of the reference fuzzy states and controls):

$$\overline{a}_1^*(\overline{S}_1) = \overline{C}_2 \qquad \overline{a}_1^*(\overline{S}_2) = \overline{C}_1 \qquad \overline{a}_1^*(\overline{S}_3) = \overline{C}_2. \tag{43}$$

Next, the subsequent iteration of (10), for $i = 2$, yields $G^0 = 0.8/\overline{S}_1 + 0.6/\overline{S}_2 + 0.6/\overline{S}_3$, and the corresponding optimal reference fuzzy control policy:

$$\overline{a}_0^*(\overline{S}_1) = \overline{C}_2 \qquad \overline{a}_0^*(\overline{S}_2) \in \{\overline{C}_1, \overline{C}_2\} \qquad \overline{a}_1^*(\overline{S}_3) \in \{\overline{C}_1, \overline{C}_2\}. \tag{44}$$

Therefore, for instance, if we start at $t = 0$ from $\overline{X}_0 = \overline{S}_1$, then $\overline{U}_0^* = \overline{a}_0^*(\overline{S}_1) = \overline{C}_2$ and we obtain $\overline{X}_1 = \overline{S}_2$. Next, at $t = 1$, $\overline{U}_1^* = \overline{a}_1^*(\overline{S}_2) = \overline{C}_1$ and $\mu_D(\overline{U}_0^*, \overline{U}_1^* \mid \overline{S}_1) = \mu_D(\overline{C}_2, \overline{C}_1 \mid \overline{S}_1) = 0.8$.

For our purposes it is expedient and illustrative to present the reference control policy at $t = 1$ as in Fig. 2, and at $t = 0$ as in Fig. **??**.

Basically, these figures show for each reference fuzzy state $\overline{X}_1$ and $\overline{X}_0$ its corresponding optimal reference fuzzy control $\overline{U}_1^*$ and $\overline{U}_0$ depicted in heavy line; since the optimal reference fuzzy controls need not obviously be unique, we obtain here $\overline{C}_1$ and $\overline{C}_2$, and they are both shown in heavy line.

We will now show the use of interpolative reasoning. Suppose that the initial (non-reference!) fuzzy state is X_0 given as a triangular fuzzy number with the mean value $\underline{X}_0 = 0.2$ and spread $2\Delta(\underline{X}_0 = 0.2$.

The consecutive steps of the interpolative-reasoning-based procedure presented in this paper are:

1. For X_0 as given above, we find using (35) that $i = 1$, i.e. $\underline{S}_i = \underline{S}_1$ and $\underline{S}_{i+1} = \underline{S}_2$.
2. We obtain the supports:
$$\begin{array}{l} \text{supp } X_0 = [0.1, 0.3] \\ \text{supp } \overline{S}_1 = [0.0, 0.2] \\ \text{supp } \overline{S}_2 = [0.4, 0.6] \end{array} \quad .$$
3. The two possible optimal reference fuzzy control policies for $\overline{S}_1$ and $\overline{S}_2$ are [see (37)]:
$$\left\{ \begin{array}{l} \text{IF } (X_0 = \overline{S}_1) \text{ THEN } U_0^*(X_0) = \overline{C}_2 \\ \text{IF } (X_0 = \overline{S}_2) \text{ THEN } U_0^*(X_0) = \overline{C}_1 \end{array} \right. \tag{45}$$
and
$$\left\{ \begin{array}{l} \text{IF } (X_0 = \overline{S}_1) \text{ THEN } U_0^*(X_0) = \overline{C}_2 \\ \text{IF } (X_0 = \overline{S}_2) \text{ THEN } U_0^*(X_0) = \overline{C}_2 \end{array} \right. \tag{46}$$

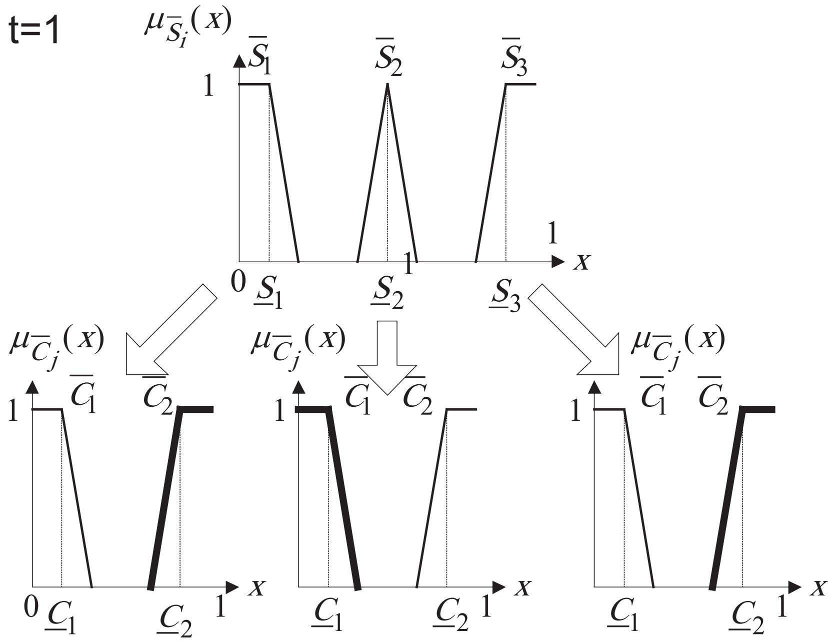

Figure 2. Pictorial representation of the optimal reference control policy at $t = 1$ in Example 2

and, for the first one, i.e. (45), we obtain:

$$\text{supp } \underline{C}_1 = [0.0, 0.4] \qquad \text{supp } \underline{C}_2 = [0.6, 1.0]$$

and analogous for the second one, i.e. (46).

4. For X_0, the support of which is supp $X_0 = [0.1, 0.3]$, and assuming the optimal reference fuzzy control policy (45), we find by using (39) and (40) the optimal fuzzy control U_0^*, the support of which is

   $$\text{supp } U_0^* = [0.75, 1.15]$$

   or, maybe more appropriately, since the fuzzy controls and states are assumed to be defined as fuzzy sets in $[0, 1]$, it is

   $$\text{supp } U_0^* = [0.75, 1.0].$$

5. The mean value of U_0 is therefore either $\underline{U}_0 = 0.95$ or $\underline{U}_0 = 0.875$, respectively, and one can notice that the result obtained is consistent with the optimal reference fuzzy control policy (44).
6. From X_0 and under the above optimal fuzzy control U_0^* as determined above, we proceed to the next state, X_1. Since X_0 is close to $\overline{S}_1$, and U_0^* is close to $\overline{C}_2$, then we can assume that by the fuzzy state transition equation of the system under control, (42), we arrive at X_1 the support of which is supp $X_1 = [0.5, 0.7]$. Notice, however, that in the determination of state transitions in the case considered, another interpolative reasoning

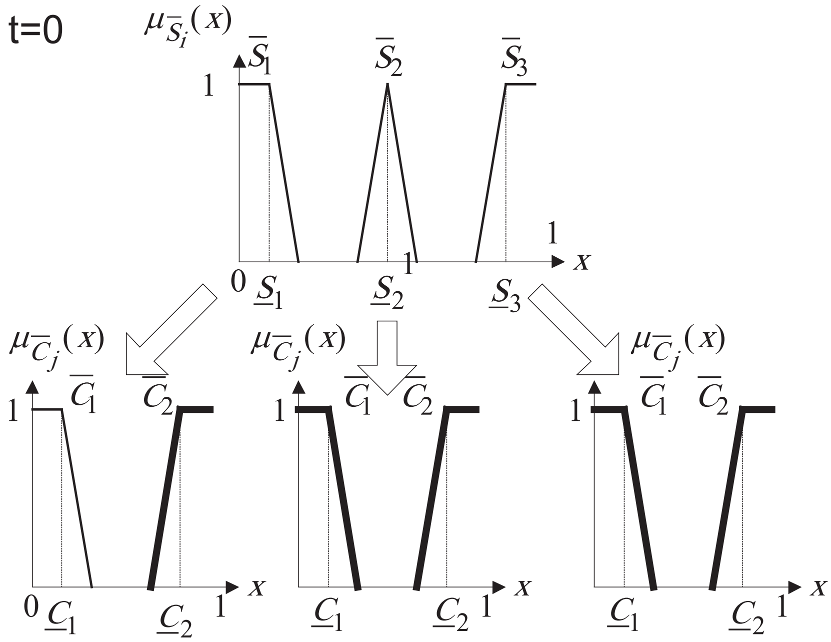

Figure 3. Pictorial representation of the optimal reference control policy at $t = 0$ in Example 2

scheme should be employed to obtain more meaningful results. This will not be, however, dealt with in this paper, and a simplified state transition scheme, as outlined above, will be used.

7. For X_1 as given above, we find by (35) that $i = 2$, i.e. $\underline{S}_i = \underline{S}_2$ and $\underline{S}_{i+1} = \underline{S}_3$.
8. We obtain the supports:
$$\text{supp } \overline{X}_1 = [0.5, 0.7] \quad \text{supp } \overline{S}_2 = [0.4, 0.6] \quad \text{supp } \overline{S}_3 = [0.8, 1.0] .$$
9. The optimal reference fuzzy control policy for $\overline{S}_2$ and $\overline{S}_3$ is [see (37)]:
$$\begin{cases} \text{IF } (X_1 = \overline{S}_2) \text{ THEN } U_0^*(X_0) = \overline{C}_1 \\ \text{IF } (X_0 = \overline{S}_3) \text{ THEN } U_0^*(X_0) = \overline{C}_2 \end{cases} \tag{47}$$
such that:
$$\text{supp } \underline{C}_1 = [0.0, 0.4] \qquad \text{supp } \underline{C}_2 = [0.6, 1.0].$$
10. For X_1, the support of which is supp $X_1 = [0.5, 0.7]$, we find by using (39) and (40) the optimal fuzzy control U_0^*, the support of which is
$$\text{supp } U_1^* = [-0.15, 0.45]$$
or, maybe more appropriately, since the fuzzy controls and states are

assumed to be defined as fuzzy sets in $[0,1]$, it is
$\text{supp}\, U_1^* = [0.0, 0.45]$

11. The mean value of U_1^* is therefore $\underline{U}_1^* = 0.3$ or $\underline{U}_1^* = 0.225$, respectively, and one may notice that the result obtained is consistent with the optimal reference fuzzy control policy (43).

5. Concluding remarks

We presented an interpolative reasoning technique meant to overcome numerical difficulties in the solution of optimal multistage fuzzy control problems with a fuzzy system under control by using dynamic programming. The method, with the roots in the early Kacprzyk's (1993a,b,c, 1995d) works, was presented in a slightly different form and perspective, implied by some recent relevant results on interpolation in fuzzy rule bases. As for future research, it seems that a fruitful area will be a more direct use of concepts and results related to sparse fuzzy rule bases as first analyzed by Kacprzyk and Fedrizzi (1995), and then further developed by many people, for instance by Wu, Mizumoto and Shi (1996), Chang, Chen and Liau (2008), Chen and Ko (2008), or Hsiao, Chen and Lee (1998). The method proposed is conceptually simple, yields a tractable numerical efficiency, and intuitive results, and may be useful in solving some practical problems exemplified by an application to regional development planning (see Kacprzyk, Romero and Gomide, 1999).

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