Scientific Issues
Jan Długosz University
in Częstochowa
Mathematics XX (2015)
15–22
DOI http://dx.doi.org/10.16926/m.2015.20.02

ANGLE – WHAT KINDS OF ANGLES WE MAKE USE OF

JACEK JĘDRZEJEWSKI, KATARZYNA TROCZKA-PAWELEC

Abstract

The idea of an angle has several meanings at school mathematics. We propose the survey of different understandings of this notion with respect to the age of pupils, the level of school. We discuss the difficulties in defining and understanding of the notion of an angle. Of course, we discuss the topic with respect to Polish programmes of school mathematics.

1. Introduction

The notion of an angle appears at the second educational level. The pupils at this level are 9–11 years old. Before that time the pupils known rather feel the notion of an angle. Every girl and every boy at this age knows what angle means. They know what means *at this angle* or *turn a right angle at the next crossing.* The children can draw a square and a triangle and so on.

After they come to the second educational level they can (according to governmental programmes), see [1]:

- point a vertex and sides of an angle;
- measure angles less than 180° with accuracy to 1°;
- draw angles which have measure less than 180°;
- recognize right angle, acute angle and obtuse one;
- compare the angles;
- recognize vertically opposite angles, supplementary angles and make use of its properties.

In the next didactic level (secondary school), the pupil:

• *Jacek Jędrzejewski* — Jan Długosz University in Częstochowa.
• *Katarzyna Troczka-Pawelec* — Jan Długosz University in Częstochowa.

- recognizes central angles;
- makes use of properties of angles and diagonals in rectangles, parallelograms, rhombuses and trapeziums;
- applies congruence principles of triangles;
- makes use of properties of similar right-angled triangles;
- recognizes perpendicular bisector of a segment and bisector of an angle;
- can construct a bisector of a segment and bisector of an angle;
- can construct angles with measure of $60°$, $30°$, $45°$;
- can construct the circle ex-scribed on a triangle and a circle inscribed in a triangle;
- recognizes regular polygons and makes use of their fundamental properties.

In the last educational level (high school) one can find the following items of governmental programme:

- definitions and values of sinus, cosine, tangent of the angles: from $0°$ up to $180°$;
- approximate values of trigonometric functions;
- relations between central angle and interior angle of a circle.

As we can see the survey of the topics on angles at school mathematics are bound to considerations of plane angles, which we can describe as convex angles.

If we want to come to consider angles on higher level of mathematics, it is not enough to discus those kinds of angles. We can observe that the notion of an angle has several meanings in mathematics. Even at school mathematics, we can notice that the angles, we were discussed, are not only geometrical figures i.e. subsets of a geometrical plane.

While considering geometrical transformations we need to see an angle as a rotation with respect to a point. Moreover one can rotate a point in one or the other direction. So we need *directed angles*. But what are such angles like that are needed to describe rotations?

In mechanics and theory of electric engines. In such a case one can notice that rotations in such machines are different from geometrical transformations of a plane.

In further considerations we shall treat angles as real numbers. Its origin is dated when we wanted to measure angles applying a unit called *radian*, which is discussed a bit later.

This short survey of what an angle is, one can understand this notion like:

- angle as a part of a plane, i.e. geometrical figure;
- directed angle, i.e. ordered pair of half-lines with a common vertex;
- angle as a measure of a rotation;
- generalized angle which measure is given by a real number.

We shall discuss the above mentioned types of angles in the following sections.

2. Angle on a plane

First idea of an angle can be found at the first step of education. It is intuitive idea of an angle which means that it is a part of a plane. There are some complications in defining angles cause this kind of the notion depends on difficult mathematical theory called topology.

Classical geometry on a plane makes us to consider geometrical objects as subsets of a plane. Before we formulate the definition of an angle it is necessary to recall some topological ideas.

Let us remind necessary definitions and properties of topology of the plane [4] and [3].

Definition 1. *A subset E of the plane is called open on a plane if for every point of the set E has a circular neighbourhood contained in E.*

Definition 2. *By an open connected set we mean an open set which has no representation as a union of two non-empty open sets.*

If we consider topology of the plane, this definition can be reformulated in the following way:

Theorem 1. *An open subset E of the plane is connected iff every two points of this set can be connected by a broken line contained in the set E.*

According to those considerations, a plane, a half-plane, a disc and all convex subsets are connected.

Definition 3. *We say that a set E cuts a plane P if the set $P \setminus E$ is not connected.*

The main property which is necessary to define an angle states as below:

Theorem 2. *The union of two (if two, it means that it is not the one considered twice) half-lines with a common vertex cuts the plane onto two disjoint non-empty open and connected subsets.*

Definition 4. *By an angle on a plane we understand a union of two half-lines with a common vertex and one (open) part of the plane on which the union of these half-lines cuts the plane. This part of the plane is called the domain of the angle.*

Geometrical angles can be measured, the most common scale of measuring angles is based on degrees. The right angle has 90° and other angles have proportional measure. It is quite obvious that angles can have measure between 0° and 360°. But there is neither angle with the measure 0° nor 360°. Geometrical ideas do not allow to consider such angle like zero angle as well as full angle.

If we want to define zero angle or full angle we have to come to a different idea of an angle.

3. Directed angle on a plane

The main disadvantage of an angle considered as subsets of the plane is caused by impossibility to consider null angle (0°) and full angle (360°) see [2].

One of the topics in mathematical education concerns to symmetry and rotation, one can say, transformations of a plane. It is a topic on transformations of some figures, but in fact it is a transformation of a plane. It is necessary to consider directed angles in defining such notion (rotation).

Both disadvantages can be repaired by introducing directed angles. But in order to define directed angles it is necessary to introduce coordinate system. Let us remind it then.

By a coordinate system on a plane we understand a pair of perpendicular real lines which have their origins as a common point of the axes. Usually, the first axis (the horizontal one) is denoted as Ox-axis, the second one (the vertical axis) is denoted as Oy-axis. The Ox-axis is directed to the right, the Oy-axis is directed upwards.

If we want to consider orientations of the whole coordinate system it is possible to see the movement from Ox-axis to Oy-axis like the movement of hands of a traditional clock or into the opposite direction. The first direction is called negative orientation, the second direction is concerned as a positive orientation of a coordinate system.

Now we are able to define directed angles.

Definition 5. *By a directed angle (on a plane) we mean an ordered pair of half-lines which have a common vertex. Such an angle is denoted by $\sphericalangle(p,q)$, where p and q are the sides of this angle.*

Saying informally, a directed angle is meant as two half-lines with a common vertex for which one the half-lines is the first one. Of course, when we have two half-lines it is possible to move the first half-line according to positive orientation or into the opposite direction.

Let us consider first the case when two sides of a directed angle (p,q) are different. Then the half-lines p and q have the common vertex and they cut the plane forming a usual (i.e. from the previous section) angle. Such a pair of half-lines forms 2 (geometrical) angles. If the interior of the angle is taken with respect to orientation agreed with orientation of the coordinate system then this angle is called to be positive. If the interior of the angle is taken against to orientation of the coordinate system then this angle is called to be negative.

By the measure of the angle $\sphericalangle(p,q)$ is meant the measure of this angle considered as a geometrical angle if it is positive and if it is negative then by the measure of $\sphericalangle(p,q)$ we mean minus measure of this angle considered as a geometrical one.

And what does happen when one and the same half-line is the first and the second side of an angle?

3.1. **Null angle, full angle.** Then an ordered pair can be consisted by one and the same element. One half-line does not cut the plane. In this case there is no interior of an angle or... the entire plane forms the interior of an angle. Such a pair does not define a geometrical angle. In the first case there is only one half-line on a plane – no angle. In the second case there is no angle as well. Considering one half-line together with the rest of the plane there is neither side nor vertex of an angle. It is (geometrically) the entire plane. It is no angle as well.

If we take the empty set as the interior of the angle $\sphericalangle(p,p)$ then such angle is called null angle. Its measure is taken to be $0°$.

If we take the whole plane except the half-line p as the interior of the angle $\sphericalangle(p,p)$ then such angle is called full angle. Its measure is taken to be $360°$ if it is positive and $-360°$ if it is negative.

Right now we have come to the next problem. Ordered pairs of half-lines are not sufficient to describe plane transformations called rotations. Several directed angles can define the same rotation. The only condition which is important in such definition is the measure of the directed angle. So in fact we should consider the set of all directed angles with the same measure as the one angle. Thus we are forced to consider equivalence relation for which equivalent angles have the same measure. In such a way we have come to abstract theory. Such considerations are not present at Polish schools.

The reader is asked to find and read some books on equivalence relation and theory of abstraction.

4. Generalized angle

4.1. **Measure of a rotation.** There is another need of considering other kinds of angles. If we rotate a plane with respect to a point, the directed angles are good enough. But it is possible to consider another rotations as movements of a half-line with respect to its vertex. Even more, this time we can rotate this half-line several times around the vertex. Of course, we can rotate the half-line in the positive or negative direction.

The first case is concerned to the rotation of a plane (transformation of a plane). The second possibility concerns to the movements of a half-line with respect to its vertex several times in positive or negative direction.

Rotation concerned as plane transformation can be defined in the following way: The rotation on the angle α with respect to a point S is defined as the transformation of the plane in which every point P of the plane is transformed to the point P' such that

$$\sphericalangle PSP' = \alpha.$$

In this relation the symbol $\sphericalangle PSP'$ denotes the directed angle between the half-line $SP^{\rightarrow}$ and the half-line $SP'^{\rightarrow}$.

Let us consider the second case.

Look at the second hand of traditional clock. Within one minute it will turn a full angle (negative one this time) i.e. 360°. Within the next minute it will turn another 360°. And so on. One can observe that the second hand of a clock will rotate of $x°$. So this hand can turn on any real number of degrees.

4.2. **Angles in trigonometry.** The angles measured by degrees are not good enough for trigonometrical functions. It is much more convenient to describe measure of angles by real numbers. To this purpose it is necessary to define a better kind of measuring angles. First let us observe that the measure of an angle is proportional to the length of the arc of the unit circle when the angle is considered as a central angle. More precisely:

Definition 6. *A central angle of a unit circle has measure* 1 *(radian) if the length of the arc of this circle cut by the angle equals* 1*.*

If the arc of the unit circle cut by circled angle has the length x, then the measure of this angle equals x.

In fact, this measure is without the name, cause it is also the quotient of the length of an arc and radius of the circle.

Hence every (geometrical) angle has measure in the interval $(0, 2\pi)$.

Every directed angle can have measure in the interval $[0, 2\pi]$ and every generalized angle can have any real number as its measure.

Thus considering trigonometrical functions sine and cosine we can define them like real functions of a real variable. Of course functions tangent and cotangent can be defined with some natural restrictions.

5. Final remarks

Summing up, we can observe that the notion of an angle depends on the purpose we need to. In each of considered cases we call the defined objects as angles, but which idea is the right one? Of course it depends on the applications to which we have to use angles. The question that arises from our considerations is like that: Is it necessary call all those objects by the same name **angle**?

References

[1] *Podstawa Programowa*, w rozporządzeniu Ministra Edukacji Narodowej z dnia 27 sierpnia 2012 r. w sprawie podstawy programowej wychowania przedszkolnego oraz kształcenia ogólnego w poszczególnych typach szkół (Poz. 977)

[2] *Encyklopedia Matematyczna*, WSiP, Warszawa, 1994.

[3] R. Engelking, *Topology*, PWN Warszawa 1980.

[4] J. Mioduszewski, *Wykłady z topologii*, Wyd. Uniwersytetu Śląskiego, Katowice 1994.

Received: July 2015

Jacek Jędrzejewski
Jan Długosz University in Częstochowa,
Institute of Mathematics and Computer Science,
Al. Armii Krajowej 13/15, 42-200 Częstochowa, Poland
E-mail address: jacek.m.jedrzejewski@gmail.com

Katarzyna Troczka-Pawelec
Jan Długosz University in Częstochowa,
Institute of Mathematics and Computer Science,
Al. Armii Krajowej 13/15, 42-200 Częstochowa, Poland
E-mail address: k.troczka@ajd.czest.pl