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Selected issues of the fractional calculus for analysis of dynamic properties of measuring transducers used in transportation facilities

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Abstract

This paper outlines the use of the fractional calculus for dynamic measurements while describing dynamic properties of measuring transducers, which the authors consider to be an original and unique achievement. The aim of this paper is to investigate how models of accelerometers based on the fractional calculus notation convey their dynamic behaviour in comparison to models represented by differential equations of integer orders and to the processing characteristics of their real counterparts. This paper presents state-of-the-art knowledge, simulation and laboratory studies of measuring transducers to measure acceleration (accelerometers), considering them a representative group of the measuring transducers used in transportation facilities. Measurement errors and comparisons of classical and fractional models in terms of dynamic properties were examined. The advantages of fractional calculus in modelling dynamic elements are also indicated. Tests are executed in the MATLAB & Simulink programme.

Introduction

The problem of measuring the dynamic properties of objects by means of fractional order differential equations (or more broadly – fractional calculus), although well known since the times of Gottfried Wilhelm Leibniz (1646–1716) and Guillaume François Antoine de l'Hospital (1661–1704), has always been ignored due to restrictions resulting from a lack of appropriate calculation methods or possibilities of their verification (among other reasons connected to the lack of calculation potential in earlier computers) (Podlubny, 1999; Ostalczyk, 2008; Kaczorek, 2011). At present, advances in technical and calculation possibilities means that the problems related to these limitations have, to a large extent, been solved. There are more and more publications dealing with the topic of fractional order differential equations. The majority of them, however, deal with theoretical aspects of the problem. There are no publications that put a strong emphasis on the practical application of fractional calculus and combining theory with real applications. This paper fills that gap.

Mathematical models describing the dynamic performance of devices (measuring devices, automation mechanisms, sensors, etc.) are widely used in different disciplines of science. Their task is to reproduce the real behaviour of the examined device in a simulation environment. They are most frequently used at early stages of research, prior to the real examination of the problem, or construction of a device as a quick tool for fast prototyping.

These models allow for simulated testing of an object's behaviour under normal and extreme working conditions. In this way onerous and costly preliminary investigations of real objects that aim at early assessment of their usefulness (the method investigated or the object) for concrete applications are avoided. What is more, these models are also the basic tools that allow us to get acquainted with the mathematical or physical foundations of a given object or phenomenon's performance.

In the classic notation, second-order measuring transducers presented in this paper are described with second-order differential equations like many other groups of measuring transducers used in transportation facilities (Luft et al., 2012), such as: RLC circuits, displacement measurement sensors, accelerometers systems including tensometric, piezoelectric transducers and mechanical vibrating systems.

The aim of this paper is to investigate how models of accelerometers based on the fractional calculus notation convey their dynamic behaviour in comparison to models represented by differential equations of integer orders and in comparison to the processing characteristics of their real counterparts (Pietruszczak, 2012). A particular aim of this paper is to combine theoretical knowledge of fractional calculus with its application for modelling dynamic properties of real accelerometers and generalize this knowledge so that it could refer to the class of objects modelled by means of differential equations.

Model of an accelerometer

The dynamic behaviour of accelerometers (or in general – objects, sensors and measuring devices for different applications using in transportation facilities) is written down in a form of differential equations or operator transmittances. In the process of determining the dynamic behaviour of a model for an accelerometer (object), dynamic behaviour and physical phenomena are taken into account, which results from external influences and specific properties of an accelerometer being an effect of their design. Thus, the accuracy of reproducing an accelerometer's real dynamic behaviour is first of all, connected with this phenomenon.

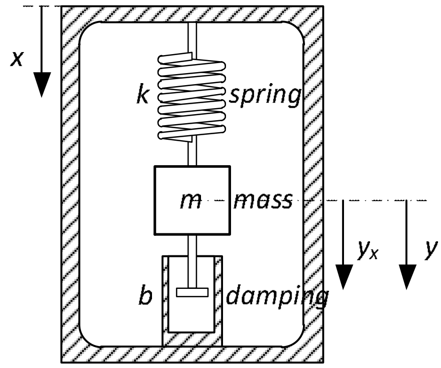

Figure 1. Diagram of an accelerometer's design: *x* – object motion relative to a fixed system of coordinates, *y* – motion of a vibrating mass relative to a fixed system of coordinates, y_x – motion of a vibrating mass relative to a vibrating object (Luft et al., 2011; Luft, Szychta & Pietruszczak, 2015)

Figure 1 depicts a mechanical model of a single-axis accelerometer. Movement *x* of the base in relation to the immovable system of coordinates entails the movement of inert mass *m*, which can be divided into movement y_x, in relation to the immovable system of coordinates and movement *y*, in relation to the base. The mass is hung on a spring having elasticity coefficient *k* and fixed to a damper of damping coefficient *b*. Movement *y* is converted into an electric signal and fed to the sensor's output. The use of piezoelectric elements is a typical conversion mechanism. Figure 2 depicts the design

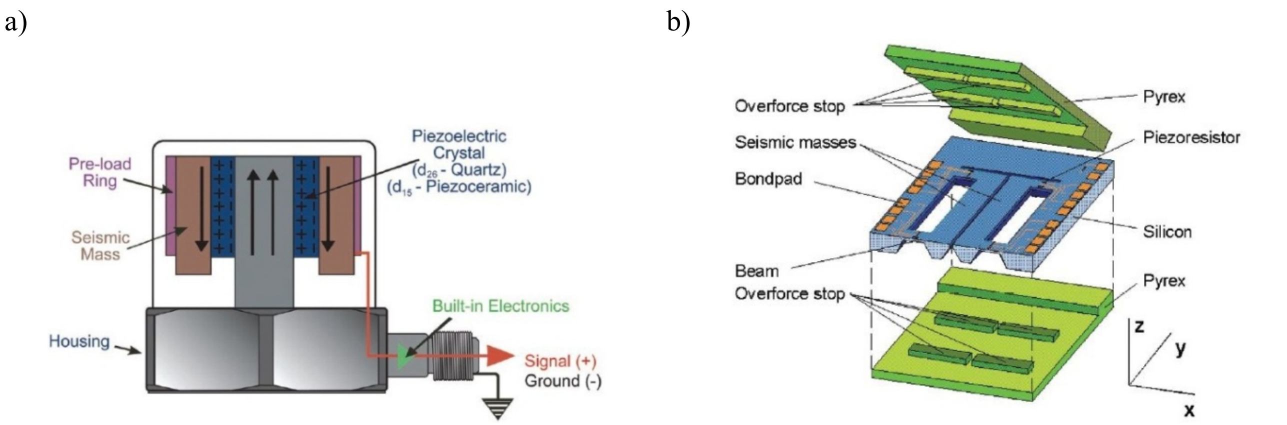

Figure 2. Piezoelectric accelerometer: a) general design, b) design of the element for seismic mass movement conversion into electric signal (Walter, 2008)

of a piezoelectric accelerometer (a) and components of the element converting the inert mass movement into an electric signal (b). In this type of sensor, the inert mass, acting on piezoelectric material, causes its deformation and generates an electric charge. The basis of the sensor's functionality is the piezoelectric phenomenon consisting of the generation of electric charges on the walls of a crystal during its elastic deformation.

Regardless of the mechanical design of accelerometers and the applied elements converting mass movement into an electric signal (piezoelectric, capacitive, piezoresistant accelerometers, MEMS) their dynamic properties are modelled on the basis of the diagram shown in Figure 1.

The dynamic behaviour of the accelerometer is written down in a form of a differential equation of the second order:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}w(t) + 2\zeta\omega_0\frac{\mathrm{d}}{\mathrm{d}t}w(t) + \omega_0^2 w(t) = -\frac{\mathrm{d}^2}{\mathrm{d}t^2}x(t) \tag{1}$$

where:

w – motion of a vibrating mass relative to a vibrating object;

x – object motion relative to a fixed system of coordinates, parameters characteristic of accelerometers:

amplification coefficient: $k = \frac{1}{k_s}$,

natural pulsation: $\omega_0 = \sqrt{\frac{k_s}{m}}$ and

damping degree: $\zeta = \frac{B_t}{2\sqrt{k_s m}}$.

Introducing a non-integral order to the measuring transducer's equation (1) converts it into (Pietruszczak, 2012):

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}w(t) + 2\zeta\omega_0\frac{\mathrm{d}^{(v)}}{\mathrm{d}t^{(v)}}w(t) + \omega_0^2 w(t) = -\frac{\mathrm{d}^2}{\mathrm{d}t^2}x(t) \tag{2}$$

where: v – fractional order derivatives.

The equation of the accelerometer's dynamic behaviour (1) refers to its design containing mechanical elements, which are the analogues of electrical elements. Thus, for example, by way of analogy mechanical elements of the accelerometer can be converted into electrical ones and such a system can be analyzed with the use of Kirchhoff laws or electric systems can be considered in this way. As the dynamic behaviour of the majority of different types of sensors is described by means of a differential equation of the first and second order, the accelerometer is a good example for further expansion of research into other types of sensors and objects.

The notation of dynamic properties of measuring sensors and instruments, automation objects, etc. (in general, the objects revealing dynamic properties) in a form of differential equations is well known, described in literature, and commonly used. However, it is restricted to the description of dynamic properties in a form of integer order equations: $\{1,2,3\ldots\}$. Despite this fact, in most cases, such equations describe real issues of the real world dynamic properties fairly well. It might seem that limiting the differential equations to integer orders should result in a simplified model that does not convey the object's real performance in full. However, in majority of applications, it is adequate for the description. Perhaps this is a result of sufficient measurement accuracy in modern technology or the degree of complexity in the modelled objects. Nevertheless, it is a fact that some basic issues are described better by fractional order equations, e.g.: properties of the so called super-condensers, permeation of liquids through porous substances, permeation of charges through real insulators, properties of viscoelastic materials, or phenomena connected with dynamic friction. Owing to the materials used for their construction (among others, piezoelectric materials) and the physical phenomena (elasticity and damping) occurring in them, the three latter issues concern accelerometers. Since these issues are described more accurately by models of fractional order equations, one can expect that the current models of the dynamic properties of accelerometers (by means of differential equations of integer orders) are not complex enough for the physical phenomena occurring in them. The aim of this paper is to compare the results of investigations for real accelerometers with those for their models based on integer order differential equations (commonly used at present) and those of fractional order so as to answer the question: *which of these ways of modelling conveys the dynamic behaviour of a real accelerometer better?*

The answer to this question may mean the first step on the way towards abandoning "classical" modelling of objects by means of differential equations of integer order in favour of those of fractional orders as the ones which better reflect the dynamic behaviour of real objects.

Selected issues of fractional calculus

In fractional calculus, a derivative of arbitrary order is treated as an interpolation of a sequence of operators of discrete orders, with operators of continuous orders. A notation introduced by H.D. Davis (Ostalczyk, 2008) is used here, in which a fractional order derivative of a function $f(t)$ is represented as:

$$ {}_{t_0}D_t^{\nu} f(t) \tag{3}$$

where t_0 and t define the integration or differentiation interval and v is the order of the derivative.
As the problem has been continually developed, there are many definitions of fractional derivative. Because describing dynamic properties of the measuring transducers requires fractional arithmetic, we can use one of three definitions (Kaczorek, 2011): Grünwald-Letnikov, Riemann-Liouville and Caputo.

The Grünwald-Letnikov Definition

The function of a real variable $f(t)$ defined in the $[t_0, t]$ interval is given. Assuming that the function increment $h > 0$ is such that: $h = (t - t_0)/k$ provided that $h \to 0$ causes that $h \to +\infty$ for the established $t - t_0$, then the Grünwald-Letnikov fractional derivative of a discrete function for $f(hi)$, $i = 0,1,2,\ldots$ is defined as:

$$ {}_{t_0}D_t^{\nu} f(t) = \lim_{\substack{h \to 0 \\ t - t_0 = kh}} \left[\frac{1}{h^{\nu}} \sum_{i=1}^{k} a_i^{(\nu)} f(t - hi) \right] \tag{4}$$

In mathematics, the Grünwald–Letnikov derivative is a basic extension of the derivative in fractional calculus that allows one to take the derivative a non-integer number of times. It was introduced by Anton Karl Grünwald (1838–1920) from Prague, in 1867, and by Aleksey Vasilievich Letnikov (1837–1888) in Moscow in 1868.

The Riemann-Liouville Definition

The Riemann-Liouville's fractional derivative is the function described by the formula:

$$ {}_0^{RL}D_t^{\alpha} f(t) = \frac{1}{\Gamma(k-\alpha)} \frac{\mathrm{d}^k}{\mathrm{d}t} \int_a^t (t-\tau)^{k-\alpha-1} f(\tau)\,\mathrm{d}\tau \tag{5}$$

where v is the order of integration within the $[t_0, t]$ interval of $f(t)$ function, $k - 1 \le \alpha \le k$, $\alpha \in R^+$, $\Gamma(x)$ is defined as:

$$ \Gamma(x) = \lim_{n \to \infty} \frac{n!\,n^x}{x(x+1)\ldots(x+n)} \tag{6}$$

for $x \in C$.

The Riemann–Liouville fractional derivative corresponding derivative is calculated using Lagrange's rule for differential operators. Computing k-th order derivative over the integral of order $(k - \alpha)$, the α order derivative is obtained. It is important to remark that k is the nearest integer bigger than α.

The Caputo's Definition

The Caputo's definition of fractional derivative is described as:

$$ {}_0^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(k-n)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}}\,\mathrm{d}\tau \tag{7}$$

where: $n - 1 \le \alpha \le n$.

The Caputo fractional derivative was introduced by M. Caputo in his paper (1967). In contrast to the Riemann-Liouville fractional derivative, when solving differential equations using Caputo's definition, it is not necessary to define the fractional order initial conditions. It is assumed that Caputo fractional derivative includes Riemman-Liouville derivatives.

Concept and research results

The concept of this work is based on a comparison between different models of an accelerometer's dynamic behaviour (based on differential equations of integer and fractional orders) with the processing characteristics of a real accelerometer so as to obtain an unambiguous answer to the question of which method of modelling is more accurate and whether there are any criteria for which a certain model is better at reproducing the dynamic behaviour of the real accelerometer (e.g. can it be related to the accelerometer's sensitivity?).

The research plan includes the following algorithm of proceedings:

1. Investigating the processing characteristics of real accelerometers over the entire range of the measuring signal processing with the highest possible measurement accuracy (determination of amplitude and phase characteristics of examined accelerometers).
2. Developing models describing the dynamic behaviour of the real accelerometers by means of differential equations of integer order on the basis of characteristics of the measuring signal processing.
3. Developing models describing the dynamic behaviour of real accelerometers by means of fractional calculus on the basis of characteristics of the measuring signal processing.
4. Comparing processing characteristics of the accelerometer models from points 2 and 3 with their real counterparts and comparing processing

characteristics of different models with each other.

5. Developing an optimum method of determining fractional order differential equations describing dynamic behaviour of real accelerometers (on the basis of research from points 1 to 4).

The research results include:

- Algorithms for determining models describing an accelerometer's dynamic behaviour based on differential equations of integer and fractional orders (for the definitions by Grünwald-Letnikov, Riemann-Liouville and Caputo).
- A comparison of amplitude and phase characteristics and responses to typical set input functions between models of integer and fractional orders.
- An assessment of the accuracy of modelling by means of integer and fractional order equations in the context of a real accelerometer behaviour, its parameters and its design (sensitivity, specific structure) and identification of conditions in which modelling dynamic behaviour by means of fractional order equations is more advantageous than integer-order models.
- A generalization of the developed theory concerning accelerometer modelling to models of other objects, the dynamic properties of which are written in the form of a differential equation.

The completed preliminary research results include:

- The determination of simulation models of the accelerometer described by means of differential equations of integer and fractional orders according to Grünwald-Letnikov definition.
- The determination of real accelerometer models described by differential equations of integer and fractional orders for selected frequencies.
- The determination of amplitude characteristics for a real accelerometer and models of integer and fractional orders for selected frequencies.

The results were presented in works: (Luft, Cioć & Pietruszczak, 2011; Luft et al., 2011; 2012; Pietruszczak, 2012; Pietruszczak & Szychta, 2013; Luft, Szychta & Pietruszczak, 2015). Table 1 includes some research results. It presents results of acceleration measurements in the measurement system shown in Figure 3.

Signals received from accelerometers of different sensitivities were compared in the measurement system. Sensitivity of accelerometer *A1*, which was adopted as a model, was ca. 30 times higher than that of the investigated accelerometer *A2*. Equations of integer and fractional orders describing dynamic behaviour of the investigated accelerometer were determined by means of the AutoRegressive with eXogenous input (ARX) identification method on the basis of the data from accelerometers. The signals from determined models were compared to the signal from the model accelerator. The relative errors of measurements were determined by adopting the signal from the model accelerator as a reference value. The median of the series of 500 successive measurement samples was adopted as the error measure. Measurements were taken separately for the following frequencies of the vibration exciter: 100 Hz, 200 Hz, 300 Hz, 400 Hz and 500 Hz.

Table 1. Values of median relative error for the transducer's model of integer and fractional order

Frequency [Hz]	Median relative error for the integer order model [%]	Median relative error for the fractional order model [%]	Difference [%]
100	30.8089	20.8040	10.0049
200	30.2997	20.8041	9.4956
300	29.5564	20.8042	8.7522
400	28.3097	20.8039	7.5058
500	26.0184	20.8040	5.2144

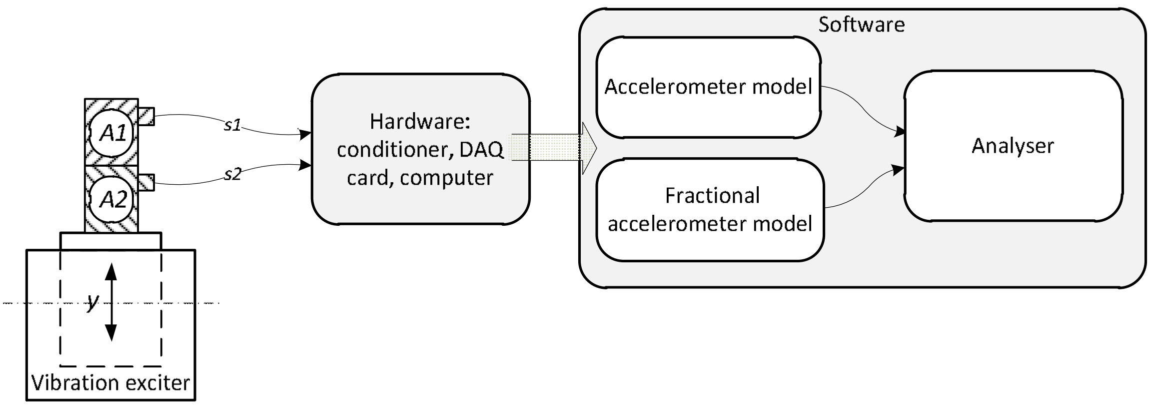

Figure 3. Measurement system (Luft, Szychta & Pietruszczak, 2015)

On the basis of preliminary investigations it was found out that in the examined cases the models described by means of fractional order equations convey the accelerometer processing characteristics more accurately than integer order equations. Depending on examined frequencies the accuracy of reproduced dynamic behaviour of an accelerometer by a model is between ca. 5% to ca. 10%. These values can be increased if we developed a more accurate model of fractional orders. Table 1 presents results of laboratory tests. Theoretical and simulation tests are included in our work. Conclusions from this research are compatible with conclusions from the laboratory tests.

Research methodology

The methodology of the first stage of research into verification of models describing dynamic behaviour of accelerometers is included:

1. The determination of a mathematical model (based on differential equations of integer orders) available for investigations of a real accelerometer. The mathematical apparatus comprises the identification of a differential equation describing dynamic behaviour of the accelerometer on the basis of data from a model (reference) accelerometer and the investigated one (the basic identification method is the ARX method) (Luft, Cioć & Pietruszczak, 2011).
2. The verification of the correctness of the accelerometer performance model obtained in point 1 via a comparison of its performance with the performance of the real accelerometer over the entire range of its working frequencies. This analysis includes amplitude and phase characteristics of the real accelerometer and the model obtained in point 1. It is assumed that the model is correct when deviations of model characteristics from those of the real accelerometer do not exceed 2% at any measurement point.
3. The determination of a mathematical model of the accelerometer from point 1 based on fractional calculus on the basis of the model from the preliminary research. The fractional order model of the accelerometer was based on Grünwald-Letnikov definition (4).
4. The comparison of performances of models from points 1 and 3 over the entire working frequency range of the accelerometer. The analysis includes amplitude and phase characteristics and the models' responses to set input functions of different courses and frequencies.
5. The comparison of performances of models from points 1 and 3 to the performance of the real accelerometer (investigations over the entire range of working frequencies of the accelerometer). Selected amplitude and phase characteristics of the real accelerometer determined in a laboratory as well as the accelerometer's responses to set input functions of different courses and frequencies will be compared with the responses of the models obtained in points 1 and 3.

The research from the first stage is to be repeated at the second stage for:

1. Different types of accelerometers (capacitive, piezoelectric, MEMS, compression and shear designs) and of different sensitivities.
2. Models based on Reimann-Liouville and Caputo's definitions of fractional derivative.

A wide scope of research in the field of different types of accelerometers aims to make the obtained results of the accelerometers' dynamic behaviour model independent, independent of the specific properties of one type of accelerometers. It will allow us to check effectiveness of dynamic behaviour modelling over a wide spectrum of cases and to grasp characteristic parameters of a model for concrete constructional solutions. Research into new models describing dynamic behaviour of objects (here – accelerometers) is connected with adopting a reference model (reference signal, reference characteristics). In the case when different models of accelerometers are compared with each other, the best reference is the amplitude and phase characteristics of the model (reference) accelerometer of very high sensitivity. As an industry standard, the accelerometer manufacturer supply only basic data concerning the accelerometer's sensitivity and its amplitude and phase characteristics in a form of a diagram showing only a linear part of processing. This limited data are insufficient when new models of dynamic behaviour, which have never been described in literature, are investigated. To avoid any doubts concerning accuracy and methodology of measurements, for the sake of comparison one must know accurate amplitude and phase characteristics of the accelerometer. It is also necessary to develop such characteristics with the use of equipment of appropriate accuracy and measurement methods adopted as a standard. In the case of accelerometers the measuring equipment must meet the standards concerning the measurement process ISO 16063-21, ISO-16063-11 and data processing – ISO-17025. All these standards are met by a complete calibration system produced by Bruel & Kjear together with Pulse software.

Conclusions

On the basis of the preliminary simulation and laboratory tests that have already been carried out, it can be concluded that these results support the continuation of this research. In order to fully confirm the preliminary research results it is necessary to generalize the results obtained over the entire processing range of the accelerometer (possibly the widest range of frequencies) and for different types of accelerometers revealing different sensitivities. As it is pioneering research, measurements must be taken with the highest accuracy possible to reduce errors resulting from inaccurate calibration. Equipment of high measurement accuracy, usually used for calibration of accelerometers is best for this purpose.

At the moment it is difficult to find a direct practical application of the knowledge we are going to gain during the course of these investigations. A common application of the theory of fractional order differential equations in the practice of modelling dynamic properties of real objects requires verification through numerous investigations of different objects. At the moment this kind of research is very limited. The proposed project, however, corresponds to the research into modelling of measuring sensors. In the future, the research results could turn out to be significant in the dissemination of this method of modelling for the description of dynamic properties of phenomena and objects.

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