Opuscula Math. 43, no. 5 (2023), 621–632
https://doi.org/10.7494/OpMath.2023.43.5.621 Opuscula Mathematica

A VIABILITY RESULT FOR CARATHÉODORY NON-CONVEX DIFFERENTIAL INCLUSION IN BANACH SPACES

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Communicated by P.A. Cojuhari

Abstract. This paper deals with the existence of solutions to the following differential inclusion: $\dot{x}(t) \in F(t, x(t))$ a.e. on $[0, T[$ and $x(t) \in K$, for all $t \in [0, T]$, where $F : [0, T] \times K \to 2^E$ is a Carathéodory multifunction and K is a closed subset of a separable Banach space E.

Keywords: viability, measurable multifunction, selection, Carathéodory multifunction.

Mathematics Subject Classification: 34A60, 28B20.

1. INTRODUCTION

Let E be a separable Banach space, K a nonempty closed subset of E, T a strictly positive real and put $I := [0, T]$. Let $F : I \times K \to 2^E$ be a multifunction measurable with respect to the first argument and uniformly continuous with respect to the second argument.

The aim of this work is to establish, for any fixed $x_0 \in K$, the existence of an absolutely continuous function $x(\cdot) : I \to K$ satisfying

$$\begin{cases} \dot{x}(t) \in F(t, x(t)) & \text{a.e. on } [0, T[, \\ x(0) = x_0, \\ x(t) \in K & \text{for all } t \in I. \end{cases} \tag{1.1}$$

Concerning this subject, we begin with recalling the pioneering work of Haddad [8], where the right-hand side is an upper semi-continuous convex and compact-valued multifunction $x \to F(x)$ in finite-dimensional space, while in [7] an existence result is established for a globally upper semi-continuous multifunction in Hilbert space, though K is convex.

The main improvement is the comparison with previous results on the same subject especially the work, of Duc Ha [6] which was the basis for several papers; see [1,2,11]. It has been proved the existence of solution to the problem (1.1), where $F(\cdot,x)$ is measurable and $F(t,\cdot)$ is $m(t)$-Lipschitz, $m(\cdot)\in L^1(I,\mathbb{R}^+)$. This result is a multivalued version of Larrieu's work [9]. More precisely, the existence of solutions of (1.1) was given under the following tangency condition:

$$\forall (t,x)\in I\times K\ :\ \liminf_{h\to 0^+}\frac{1}{h}e\Bigg(x+\int\limits_t^{t+h}F(s,x)ds,K\Bigg)=0,$$

where $e(\cdot,\cdot)$ denotes the Hausdorff excess and $\int_t^{t+h}F(s,x)ds$ stands for the Aumann integral of the multifunction $t\to F(t,x)$. Note that the convergence to zero of the above tangency condition depend on the t. Here techniques of existence of selections have been introduced, notably a Lemma given by Zhu [13], that will given another proof in this paper.

Different extensions of the result of Duc Ha [6] have been investigated by many authors in the case of functional differential inclusions or semilinear differential inclusions. See Aitalioubrahim [2], Lupulescu and Necula [10–12] and the references therein.

In current literature, regarding the differential inclusion without Lipschitz condition we refer the reader to the work of Fan and Li [5]. They considered the following differential inclusion:

$$\dot{u}(t)\in A(t)u(t)+F(t,u(t)), \tag{1.2}$$

where $A(t)$ is a family of unbounded linear operators generating an evolution operator and $F(t,\cdot)$ is lower semicontinuous. However $\chi(F(t,D))\le k(t)\chi(D)$ for every bounded subset D, where χ is the measure of noncompactness and $k(\cdot)\in L^1(I,\mathbb{R}^+)$. Dong and Li [4] have established a viable solution to (1.2) when $A(t)=A$ and F is a Carathéodory single-valued map.

In this paper, we consider the existence of solutions to the problem (1.1) in general situation supposing that the right-hand side $(t,x)\to F(t,x)$ is measurable with respect to the first argument and uniformly continuous with respect to the second argument in the sense that

$$\begin{aligned}&\forall\varepsilon>0\,\exists\delta(\varepsilon)>0\,\forall(t,x,y)\in I\times K\times K:\\&\qquad\qquad\|x-y\|\le\delta(\varepsilon)\Rightarrow d_H(F(t,x),F(t,y))\le\varepsilon,\end{aligned}$$

where d_H denotes the Hausdorff distance.

This condition is weaker than the one adopted by Duc Ha [6] in the spatial case when the Lipschitz coefficient $m(t)$ is a constant $L>0$.

The following case deserves mentioning: F is a time-independant continuous multifunction and K is compact. In this case the above hypothesis is satisfied.

Our approach is based on Euler's method, it consists of constructing a sequence of approximate solutions by using Lebesgue's Differentiation Theorem and selection techniques.

2. NOTATIONS, DEFINITIONS AND THE MAIN RESULT

In all paper, E is a separable Banach space with the norm $\|\cdot\|$. For $x \in E$ and $r > 0$, let $B(x,r) := \{y \in E : \|y-x\| < r\}$ be an open ball centered at x with radius r and $\overline{B}(x,r)$ be its closure, and put $B = B(0,1)$. For $x \in E$ and for nonempty bounded subsets A, B of E, we denote by $d_A(x)$ or $d(x,A)$ the real value $\inf\{\|x-y\| : y \in A\}$,

$$e(A,B) := \sup\{d_B(x) : x \in A\} \quad \text{and} \quad d_H(A,B) = \max\{e(A,B), e(B,A)\}.$$

We denote by $\mathcal{L}(I)$ the σ-algebra of Lebesgue measurable subsets of I, and $B(E)$ is the σ-algebra of Borel subsets of E for the strong topology. A multifunction is said to be measurable if its graph belongs to $\mathcal{L}(I) \otimes B(E)$. For more details on measurability theory, we refer the reader to the book by Castaing and Valadier [3].

Let $F : I \times K \to 2^E$ be a multifunction with nonempty closed values in E.

On F we make the following hypotheses:

(H_1) For each $x \in K$, $t \to F(t,x)$ is measurable.

(H_2) For all $t \in I$, $x \to F(t,x)$ is uniformly continuous as follows:

$$\forall \varepsilon > 0\, \exists \delta(\varepsilon) > 0\, \forall (t,x,y) \in I \times K \times K : \\ \|x-y\| \le \delta(\varepsilon) \Rightarrow d_H(F(t,x), F(t,y)) \le \varepsilon.$$

(H_3) There exists $M > 0$, for all $(t,x) \in I \times K$,

$$\|F(t,x)\| := \sup_{z \in F(t,x)} \|z\| \le M.$$

(H_4) For all $t \in I$ and $x \in K$, for every measurable selection $\sigma(\cdot)$ of the multifunction $t \to F(t,x)$

$$\liminf_{h \to 0^+} \frac{1}{h} d_K\left(x + \int_t^{t+h} \sigma(s)ds\right) = 0,$$

which is equivalent to

$$\liminf_{h \to 0^+} \frac{1}{h} e\left(x + \int_t^{t+h} F(s,x)ds, K\right) = 0.$$

Let $x_0 \in K$. Under hypotheses (H_1)–(H_4) we shall prove the following result:

Theorem 2.1. *There exists an absolutely continuous function $x(\cdot) : I \to E$ such that*

$$\begin{cases} \dot{x}(t) \in F(t, x(t)) & \textit{a.e. on } [0,T[, \\ x(0) = x_0, & \\ x(t) \in K, & \textit{for all } t \in I. \end{cases}$$

3. PRELIMINARY RESULTS

To begin with, let us recall the following lemmas that will be used in the sequel.

Lemma 3.1 ([13]). *Let Ω be a nonempty set in E. Let $G : [a,b] \times \Omega \to 2^E$ be a multifunction with nonempty closed values satisfying:*

(i) *for every $x \in \Omega$, $G(\cdot, x)$ is measurable on $[a,b]$,*
(ii) *for every $t \in [a,b]$, $G(t,\cdot)$ is (Hausdorff) continuous on Ω.*

Then for any measurable function $x(\cdot) : [a,b] \to \Omega$ the multifunction $G(\cdot, x(\cdot))$ is measurable on $[a,b]$.

Lemma 3.2 ([3]). *Let $R : I \to 2^E$ be a measurable multifunction with nonempty closed values in E. Then R admits a measurable selection: there exists a measurable function $r : I \to E$ that is $r(t) \in R(t)$ for all $t \in I$.*

We need also the following lemma, due to Zhu [13], established for a multifunction (not necessarily closed values) in Banach spaces (not necessarily separable). However, the result was proven for almost everywhere on I, because the measurability of a multifunction Γ adopted by this author is defined as follows: there exists a sequence $(\sigma_n(\cdot))_{n\in\mathbb{N}}$ of measurable functions that is $\Gamma(t) \subset \overline{\{\sigma_n(t) : n \in \mathbb{N}\}}$ a.e.on I. Here, we are concerned with this Lemma in the context of measurable closed-values multifunction in separable Banach spaces. This result is obtained at every element of I by a different method.

Lemma 3.3. *Let $G : I \to 2^E$ be a measurable multifunction with nonempty closed values and $z(\cdot) : I \to E$ a measurable function. Then for any positive measurable function $r(\cdot) : I \to \mathbb{R}^+$, there exists a measurable selection $g(\cdot)$ of G such that for all $t \in I$,*

$$\|g(t) - z(t)\| \le d(z(t), G(t)) + r(t).$$

Proof. Let $t \in I$. By the characterization of the lower bound, there exists $x \in G(t)$ such that

$$\|x - z(t)\| \le d(z(t), G(t)) + r(t).$$

Consider the following multifunction

$$t \to Q(t) = \big\{x \in E : \|x - z(t)\| \le d(z(t), G(t)) + r(t)\big\}.$$

Obviously, Q is measurable with nonempty closed values. On the other hand, since G is measurable with closed values, then $Q(t) \cap G(t)$ is a measurable multifunction with nonempty closed values, hence by Lemma 3.2, admits a measurable selection $t \to g(t)$. This completes the proof. □

4. PROOF OF THE MAIN RESULT

The proof is based on two steps. It consists of the construction of a sequence of approximants in the first one, while in the second step we establish the convergence of such approximate solutions.

Step 1. Construction of approximants.

For each integer $n > \max(T;1)$, put $\tau_n := \frac{T}{n}$ and consider the following partition of the interval I with the points

$$t_i^n = i\tau_n, \quad i = 0, 1, \dots, n.$$

Remark that $I = \bigcup_{i=0}^{n-1} [t_i^n, t_{i+1}^n]$. Since $t \to F(t, x_0)$ is measurable with closed values, then by Lemma 3.2, there exists a measurable function $f_0(\cdot)$ such that for all $f_0(t) \in F(t, x_0)$. Note that by (H_3), $f_0(\cdot) \in L^1(I, E)$.

For all $n \in \mathbb{N}^*$, put $f_0^n(\cdot) = f_0(\cdot)$. We shall prove the following theorem:

Theorem 4.1. *For all $n \in \mathbb{N}^*$, there exist $\varphi_0(n) \in \mathbb{N}^*$, $x_1^n \in K$, $u_0^n(\cdot)$, $f_1^n(\cdot) \in L^1(I, E)$ such that for all $t \in I$,*

$$f_1^n(t) \in F(t, x_1^n), \quad \|f_1^n(t) - f_0^n(t)\| \le \frac{1}{2^{n+1}},$$

and for almost every $t \in I$,

$$u_0^n(t) \in F(t, x_0) + \frac{1}{2^n}\overline{B}, \quad \|u_0^n(t) - f_0(t)\| \le \frac{1}{2^n},$$

and

$$x_1^n = x_0 + \tau_{\varphi_0(n)} u_0^n(0) \in K.$$

Proof. By (H_4), for all $t \in [0, T[$,

$$\liminf_{n \to +\infty} \frac{1}{\tau_n} d_K\left(x_0 + \int_t^{t+\tau_n} f_0(s)ds \right) = 0.$$

Then for all $t \in [0, T[$, there exists an integer $\varphi_t(n) > n$ such that

$$\frac{1}{\tau_{\varphi_t(n)}} d_K\left(x_0 + \int_t^{t+\tau_{\varphi_t(n)}} f_0(s)ds \right) \le \frac{1}{2^{n+2}}.$$

Hence, by the characterization of the lower bound, there exists $x_1(t) \in K$ such that

$$\frac{1}{\tau_{\varphi_t(n)}} \left\| x_1(t) - x_0 - \int_t^{t+\tau_{\varphi_t(n)}} f_0(s)ds \right\| \le \frac{\tau_{\varphi_t(n)}}{2^{n+2}} + \frac{1}{2^{n+2}}.$$

Then

$$\left\| \frac{x_1(t) - x_0}{\tau_{\varphi_t(n)}} - \frac{1}{\tau_{\varphi_t(n)}} \int_t^{t+\tau_{\varphi_t(n)}} f_0(s)ds \right\| \le \frac{1}{2^{n+1}}.$$

On the other hand, in view of Lebesgue's Differentiation Theorem, we can suppose

$$\left\| \frac{1}{\tau_{\varphi_t(n)}} \int_t^{t+\tau_{\varphi_t(n)}} f_0(s)ds - f_0(t) \right\| \le \frac{1}{2^{n+1}} \quad \text{a.e. on } I.$$

Therefore,

$$\left\| \frac{x_1(t) - x_0}{\tau_{\varphi_t(n)}} - f_0(t) \right\| \le \frac{1}{2^n} \quad \text{a.e. on } I.$$

Set

$$u_0^n(t) = \frac{x_1(t) - x_0}{\tau_{\varphi_t(n)}}.$$

Then for all $t \in [0, T[$,

$$x_1(t) = x_0 + \tau_{\varphi_t(n)} u_0^n(t) \in K,$$

and

$$\|u_0^n(t) - f_0(t)\| \le \frac{1}{2^n} \quad \text{a.e. on } I$$

from which we deduce that

$$u_0^n(t) \in F(t, x_0) + \frac{1}{2^n}\overline{B}.$$

Particularly

$$x_0 + \tau_{\varphi_t(n)} u_0^n(t) \in K, \quad \text{for all } t \in [t_0^n, t_1^n],$$

and

$$u_0^n(t) \in F(t, x_0) + \frac{1}{2^n}\overline{B} \quad \text{a.e. on } [t_0^n, t_1^n[.$$

Let $\delta_n = \delta(\frac{1}{2^{n+2}})$ be the real given by (H_2). Choose $\varphi_0(n) > \frac{T(M+1)}{\delta_n}$, and set

$$x_1^n := x_1(t_0^n) = x_0 + \tau_{\varphi_0(n)} u_0^n(0) \in K.$$

Since

$$\|x_1^n - x_0\| = \frac{T}{\varphi_0(n)} \|u_0^n(0)\| \le \frac{T}{\varphi_0(n)}(M+1) \le \delta_n,$$

then, by (H_2),

$$d_H(F(t, x_1^n), F(t, x_0)) \le \frac{1}{2^{n+2}}, \quad \text{for all } t \in I,$$

thus

$$d(f_0(t), F(t, x_1^n)) \le \frac{1}{2^{n+2}}, \quad \text{for all } t \in I.$$

In view of Lemma 3.3, there exists a measurable function $f_1^n(\cdot) \in L^1(I, E)$ such that $f_1^n(t) \in F(t, x_1^n)$ and for all $t \in I$,

$$\|f_1^n(t) - f_0(t)\| \leq d(f_0(t), F(t, x_1^n) + \frac{1}{2^{n+2}} \leq \frac{1}{2^{n+1}}. \qquad \square$$

By induction, for $p \in \{2, \dots, n\}$, assume that have been constructed $\varphi_{_{p-2}}(n) \in \mathbb{N}^*$, $x_{p-1}^n \in K$, $f_{p-1}^n(t) \in F(t, x_{p-1}^n)$ and $u_{p-2}^n(\cdot)$, satisfying the following relations:

$$u_{p-2}^n(t) \in F(t, x_{p-2}^n) + \frac{1}{2^n}\overline{B} \quad \text{a.e. on } [t_{p-2}^n, t_{p-1}^n[,$$

$$\|u_{p-2}^n(t) - f_{p-2}^n(t)\| \leq \frac{1}{2^n} \quad \text{a.e. on } [t_{p-2}^n, t_{p-1}^n[,$$

$$x_{p-1}^n := x_p(t_{p-2}^n) = x_{p-2}^n + \tau_{\varphi_{_{p-2}(n)}} u_{p-2}^n(t_{p-2}^n) \in K,$$

and

$$\|f_{p-1}^n(t) - f_{p-2}^n(t)\| \leq \frac{1}{2^{n+1}}, \quad \text{for all } t \in I.$$

Let us define x_p^n, $f_p^n(\cdot)$, $u_{p-1}^n(\cdot)$ and $\varphi_{_{p-1}}(n)$, that is, $\varphi_{_{p-1}}(n) > \varphi_{_{p-2}}(n)$. Indeed, for all $t \in I$, $f_{p-1}^n(t) \in F(t, x_{_{p-1}}^n)$. Then, by (H_4),

$$\liminf_{n \to +\infty} \frac{1}{\tau_n} d_K\left(x_{p-1}^n + \int_t^{t+\tau_n} f_{p-1}^n(s)ds\right) = 0, \quad \text{for all } t \in [0, T[.$$

Then for all $t \in [0, T[$, there exists $\varphi_{_t}^{p-1}(n) \in \mathbb{N}$ such that $\varphi_{_t}^{p-1}(n) > \varphi_{_t}^{p-2}(n)$,

$$\frac{1}{\tau_{\varphi_t^{p-1}(n)}} d_K\left(x_{p-1}^n + \int_t^{t+\tau_{\varphi_t^{p-1}(n)}} f_{p-1}^n(s)ds\right) \leq \frac{1}{2^{n+2}},$$

Hence, by the characterization of the lower bound, there exists $x_p(t) \in K$ such that

$$\frac{1}{\tau_{\varphi_t^{p-1}(n)}} \left\| x_p(t) - x_{p-1}^n - \int_t^{t+\tau_{\varphi_t^{p-1}(n)}} f_{p-1}^n(s)ds \right\| \leq \frac{\tau_{\varphi_t^{p-1}(n)}}{2^{n+2}} + \frac{1}{2^{n+2}}.$$

Then

$$\left\| \frac{x_p(t) - x_{p-1}^n}{\tau_{\varphi_t^{p-1}(n)}} - \frac{1}{\tau_{_{\varphi_t^{p-1}(n)}}} \int_t^{t+\tau_{\varphi_t^{p-1}(n)}} f_{p-1}^n(s)ds \right\| \leq \frac{1}{2^{n+1}}.$$

On the other hand, in view of Lebesgue's Differentiation Theorem, we can suppose

$$\left\| \frac{1}{\tau_{\varphi_t^{p-1}(n)}} \int_t^{t+\tau_{\varphi_t^{p-1}(n)}} f_{p-1}^n(s)ds - f_{p-1}^n(t) \right\| \leq \frac{1}{2^{n+1}} \quad \text{a.e. on } I.$$

Therefore,

$$\left\| \frac{x_p(t) - x_{p-1}^n}{\tau_{\varphi_t^{p-1}(n)}} - f_{p-1}^n(t) \right\| \leq \frac{1}{2^n} \quad \text{a.e. on } I.$$

Set

$$u_{p-1}^n(t) = \frac{x_p(t) - x_{p-1}^n}{\tau_{\varphi_t^{p-1}(n)}},$$

then for all $t \in [0, T[$

$$x_p(t) = x_{p-1}^n + \tau_{\varphi_t^{p-1}(n)} u_{p-1}^n(t) \in K,$$

and

$$\|u_{p-1}^n(t) - f_{p-1}^n(t)\| \leq \frac{1}{2^n} \quad \text{a.e. on } I,$$

from which, we get

$$u_{p-1}^n(t) \in F(t, x_{p-1}^n) + \frac{1}{2^n}\overline{B}.$$

Then we have

$$x_{p-1}^n + \tau_{\varphi_t^{p-1}(n)} u_{p-1}^n(t) \in K, \quad \text{for all } t \in [t_{p-1}^n, t_p^n],$$

and

$$u_{p-1}^n(t) \in F(t, x_{p-1}^n) + \frac{1}{2^n}\overline{B} \quad \text{a.e. on } [t_{p-1}^n, t_p^n[.$$

Choose $\varphi_{p-1}(n) > \max(\varphi_{t_{p-1}^n}^{p-1}(n); \varphi_{p-2}(n))$. Then $\varphi_{p-1}(n) > \frac{T(M+1)}{\delta_n}$. We set

$$x_p^n := x_p(t_{p-1}^n) = x_{p-1}^n + \tau_{\varphi_{p-1}(n)} u_{p-1}^n(t_{p-1}^n) \in K.$$

Then

$$\|x_p^n - x_{p-1}^n\| = \frac{T}{\varphi_{p-1}(n)} \|u_{p-1}^n(t_{p-1}^n)\| \leq \frac{T}{\varphi_{p-1}(n)}(M+1) \leq \delta_n,$$

hence, by (H$_2$),

$$d_H(F(t, x_p^n), F(t, x_{p-1}^n)) \leq \frac{1}{2^{n+2}}, \quad \text{for all } t \in I.$$

By Lemma 3.3, there exists a measurable function $f_p^n(\cdot) \in L^1(I,E)$ such that $f_p^n(t) \in F(t,x_p^n)$ and for all $t \in I$,

$$\|f_p^n(t) - f_{p-1}^n(t)\| \leq d(f_{p-1}^n(t), F(t,x_p^n)) + \frac{1}{2^{n+2}}.$$

Then

$$\|f_p^n(t) - f_{p-1}^n(t)\| \leq \frac{1}{2^{n+1}}. \tag{4.1}$$

Put $k_n = \varphi_n(n)$. Remark that the previous properties are satisfied for k_n.

Now, let us define the step functions.

For all $n \geq 1$, for all $p = 1,2,\ldots,n$, for all $t \in [0,T[$, set $\theta_n(t) = t_{p-1}^n$, whenever $t \in [t_{p-1}^n, t_p^n[$, $f_n(t) = \sum_{p=1}^{n} \chi_{[t_{p-1}^n, t_p^n]}(t) f_{p-1}^n(t)$ and $u_n(t) = \sum_{p=1}^{n} \chi_{[t_{p-1}^n, t_p^n]}(t) u_{p-1}^n(t)$.

On each interval $[t_{p-1}^n, t_p^n]$ consider

$$x_n(t) = x_{p-1}^n + \int_{t_{p-1}^n}^{t} u_{p-1}^n(s)ds.$$

Then

$$\begin{cases} x_n(\theta_n(t)) = x_{p-1}^n \in K, & \text{for all } t \in [0,T[, \\ \dot{x}_n(t) = u_n(t) \in F(t, x_n(\theta_{k_n}(t))) + \frac{1}{2^n}\overline{B} & \text{a.e. on } I, \\ \|u_n(t) - f_n(t)\| \leq \frac{1}{2^n} & \text{a.e. on } I. \end{cases}$$

Step 2. The convergence of $(x_n(\cdot))$

By construction for all $t \in I$,

$$f_n(t) \in F(t, x_n(\theta_{k_n}(t))).$$

On the other hand, let $t \in I$ and $p = 1,2,\ldots,n$, by relation (4.1),

$$\|f_p^n(t) - f_{p-1}^n(t)\| \leq \frac{1}{2^{n+1}}.$$

Then, by induction,

$$\|f_p^n(t) - f_0(t)\| \leq \frac{p}{2^{n+1}},$$

which implies

$$\|f_n(t) - f_0(t)\| \leq \frac{n}{2^{n+1}}.$$

Then

$$\begin{aligned} \|f_{n+1}(t) - f_n(t)\| &\leq \|f_{n+1}(t) - f_0(t)\| + \|f_n(t) - f_0(t)\| \\ &\leq \frac{n+1}{2^{n+2}} + \frac{n}{2^{n+1}} \leq \frac{3(n+1)}{2^{n+2}}. \end{aligned}$$

Let $t \in I$ and $(m,n) \in \mathbb{N}^* \times \mathbb{N}^*$ with $m > n$. Then

$$\begin{aligned}\|f_m(t) - f_n(t)\| &\le \|f_m(t) - f_{m-1}(t)\| + \|f_{m-1}(t) - f_{m-2}(t)\| \dots \|f_{n+1}(t) - f_n(t)\| \\ &\le \frac{3m}{2^{m+1}} + \frac{3(m-1)}{2^m} + \dots + \frac{3(n+1)}{2^{n+2}} \\ &\le \frac{3}{2}\Big(\frac{m}{2^m} + \frac{m-1}{2^{m-1}} + \dots + \frac{n+1}{2^{n+1}}\Big).\end{aligned}$$

Put $v_n = \frac{n}{2^n}$. Then according to a classical argument (the d'Alembert criterion), the numerical series $\sum_{i=0}^{+\infty} v_i$ converges, hence $(S_n) = (\sum_{i=0}^{n} v_i)$ is a Cauchy sequence.

Since

$$\|f_m(t) - f_n(t)\| \le S_m - S_n,$$

then $(f_n(\cdot))_{n\ge 1}$ is a Cauchy sequence in $L^1(I,E)$. We denote by $f(\cdot)$ its limit.

Moreover, by relations

$$x_n(t) = x_0 + \int_0^t u_n(s)ds$$

and

$$\|u_n(t) - f_n(t)\| \le \frac{1}{2^n}, \quad \text{a.e. on } I,$$

it follows that the subsequence $(x_n(\cdot))_n$ converges almost everywhere on I to an absolutely continuous function, namely $x(\cdot)$.

Recall that

$$|\theta_{k_n}(t)) - t| < \frac{T}{n}$$

for all $n \ge 1$. Since

$$\begin{aligned}\|x_n(\theta_{k_n}(t)) - x(t)\| &\le \|x_n(\theta_{k_n}(t)) - x_n(t)\| + \|x_n(t) - x(t)\| \\ &\le \int_{\theta_{k_n}(t)}^t (M+1)ds + \|x_n(t) - x(t)\|,\end{aligned}$$

then

$$\lim_{n\to\infty} x_n(\theta_{k_n}(t)) = x(t), \quad \text{for all } t \in [0,T[.$$

Hence, by dominated convergence theorem, for all $t \in I$

$$x(t) = \lim_{n\to\infty} x_n(t) = \lim_{n\to\infty}\Big(x_0 + \int_0^t u_n(s)ds\Big) = x_0 + \int_0^t f(s)ds,$$

so $f(t) = \dot{x}(t)$ a.e. on I.

In addition, for every $t \in [0,T[$ we have $x_n(\theta_{k_n}(t)) \in K$. Since K is closed, then $x(t) \in K$. Moreover, as $x(\cdot)$ is $(M+1)$-Lipschitz, then $x(t) \in K$, for all $t \in [0,T]$.

Furthermore, observe that

$$d(f(t), F(t, x(t)) \leq \|f(t) - f_n(t)\| + d_H(F(t, x_n(\theta_{k_n}(t))), F(t, x(t))),$$

since $(f_n(\cdot))$ converges to $f(\cdot)$ a.e. on $[0, T[$ and $x \to F(t, x)$ is continuous, then $\dot{x}(t) = f(t) \in F(t, x(t))$ for a.e. $t \in I$. This completes the proof of Theorem 2.1.

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Received: March 17, 2023.
Revised: April 24, 2023.
Accepted: May 14, 2023.