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EXACT SOLUTION FOR HEAT CONDUCTION INSIDE A SPHERE WITH HEAT ABSORPTION USING THE REGULARIZED HILFER-PRABHAKAR DERIVATIVE

Hager Elhadedy[1], Mohamed S. Abdel Latif[2,1], Hamed M. Nour[1], Abbas H. Abdel Kader[1]

*[1] Mathematics and Engineering Physics Department, Engineering Faculty
Mansoura University, Mansoura, Egypt*
*[2] Department of Mathematics, Faculty of Science, New Mansoura University
New Mansoura City, Egypt*
m_gazia@mans.edu.eg

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Abstract. In this article, we utilize the finite Sine-Fourier transform and the Laplace transform for solving fractional partial differential equations with regularized Hilfer-Prabhakar derivative. These transforms are used to get analytical solutions for the time fractional heat conduction equation (TFHCE) with the regularized Hilfer-Prabhakar derivative associated with heat absorption in spherical coordinates. Two cases of Dirichlet boundary conditions are considered by obtaining an analytical solution in each case. The effect of the parameters of the regularized Hilfer-Prabhakar derivative on the heat transfer inside the sphere is discussed using some figures.

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1. Introduction

Nowadays, the fractional calculus is considered as an important subfield of mathematics, it is widely used in modeling many physical and engineering phenomena [1-6]. More attention has been placed on the research of fractional differential equations (FDEs) because they can model many physical phenomena more accurately than the classical integer-order DEs [3-9]. FDEs are widely used in modeling many real-world physical processes, such as the anomalous diffusion process [10], the continuous time random walk problem [10], non-exponential relaxation processes [11], and the generalized Langevin problem [12].

Many definitions for the fractional derivatives operators with singular and nonsingular kernels are proposed in the literature, these definitions include Riemann--Liouville operator [13], Caputo operator [13], Hadamard operator [13], Atangana-

-Baleanu operator [6], and the regularized Hilfer-Prabhakar derivative [14-17]. Many methods have been used for finding exact solutions of fractional partial differential equations (FPDEs) [18-21]. The integral transform methods are widely used in solving linear FPDEs. Some examples of these integral transforms are the Fourier transform, the Sine-Fourier transform, the finite Sine-Fourier transform and the Laplace transform [22, 23].

Fourier's law is a fundamental law in the classical heat transfer theory which gives the parabolic partial differential equation of the heat conduction. As a result of the Fourier's law, the speed of heat flow in the medium is impractical. A generalization of the Fourier's law that results in a fractional heat conduction equation can be utilized to avoid this problem [22]. These fractional heat conduction equations are investigated in many papers [9, 18, 22]. When considering heat transfer in a bounded medium, the heat equation will be associated with some boundary conditions. The boundary conditions of Dirichlet, Neumann and Robin are typically used in heat transfer problems [22].

In this article, we use the finite Sine-Fourier transform and Laplace transform to solve the TFHCE with a heat absorption term in spherical coordinates in the case of central symmetry [22] which is given by

$$ {}^{C}D^{\gamma,\mu}_{\rho,\omega,0^+}T(r,t) = a\left(\frac{\partial^2 T(r,t)}{\partial r^2} + \frac{2}{r}\frac{\partial T(r,t)}{\partial r}\right) - bT(r,t),\ \ t>0,\ \ 0 \le r < R, \tag{1} $$

where T is the temperature, t is the time, $a > 0$ is the coefficient of thermal diffusivity, b denotes the heat absorption and ${}^{C}D^{\gamma,\mu}_{\rho,\omega,0^+}T(r,t)$ is the regularized Hilfer-Prabhakar derivative of order μ defined as [15]

$$ {}^{C}D^{\gamma,\mu}_{\rho,\omega,0^+}f(t) = \int_0^t (t-y)^{-\mu} E^{-\gamma}_{\rho,1-\mu}(\omega(t-y)^{\rho}) f'(y)dy, \tag{2} $$

where $\gamma, \omega \in R$, $\rho > 0$, $\mu \in (0,1)$ and $E^{\gamma}_{\rho,\mu}(z)$ is the three-parameter Mittag-Leffler function which is given by [15]

$$ E^{\gamma}_{\rho,\mu}(z) = \sum_{k=0}^{\infty} \frac{\Gamma(\gamma+k)}{\Gamma(\gamma)\Gamma(\rho k+\mu)} \frac{z^k}{k!}. \tag{3} $$

This Mittag-Leffler function (Eq. (3)) has the following property [16]:

$$ E^{0}_{\rho,\mu}(z) = E^{\gamma}_{\rho,\mu}(0) = \frac{1}{\Gamma(\mu)}. \tag{4} $$

The Caputo derivative can be obtained as a special case of the regularized Hilfer-Prabhakar derivative (2) [17] in the following two cases:

Case 1: when $\gamma = 0$, in this case Eq. (2) will be transformed into (using property (4))

$$ {}^{C}D^{0,\mu}_{\rho,\omega,0^+}f(t) = \frac{1}{\Gamma(1-\mu)}\int_0^t (t-y)^{-\mu} f'(y)dy = {}^{C}D^{\mu}_{0}f(t), \tag{5} $$

where ${}^{C}D^{\mu}_{0}f(t)$ is Caputo fractional derivative.

Case 2: when $\omega = 0$, in this case Eq. (2) will be transformed into (using property (4))

$$ {}^{C}D^{\gamma,\mu}_{\rho,0,0^+}f(t) = \frac{1}{\Gamma(1-\mu)}\int_0^t (t-y)^{-\mu} f'(y)dy = {}^{C}D^{\mu}_{0}f(t). \tag{6} $$

So, the regularized Hilfer-Prabhakar derivative (2) can be considered as a generalized derivative. Using the regularized Hilfer-Prabhakar derivative in investigating physical and engineering phenomena enables us to obtain generalized solutions with many arbitrary parameters $(\gamma, \mu, \rho, \omega)$ [16, 17, 24, 25].

Equation (1) will be considered under the following conditions:

1. The value of the temperature T along the radial axis r at $t = 0$ (initial condition) is given by

$$ T(r, 0) = 0, \tag{7} $$

2. The value of the temperature T at $r = R$ for any time (Dirichlet boundary conditions) will be given in two cases

Case 1

$$ T(R, t) = p\delta(t), \tag{8} $$

where $\delta(t)$ is the impulse function, whereas p is an arbitrary constant.

Case 2

$$ T(R, t) = pt^{\beta}, \tag{9} $$

where β is a positive constant.

The regularized Hilfer-Prabhakar derivatives (2) naturally come from physical models such as in dielectric relaxation phenomena [26, 27], in the fractional Poisson process [15], in fractional diffusion equation [14], in linear viscoelasticity [17, 28], in modeling filtration dynamics [29], as well as in the generalized Langevin equation [30].

The time-fractional diffusion equation (1) describes many important physical phenomena in dielectrics, semiconductors, biological systems, polymers, amorphous, colloids, porous and disordered media. Equation (1) is a result of the time-nonlocal generalization of the Fourier law associated with the “long-tail” power kernel [22]. This generalization can be interpreted in terms of non-integer order derivatives and integrals. Equation (1) takes into consideration the memory effects with respect to time.

2. Basic definitions

Lemma 1

1. Following [15], the Laplace transform of Hilfer-Prabhakar derivative (2) is given by

$$L\{{}^{C}D^{\gamma,\mu}_{\rho,\omega,0^+}f(t)\} = s^{\mu}(1-\omega s^{-\rho})^{\gamma}L\{f(t)\} - s^{\mu-1}(1-\omega s^{-\rho})^{\gamma}f(0^+). \quad (10)$$

2. Following [15], the Laplace transform of $t^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega t^{\rho})$ is given by

$$L\{t^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega t^{\rho})\} = \frac{s^{\rho\gamma-\mu}}{(s^{\rho}-\omega)^{\gamma}}. \quad (11)$$

Lemma 2 [22] The finite Sine-Fourier transform applied to the coordinate r in the domain $r \in [0, R]$ is given by

$$\mathcal{F}\{f(r)\} = \tilde{f}(\xi_k) = \int_0^R rf(r)\frac{\sin(r\xi_k)}{\xi_k}dr, \quad (12)$$

with

$$\xi_k = \frac{k\pi}{R}. \quad (13)$$

Following [22], we get

$$\mathcal{F}\left\{\frac{d^2}{dr^2}f(r) + \frac{2}{r}\frac{d}{dr}f(r)\right\} = -\xi_k^2\tilde{f}(\xi_k) + (-1)^{k+1}Rf(R). \quad (14)$$

Lemma 3 [22] The inverse of the finite sin-Fourier transform can be expressed as

$$\mathcal{F}^{-1}\{\tilde{f}(\xi_k)\} = f(r) = \frac{2}{R}\sum_{k=1}^{\infty}\xi_k\tilde{f}(\xi_k)\frac{\sin(r\xi_k)}{r}. \quad (15)$$

3. Exact solution of the fractional heat conduction problem

In this section we derive the exact solution of Eq. (1) in the two cases of the boundary conditions given by Eq. (8) and Eq. (9).
Performing the Laplace transform to Eq. (1) and utilizing Eq. (10), we obtain

$$\begin{aligned} s^{\mu}(1-\omega s^{-\rho})^{\gamma}\tilde{T}(r,s) - s^{\mu-1}(1-\omega s^{-\rho})^{\gamma}T(r,0) \\ = a\left(\frac{\partial^2}{\partial r^2}\tilde{T}(r,s) + \frac{2}{r}\frac{\partial}{\partial r}\tilde{T}(r,s)\right) - b\tilde{T}(r,s). \end{aligned} \quad (16)$$

Using the initial condition (7), Eq. (16) becomes

$$s^{\mu}(1-\omega s^{-\rho})^{\gamma}\tilde{T}(r,s)=a\left(\frac{\partial^2}{\partial r^2}\tilde{T}(r,s)+\frac{2}{r}\frac{\partial}{\partial r}\tilde{T}(r,s)\right)-b\tilde{T}(r,s). \quad (17)$$

Applying the finite Sine-Fourier transform to Eq. (17) and using Eq. (14), we get

$$s^{\mu}(1-\omega s^{-\rho})^{\gamma}\tilde{T}^{*}(\xi_k,s)=a\left[-\xi_k^2\,\tilde{T}^{*}(\xi_k,s)+(-1)^{k+1}R\tilde{T}(R,s)\right]-b\tilde{T}^{*}(\xi_k,s). \quad (18)$$

Now Eq. (18) will be investigated in the two cases as follows:

Case 1 Taking Laplace transform of the boundary condition (8) gives

$$\tilde{T}(R,s)=p. \quad (19)$$

Substituting Eq. (19) into Eq. (18) gives

$$\begin{aligned}\tilde{T}^{*}(\xi_k,s)&=\frac{a(-1)^{k+1}Rp}{s^{\mu}(1-\omega s^{-\rho})^{\gamma}+a\xi_k^2+b}\\&=\frac{a(-1)^{k+1}Rps^{\rho\gamma-\mu}}{(s^{\rho}-\omega)^{\gamma}}\left(\frac{1}{1+(a\xi_k^2+b)s^{\rho\gamma-\mu}(s^{\rho}-\omega)^{-\gamma}}\right)\\&=a(-1)^{k+1}Rp\sum_{n=0}^{\infty}(-1)^n(a\xi_k^2+b)^n\frac{s^{(n+1)(\rho\gamma-\mu)}}{(s^{\rho}-\omega)^{(n+1)\gamma}}.\end{aligned} \quad (20)$$

After using Eq. (11), the inverse Laplace transform of Eq. (20) is

$$T^{*}(\xi_k,t)=a(-1)^{k+1}Rp\sum_{n=0}^{\infty}(-1)^n(a\xi_k^2+b)^n t^{\mu(n+1)-1}E_{\rho,(n+1)\mu}^{(n+1)\gamma}(\omega t^{\rho}). \quad (21)$$

The inverse sin-Fourier transform of Eq. (21) gives the following solution of Eq. (1)

$$T(r,t)=\frac{2ap}{r}\sum_{k=1}^{\infty}\xi_k\sum_{n=0}^{\infty}(-1)^{n+k+1}\left(a\xi_k^2+b\right)^n t^{\mu(n+1)-1}E_{\rho,(n+1)\mu}^{(n+1)\gamma}(\omega t^{\rho})\sin(r\xi_k). \quad (22)$$

To study the asymptotic behavior of the solution (22) at large t, we use the relation [16]

$$E_{\alpha,\beta}^{\gamma}(z)\sim\frac{(-z)^{\gamma}}{\Gamma(\beta-\alpha\gamma)},\quad z\gg 1. \quad (23)$$

Using Eq. (23), solution (22) takes the form

$$T(r,t)=\frac{2ap}{r}\sum_{k=1}^{\infty}(-1)^{k+1}\xi_k\sum_{n=0}^{\infty}(-1)^{n-\gamma}\,t^{\mu-\gamma\rho-1}E_{\mu-\gamma\rho,\mu-\gamma\rho}(\psi)\sin(r\xi_k), \quad (24)$$

where, $\psi=-\frac{a\xi_k^2+b}{\omega^\gamma}t^{\mu-\gamma\rho}$. The two-parameter Mittag-Leffler function appears in Eq. (24) converges to zero as $t\to\infty$ [16]. So, solution (24) converges to zero as $t\to\infty$. From Figure 2, we can also realize the convergence of Solution (22) to zero as $t\to\infty$.

Remark 1: When $\gamma=0$ or $\omega=0$, the solution (22) will be transformed into

$$T(r,t)=\frac{2apt^{\mu-1}}{r}\sum_{k=1}^{\infty}\xi_k\,(-1)^{k+1}E_{\mu,\mu}\left(-\left(a\xi_k^2+b\right)t^{\mu}\right)\sin(r\xi_k), \quad (25)$$

which is the solution of Eq. (1) with conditions (7) and (8) when the fractional derivative is in the Caputo sense. Solution (25) is obtained in [22]. If we put $\mu=1$ in Eq. (25), we obtain the following solution of Eq. (1) with standard derivative [22]

$$T(r,t)=\frac{2ap}{r}\sum_{k=1}^{\infty}(-1)^{k+1}\xi_k\sin(r\xi_k)\exp\left(-\left(a\xi_k^2+b\right)t\right).$$

Case 2 Taking the Laplace transform of the Dirichlet boundary condition (9) gives

$$\tilde{T}(R,s)=p\frac{\Gamma(\beta+1)}{s^{\beta+1}},\ \ \beta\in R,\ \ s>0. \quad (26)$$

Substituting Eq. (26) into Eq. (18) gives

$$\begin{aligned}\tilde{T}^*(\xi_k,s)&=\frac{a(-1)^{k+1}Rp\Gamma(\beta+1)s^{-(\beta+1)}}{s^{\mu}(1-\omega s^{-\rho})^{\gamma}+a\xi_k^2+b}\\&=\frac{a(-1)^{k+1}Rp\Gamma(\beta+1)s^{\rho\gamma-\mu-\beta-1}}{(s^{\rho}-\omega)^{\gamma}}\left(\frac{1}{1+(a\xi_k^2+b)s^{\rho\gamma-\mu}(s^{\rho}-\omega)^{-\gamma}}\right)\\&=a(-1)^{k+1}Rp\Gamma(\beta+1)\sum_{n=0}^{\infty}(-1)^n(a\xi_k^2+b)^n\frac{s^{(n+1)(\rho\gamma-\mu)-\beta-1}}{(s^{\rho}-\omega)^{(n+1)\gamma}}.\end{aligned} \quad (27)$$

The inverse Laplace transform of Eq. (27) can be performed using Eq. (11) to get

$$T^*(\xi_k,t)=a(-1)^{k+1}Rp\Gamma(\beta+1)\sum_{n=0}^{\infty}(-1)^n(a\xi_k^2+b)^n t^{\mu(n+1)+\beta}E^{(n+1)\gamma}_{\rho,\mu(n+1)+\beta+1}(\omega t^{\rho}). \quad (28)$$

The inverse Sine-Fourier transform of Eq. (28) gives the following solution of Eq. (1)

$$T(r,t)=\frac{2ap\Gamma(\beta+1)}{r}\sum_{k=1}^{\infty}\xi_k\sum_{n=0}^{\infty}(-1)^{n+k+1}\left(a\xi_k^2+\right.$$
$$\left.+\,b\right)^n t^{\mu(n+1)+\beta}E_{\rho,\mu(n+1)+\beta+1}^{(n+1)\gamma}(\omega t^{\rho})\sin(r\xi_k) \quad (29)$$

The obtained solutions (22) and (29) of Eq. (1) for the two cases of the boundary conditions are obtained in the form of the series of the three parameter Mittag-Leffler function.

Remark 2: When $\gamma=0$ or $\omega=0$, the solution (29) will be transformed into

$$T(r,t)=\frac{2ap\Gamma(\beta+1)t^{\beta+\mu}}{r}\sum_{k=1}^{\infty}\xi_k\,(-1)^{k+1}E_{\mu,\mu+\beta+1}\left(-\left(a\xi_k^2+b\right)t^{\mu}\right)\sin(r\xi_k), \quad (30)$$

which is the solution of Eq. (1) with conditions (7) and (9) when the fractional derivative is in the Caputo sense. A special case of solution (30) is obtained in [22] when $\beta=0$. If we put $\mu=1,\beta=0$ in Eq. (30), we obtain the following solution of Eq. (1) with standard derivative [22]

$$T(r,t)=\frac{2ap}{r}\sum_{k=1}^{\infty}(-1)^{k+1}\,\xi_k\sin(r\xi_k)\left(\exp\left(-\left(a\xi_k^2+b\right)t\right)-1\right),$$

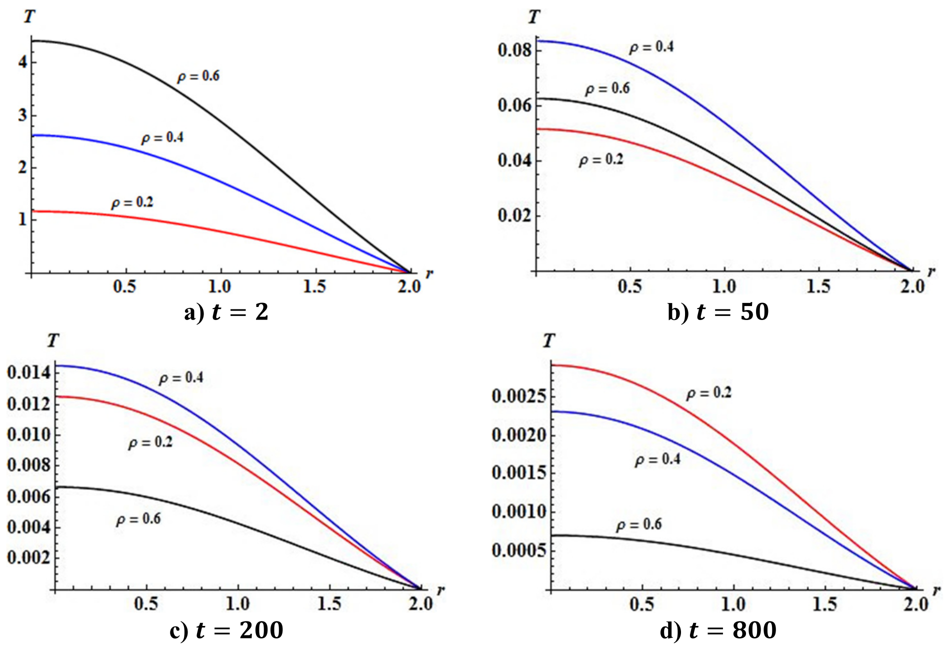


Fig. 1. Plot of the solution (22) when $a=b=\gamma=1$, $R=2$, $p=5$, $\omega=3$, $\mu=\rho$ for different values ρ and t

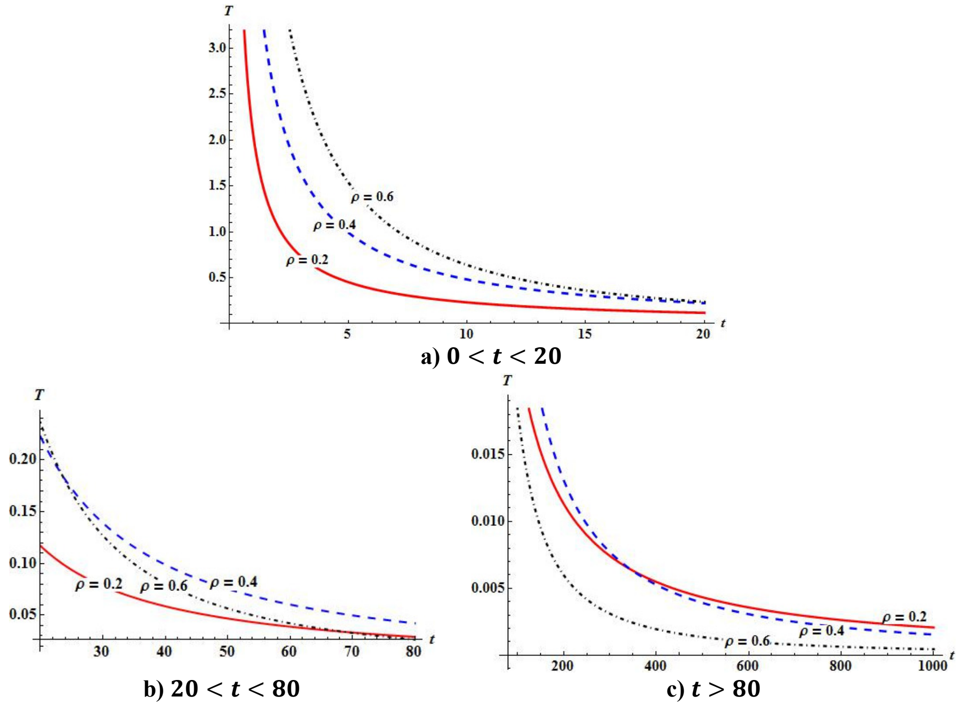


Fig. 2. Plot of the solution (22) when $a = b = \gamma = 1$, $R = 2$, $p = 5$, $\omega = 3$, $\mu = \rho$, $r = 0.5$ for different values ρ

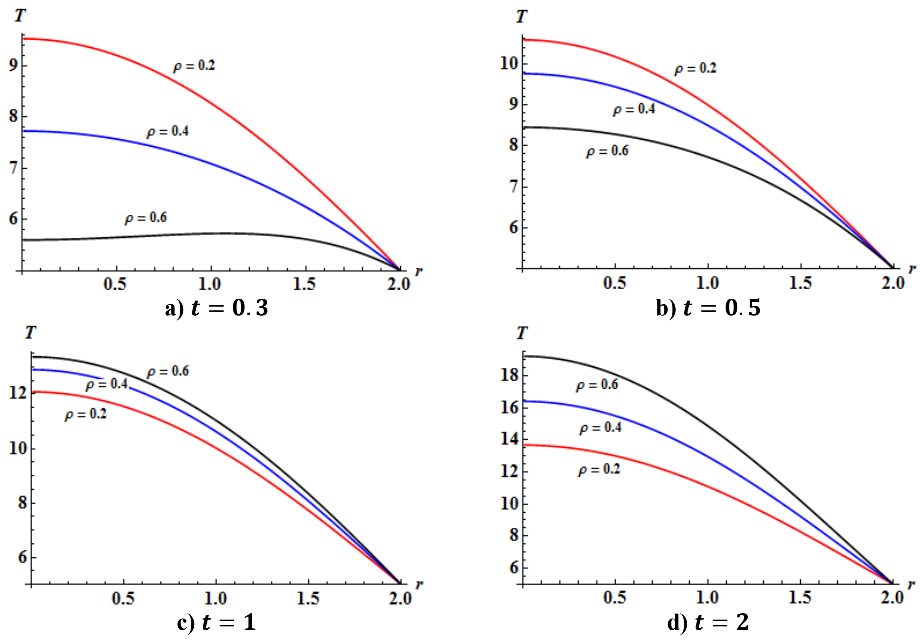
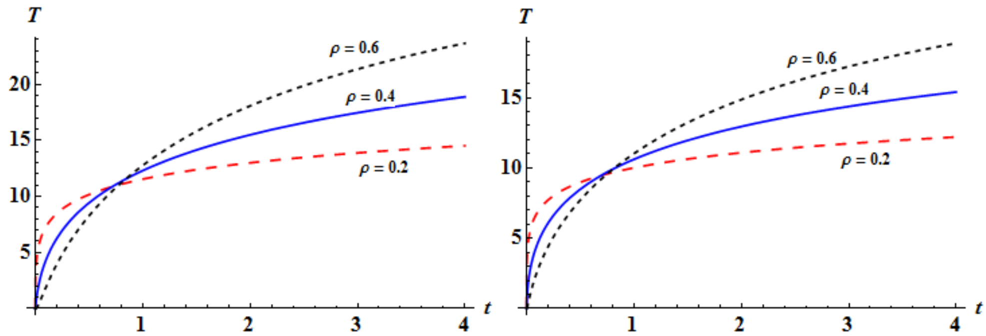


Fig. 3. Plot of the solution (29) when $a = b = \gamma = 1$, $R = 2$, $\beta = 0$ $p = 5$, $\omega = 3$, $\mu = \rho$ for different values ρ and t

Figures 1 and 2 illustrate the profile of the temperature T at some values of ρ and the time t when the boundary condition is taken in the form of the Dirac delta function. In general, we can notice that temperature T decays with increasing time. Also, we can realize that, when t is small, the temperature T increases with increasing the value of ρ. When t is large, the temperature T decreases with increasing the value of ρ.



a) $r = 0.5$ b) $r = 1.5$

Fig. 4. Plot of the solution (29) when $a = b = \gamma = 1$, $R = 2$, $\beta = 0$ $p = 5$, $\omega = 3$, $\mu = \rho$ for different values ρ and r

Figures 3 and 4 illustrate the distribution of the temperature T at different values of ρ and the time t when the boundary condition is taken as a constant. In this case, when t is small, the temperature T decreases with increasing the value of ρ. When t is large, the temperature T increases with increasing the value of ρ.

4. Conclusions

The finite Sine-Fourier transform and the Laplace transform are considered as powerful tools in solving FDEs with regularized Hilfer-Prabhakar derivative. New exact solutions of the TFHCE with regularized Hilfer-Prabhakar derivative of order μ are obtained in two different cases of Dirichlet boundary conditions. The results are obtained in terms of the threeparameter Mittag-Leffler function. The solutions are represented graphically in different values of ρ at certain values of the time and the coefficients ω and μ. The solutions obtained in this paper generalize the results obtained in [22].

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