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SOLUTIONS OF THE DHOMBRES-TYPE TRIGONOMETRIC FUNCTIONAL EQUATION

Iwona Tyrala

Institute of Mathematics and Computer Science
Jan Długosz University in Częstochowa
al. Armii Krajowej 13/15, 42-200 Częstochowa, Poland
e-mail: i.tyrala@ajd.czest.pl

Abstract. Let $(G,+)$ be a uniquely 2-divisible Abelian group. In the present paper we will consider the solutions of functional equation
$[f(x+y)]^2-[f(x-y)]^2+f(2x+2y)+f(2x-2y)=f(2x)[f(2y)+2g(2y)],\ x,y\in G$, where f and g are complex-valued functions defined on G.

1. Introduction

We know many trigonometric identities. To us, important will be the following:

$$\left[\sin\left(\frac{x+y}{2}\right)\right]^2-\left[\sin\left(\frac{x-y}{2}\right)\right]^2=\sin(x)\sin(y),\quad x,y\in\mathbb{R},\tag{1}$$

$$\sin(x+y)+\sin(x-y)=2\sin(x)\cos(y),\quad x,y\in\mathbb{R},\tag{2}$$

$$\sinh(x-y)=\sinh(x)\cosh(y)-\cosh(x)\sinh(y),\quad x,y\in\mathbb{R}.\tag{3}$$

Let $(G,+)$ be a uniquely 2-divisible Abelian group and $f,g\colon G\to\mathbb{C}$. Equation (1) translates into the well known *sine functional equation* [1, 8]

$$\left[f\left(\frac{x+y}{2}\right)\right]^2-\left[f\left(\frac{x-y}{2}\right)\right]^2=f(x)f(y)\quad\text{for all}\quad x,y\in G,\tag{4}$$

and (2) gives rise to the familiar *cosine functional equation* [1, 6, 7]

$$f(x+y)+f(x-y)=2f(x)g(y) \quad \text{for all} \quad x,y\in G, \tag{5}$$

and (3) leads to the *Aczel-Dhombres functional equation* [1]

$$f(x-y)=f(x)g(y)-g(x)f(y) \quad \text{for all} \quad x,y\in G. \tag{6}$$

From now on, f_o and f_e stand for the odd and the even part of a function f.

Theorem 1 (Aczél and Dhombres [1]). *Let* $(G,+)$ *be a uniquely 2-divisible Abelian group. Then* $f,g:G\to\mathbb{C}$ *satisfy equation* (6) *if and only if*

(i) $f=0$ *and* g *is arbitrary; or*

(ii) *there exists an additive function* $A:G\to\mathbb{C}$ *and a constant* $\alpha\in\mathbb{C}$ *such that* $f(x)=A(x),\ g(x)=\alpha A(x)+1, x\in G$*; or*

(iii) *there exists an exponential function* $m:G\to\mathbb{C}$ *and constants* $\beta,\gamma\in\mathbb{C}$ *such that* $f(x)=\beta m_o(x),\ g(x)=\gamma m_o(x)+m_e(x),\ x\in G.$

From the system of equations

$$\begin{cases} f(x+y)=f(x)+f(y),\\ f(xy)=f(x)f(y),\end{cases}$$

we get *the Dhombres functional equation* (see [2])

$$f(x+y)+f(xy)=f(x)+f(y)+f(x)f(y)$$

for functions f mapping a given ring into another one. A different system of the functional equations has been studied by Ger [3, 4, 5]. Here we consider the sum of equations (4) and (5).

2. Main results

We replace x by $2x$ and y by $2y$ in (4) and (5). Summing up these functional equations side by side, for all $x,y\in G$, we get

$$[f(x+y)]^2-[f(x-y)]^2+f(2x+2y)+f(2x-2y)=f(2x)[f(2y)+2g(2y)]. \tag{7}$$

Remark 1. *Put* $x=y=0$ *in* (7)*, so that we have* $f(0)=0\vee f(0)=2-2g(0)$.

Lemma 1. *Let* $(G,+)$ *be a uniquely 2-divisible Abelian group and let functions* $f,g:G\to\mathbb{C}$ *satisfy equation* (7)*. In this case*

(i) *if* $f=0$*, then* g *is arbitrary;*

(ii) *if* $g=0$*, then* $f=0$ *or* $f=2$.

Proof. Ad (ii). For $g = 0$, putting $y = x$ in (7), we get

$$f(x) = [f(0)]^2 - f(0) = f(0)[f(0) - 1] = \gamma, \quad x \in G. \tag{8}$$

From equation (7) we obtain

$$\gamma^2 - \gamma^2 + 2\gamma = \gamma^2,$$

whence $\gamma = 0 \vee \gamma = 2$. By (8), we conclude that $f = 0 \vee f = 2$. □

Lemma 2. *Let $(G, +)$ be a uniquely 2-divisible Abelian group and let nonzero functions $f, g : G \to \mathbb{C}$ be functions defined by*

$$f(x) = aA(x) + f(0), \ g(x) = bA(x) + g(0), \quad x \in G, \tag{9}$$

with some additive function $A : G \to \mathbb{C}$ and $a, b \in \mathbb{C}$ satisfy equation (7). In this case we have the following possibilities:

(i) *If $f(0) = 0$, then $f(x) = aA(x)$, $g(x) = 1$, $x \in G$.*

(ii) *If $f(0) \neq 0$, then $f(x) = f(0)$, $g(x) = 1 - \frac{1}{2}f(0)$, $x \in G$.*

Proof. Applying (9) to (7), we have

$$[aA(x+y)+f(0)]^2 - [aA(x-y)+f(0)]^2 + aA(2x+2y) + 2f(0) + aA(2x-2y)$$
$$= [aA(2x) + f(0)][(a+2b)A(2y) + f(0) + 2g(0)], \ x, y \in G.$$

From the properties of additive function A for all $x, y \in G$, we infer that

$$[2 - f(0) - 2g(0)][2aA(x) + f(0)] + 2f(0)[a - 2b]A(y) = 8abA(x)A(y). \tag{10}$$

Case 1. Assume that $f(0) = 0$. Then equation (10) has a form

$$aA(x)[1 - g(0) - 2bA(y)] = 0, \quad x, y \in G. \tag{11}$$

If $a = 0 \vee A = 0 \vee (a \neq 0 \wedge A \neq 0 \wedge b \neq 0)$, then we get $f = 0$, a contradiction. Hence we have only one possibility $(a \neq 0 \wedge A \neq 0 \wedge b = 0)$. Consequently, equation (11) gives $g(0) = 1$. From (9) we obtain (i).

Case 2. Let $f(0) \neq 0$. By Remark 1 and equation (10), we get the relation

$$A(y)[f(0)(a - 2b) - 4abA(x)] = 0, \quad x, y \in G.$$

If $A = 0$, then (ii). Assume that $A \neq 0$. From above we have

$$f(0)(a - 2b) = 4abA(x), \quad x \in G.$$

Therefore $(a = 0 \Rightarrow b = 0) \vee (b = 0 \Rightarrow a = 0)$, the case (ii). If $a \neq 0 \wedge b \neq 0$, then $A = 0$, a contradiction. □

Now, we formulate some properties of the exponential function without a proof.

Lemma 3. *Let $(G,+)$ be a uniquely 2-divisible Abelian group. Then a nonzero exponential function $m : G \to \mathbb{C}$ has the following properties*

(i) $m_e(x+y) + m_e(x-y) = 2m_e(x)m_e(y), \quad x, y \in G;$

(ii) $[m_o(x+y)]^2 - [m_o(x-y)]^2 = m_o(2x)m_o(2y), \quad x, y \in G;$

(iii) $m_o(x+y) + m_o(x-y) = 2m_o(x)m_e(y), \quad x, y \in G;$

(iv) $m_o(2x) = 2m_o(x)m_e(x), \quad x \in G;$

(v) $[m_e(x+y)]^2 - [m_e(x-y)]^2 = m_o(2x)m_o(2y), \quad x, y \in G;$

(vi) $m_o(x+y) - m_o(x-y) = 2m_e(x)m_o(y), \quad x, y \in G.$

Lemma 4. *Let $(G,+)$ be a uniquely 2-divisible Abelian group and let nonzero functions $f, g : G \to \mathbb{C}$ be functions defined by*

$$f(x) = am_o(x) + bm_e(x),\ g(x) = cm_o(x) + dm_e(x),\ x \in G, \tag{12}$$

with some exponential function $m : G \to \mathbb{C}$ and $a, b, c, d \in \mathbb{C}$ satisfying equation (7). Then we have the following possibilities:

(i) $f(x) = am_o(x),\ g(x) = m_e(x),\ x \in G;$ *or*

(ii) $f(x) = b \neq 0,\ g(x) = 1 - \frac{1}{2}b,\ x \in G;$ *or*

(iii) $f(x) = bm_o(x) + bm_e(x), g(x) = \frac{1}{2}bm_o(x) + (1 - \frac{1}{2}b)m_e(x), x \in G;$ *or*

(iv) $f(x) = -bm_o(x) + bm_e(x), g(x) = -\frac{1}{2}bm_o(x) + (1 - \frac{1}{2}b)m_e(x), x \in G.$

Proof. Inserting functions (12) into equation (7), for all $x, y \in G$, we obtain

$$[am_o(x+y) + bm_e(x+y)]^2 - [am_o(x-y) + bm_e(x-y)]^2 + a[m_o(2x+2y)$$

$$+m_o(2x-2y)] + b[m_e(2x+2y) + m_e(2x-2y)]$$

$$= [am_o(2x) + bm_e(2x)][(a+2c)m_o(2y) + (b+2d)m_e(2y)].$$

From above and Lemma 3, we get

$$[b^2 - 2ac]m_o(x)m_o(y) + b[a-2c]m_e(x)m_o(y)$$

$$+a[2-b-2d]m_o(x)m_e(y) + b[2-b-2d]m_e(x)m_e(y) = 0,\ x, y \in G. \tag{13}$$

Directly from the definition (12), we see that $f(0) = b$ and $g(0) = d$. Moreover, from Remark 1 we infer that $b = 0$ or $b = 2(1-d)$.

Now we shall distinguish two cases regarding the value of function f at zero.

Case 1. Let $f(0) = b = 0$. Then, by (13), we conclude that

$$am_o(x)[-cm_o(y) + (1-d)m_e(y)] = 0,\ x, y \in G.$$

If $a = 0$ or $m_o = 0$, then also $f = 0$. Hence

$$-cm_o(y) + (1-d)m_e(y) = 0,\ y \in G. \tag{14}$$

Putting $y = 0$ in (14) and using $m_o(0) = 0$, we have $d = 1$. Jointly with (14), for all $y \in G$, this implies that $-cm_o(y) = 0$, whence $c = 0$, which ends the proof of (i).

Case 2. Assume that $f(0)=b\neq 0$. Set $b=2(1-d)$ in (13). Then, we get

$$m_o(y)[(b^2 - 2ac)m_o(x) + b(a-2c)m_e(x)] = 0,\ x, y \in G. \tag{15}$$

Subcase 2.1. Let $m_o = 0$. By equation (12), we conclude that $f = bm_e$, $g = dm_e$. Replacing y by $-y$ in (7), we arrive at

$$[f(x-y)]^2-[f(x+y)]^2+f(2x-2y)+f(2x+2y)=f(2x)[f(2y)+2g(2y)]. \tag{16}$$

Subtracting (7) and (16), we get

$$[f(x+y)]^2 = [f(x-y)]^2, \quad x, y \in G.$$

Putting here $y = x$ and replacing x by $\frac{x}{2}$, we obtain $f^2 = b^2$. The case $f = -b$ is impossible. In other words, we have (ii): $f = b,\ m_e = 1,\ g = d = \frac{2-b}{2} = 1 - \frac{1}{2}b$.

Subcase 2.2. Suppose $m_o \neq 0$. Then (15) yields

$$(b^2 - 2ac)m_o(x) + b(a-2c)m_e(x) = 0, \quad x \in G. \tag{17}$$

Putting $x = 0$, we have $a = 2c$. From (17), for all $x \in G$, we get $(b^2 - 4c^2)m_o(x) = 0$, i.e. $b^2 = 4c^2$. If $a = 2c \wedge b = 2c$, then we have the case (iii). However, $a = 2c \wedge b = -2c$ yields (iv). $\square$

Theorem 2. *Let $(G, +)$ be a uniquely 2-divisible Abelian group. Then functions $f, g : G \to \mathbb{C}$ satisfy equation (7) if and only if*

(i) $f = 0$ *and g is arbitrary; or*

(ii) $f(x) = \alpha \neq 0,\ g(x) = 1 - \frac{1}{2}\alpha,\ x \in G$*; or*

(iii) *there exists an additive function $A : G \to \mathbb{C}$ such that $f = A, g = 1$; or*

(iv) *there exists an exponential function $m : G \to \mathbb{C}$ and some constant $\beta \in \mathbb{C}$ such that $f = \beta m_o,\ g = m_e$; or*

(v) *there exists an exponential function* $m\colon G\to\mathbb{C}$ *such that* $f(x)=f(0)m_o(x)+f(0)m_e(x)$, $g(x)=\frac{f(0)}{2}m_o(x)+\left(1-\frac{f(0)}{2}\right)m_e(x)$, $x\in G$; *or*

(vi) *there exists an exponential function* $m : G \to \mathbb{C}$ *such that* $f(x) = -f(0)m_o(x)+f(0)m_e(x)$, $g(x)=-\frac{f(0)}{2}m_o(x)+\left(1-\frac{f(0)}{2}\right)m_e(x)$, $x\in G$.

Proof. From Lemma 1 we obtain (i) and (ii) for $\alpha = 2$. Assume that $f \neq 0$ and $g \neq 0$. Putting $x = 0$ in (7), we get

$$[f(y)]^2-[f(-y)]^2+f(2y)+f(-2y)=f(0)[f(2y)+2g(2y)],\quad y\in G.$$

Let $2C := f(0)$. Thus, from above

$$f(2y)+f(-2y)-2C[f(2y)+2g(2y)]=[f(-y)]^2-[f(y)]^2,\quad y\in G. \tag{18}$$

Interchanging the roles of x and y in (7), we obtain

$$\begin{aligned}&[f(y+x)]^2-[f(y-x)]^2+f(2y+2x)+f(2y-2x)\\&=f(2y)f(2x)+2f(2y)g(2x),x,y\in G.\end{aligned}\tag{19}$$

Subtracting (7) and (19), we get

$$\begin{aligned}&[f(y-x)]^2-[f(x-y)]^2+f(2x-2y)-f(2y-2x)\\&=2f(2x)g(2y)-2f(2y)g(2x),x,y\in G.\end{aligned}\tag{20}$$

Applying (18) for y equal $x-y$, we recive

$$\begin{aligned}&f(2x-2y)+f(-2x+2y)-2C[f(2x-2y)+2g(2x-2y)]\\&=[f(-x+y)]^2-[f(x-y)]^2,x,y\in G.\end{aligned}\tag{21}$$

By (20) and (21), we get the relation

$$(1-C)f(2x-2y)-2Cg(2x-2y)=f(2x)g(2y)-f(2y)g(2x),\quad x,y\in G.$$

Replacing x by $\frac{x}{2}$ and y by $\frac{y}{2}$, we obtain

$$(1-C)f(x-y)-2Cg(x-y)=f(x)g(y)-f(y)g(x),\quad x,y\in G. \tag{22}$$

Case 1. Let $f(0)=0\Rightarrow C=0$. Thus, from (22) we get

$$f(x-y)=f(x)g(y)-f(y)g(x),\quad x,y\in G,$$

By Theorem 1 (ii), we infer that $f(x) = A(x), g(x) = \alpha A(x) + 1$ for some additive function A and some constant α. In view of Lemma 2 for $a = 1$, $b = \alpha$, $f(0) = 0$, $g(0) = 1$, we deduce that

$$f(x) = A(x), \quad g(x) = 1, \quad x \in G.$$

This is the case (iii) of our theorem. By Theorem 1 (iii), we get

$$f(x) = \beta m_o(x), \quad g(x) = \gamma m_o(x) + m_e(x), \quad x \in G.$$

For $a = \beta,\ b = 0,\ c = \gamma,\ d = 1$ in Lemma 4 (i) we have the case (iv), i.e.

$$f(x) = \beta m_o(x), \quad g(x) = m_e(x), \quad x \in G.$$

Case 2. Assume that $f(0) \neq 0$. Then $g(0) = 1 - C$, and (22) gives

$$g(0)f(x-y) - f(0)g(x-y) = f(x)g(y) - f(y)g(x), \quad x, y \in G. \tag{23}$$

Subcase 2.1. If $g(0) = 0$, then $f(0) = 2$. By (23), we infer that

$$g(x-y) = g(x)\frac{f(y)}{2} - g(y)\frac{f(x)}{2}, \quad x, y \in G. \tag{24}$$

Theorem 1 (ii) yields $g(x) = A(x)$, $\frac{f(x)}{2} = \alpha A(x) + 1$ for some additive function A and some constant α. Thus

$$f(x) = 2\alpha A(x) + 2, \quad g(x) = A(x), \quad x \in G.$$

By Lemma 2 for $a = 2\alpha$, $b = 1$, $f(0) = 2$, $g(0) = 0$, we get $f = 2$, $g = 0$. This is the case (ii). Theorem 1 (iii) leads us to

$$f(x) = 2\gamma m_o(x) + 2m_e(x), \quad g(x) = \beta m_o(x), \quad x \in G.$$

From Lemma 4 (ii) for $a = 2\gamma$, $b = 2, c = \beta, d = 0$, we get (ii) of the theorem. The case (iii) for $f(0) = 2$ gives (v), and (iv) gives (vi).
Subcase 2.2. Let $f(0) \neq 0$ and $g(0) \neq 0$. Thus, from (23) for

$$F(x) := g(0)f(x) - f(0)g(x), \quad G(x) := \frac{g(x)}{g(0)}, \quad x \in G,$$

we conclude that

$$F(x-y) = F(x)G(y) - F(y)G(x), \quad x, y \in G. \tag{25}$$

Again, by Theorem 1 (ii), we obtain

$$g(x) = g(0)\alpha A(x) + g(0), \quad f(x) = \frac{1 + f(0)g(0)\alpha}{g(0)}A(x) + f(0), \quad x \in G.$$

By Lemma 2 (ii) for $a = \frac{1+f(0)g(0)\alpha}{g(0)}$, $b = g(0)\alpha$, we get (ii) of the theorem. Further, Theorem 1 (iii) yields

$$F(x) := g(0)f(x) - f(0)g(x) = \beta m_o(x), \quad x \in G,$$

$$G(x) := \frac{g(x)}{g(0)} = \gamma m_o(x) + m_e(x), \quad x \in G,$$

or, equivalently,

$$g(x) = g(0)\gamma m_o(x) + g(0)m_e(x), \quad x \in G,$$

$$f(x) = \frac{\beta + \gamma f(0)g(0)}{g(0)} m_o(x) + f(0)m_e(x), \quad x \in G.$$

Now, using Lemma 4 for $a = \frac{\beta+\gamma f(0)g(0)}{g(0)}$, $b = f(0)$, $c = \gamma g(0)$, $d = g(0)$, the case (ii) gives (ii) of our theorem, however (iii) yields (v), and (iv) gives (vi). □

References

[1] J. Aczél, J. Dhombres. *Functional Equations in Several Variables.* Cambridge University Press, Cambridge 1989.

[2] J. Dhombres. Relations de dépendance entre les équations fonctionnelles de Cauchy. *Aequat. Math.*, **35**, 186–212, 1988.

[3] R. Ger. On an equation of ring homomorphisms. *Publ. Math. Debrecen*, **52**, 397–417, 1998.

[4] R. Ger. Ring homomorphisms equation revisited. *Rocznik Nauk.-Dydakt. Prace Mat.*, **17**, 101–115, 2000.

[5] R. Ger. Additivity and exponentiality are alien to each other. *Aequat. Math.*, **80**, 111–118, 2010.

[6] Pl. Kannappan. The functional equations $f(xy)+f(xy)^{-1} = 2f(x)f(y)$ for groups. *Proc. Amer. Math. Soc.*, **19**, 69–74, 1968.

[7] I. Tyrala. A characterization of the sine function by functional inequalities. *Mathematical Inequalities & Applications*, **14**, 13–33, 2011.

[8] W.H. Wilson. On certain related functional equations. *Bull. Amer. Math. Soc.*, **26**, 300–312, 1919.