Vol. 32, No. 3 (2022) | DOI: 10.37190/ord220304

OPEN ACCESS

Operations Research and Decisions

www.ord.pwr.edu.pl

OPERATIONS RESEARCH AND DECISIONS QUARTERLY

Barter exchange as the way to deal with excess inventory: newsvendor problem with multiplicative demand

Milena Bieniek*[1]

[1] *Faculty of Economics, Institute of Management and Quality Sciences, Maria Curie-Sklodowska University, Lublin, Poland*
* *Corresponding author: milena.bieniek@umcs.lublin.pl*

Abstract

Barter exchange has been growing in popularity during the coronavirus pandemic. This article considers bartering introduced to the newsvendor model with multiplicative demand. The objective of the model is to specify the order quantity and retail price to maximize the expected profit. We distinguish cases with the co-movement of prices of exchanged products and without it. In the first case, we calculate a precise optimal solution to the problem. In the latter case, we prove the existence of an optimal solution and give the conditions under which it is unique. We examine the sensitivity analysis of the results which is illustrated in numerical examples. The analysis revealed that the greater the commission, the lower the optimal profit. We make a conclusion that barter exchange can help the retailer to improve the profit.

Keywords: *barter exchange, inventory, newsvendor, pricing, multiplicative demand*

1. Introduction

Barter exchange is defined as a direct exchange of services or goods without the use of money [11]. The limitations of barter have led to the emergence of money. However, trading with money also produces the problems of monetary economy such as inflation, deflation, currency devaluation and exchange fluctuation. Three main problems were identified in connection with barter: the problem of the coincidence of wants, the problem of non-feasibility of multilateral barter, and finally, the problem with inefficient shopping experience. The coincidence of wants in barter is described by [22] in the following manner "the first difficulty in barter is to find two persons whose disposable possessions mutually suit each other's wants. There may be many people wanting, and many possessing those things wanted; but to allow of an act of barter, there must be a double coincidence, which will rarely happen." The problem of the coincidence of wants is a transaction cost which imposes limitations on trade under a barter system. Along with the increasing number of parties involved in a swap, the problem of the coincidence of wants aggregates, as a result making multilateral trade less feasible than bilateral trade. This is a major inefficiency of barter exchange. The third problem of barter is inefficient shopping experience. In the

Received 11 January 2022, accepted 8 July 2022, published online 22 November 2022
ISSN 2391-6060 (Online)/

case of using money, the shopping flow is separated into two modules. In the first module, the customer browses all of the products available in a shop and chooses what it wants. In the second module, the shopper makes a transaction by paying with money. In barter exchange the shopping experience can be also divided into two modules, namely browsing or picking items and making a transaction. However, during the exchange, these modules are repeated every time the shopper visits a different seller. Another inefficient shopping experience is that in a physical market, sellers can communicate with another seller one at the time. This implies that in a market full of sellers, many people might have to wait in line until a person becomes available. Finally, another inefficient shopping experience is that items are displayed per seller, and not based on their functionality. All of these problems cause barter to be time consuming [2, 32, 29].

Barter exchange remained unexplored as money came to replace bartering as the prevalent infrastucture of trade. The emergence of technology and the digital age has given new methods of solving barter issues [32]. The authors of [32] stated that online barter is mainly used by small businesses and the market gap is caused by the unsolved inefficiencies of barter. They proposed e-barter system which might overcome all of the three identified obstacles. Additionally, the authors stated that since 2016 online barter solutions are rare on the market. When considering the present reality, a strong argument can be made that the situation is evolving. Barter is a trade system which has re-emerged with a modern twist. This trend is particularly evident in the time of the coronavirus pandemic. In this difficult moment, social networks such as Facebook are flooded with posts from users who seek to swap things or services. [5] examined 5800 small businesses in the beginning of Covid-19 outbreak and stated that: "The median firm with monthly expenses over $\$10,000$ had only enough cash on hand to last roughly 2 wk [weeks]. Three-quarters of respondents only had enough cash on hand to last 2 mo [months] or less.". It might be a reason that small and medium businesses whose cash trade has strongly declined quite more often decide to turn to e-barter exchanges [25]. The International Reciprocal Trade Association (IRTA) is a global association whose members include 100 barter exchange organizations. During the pandemic, some of them have reported an increase of $20 - 35\%$ in member signups. According to the IRTA, recent data shows approximately $\$12 - 14$ billion in barter trade per annum [21].

Bartering is not exclusively limited to physical objects. While combating the Covid-19 outbreak, many companies seek alternative moneyless ways of working with their business partners. The exchange of skills and services, referred to as business bartering, has become more widespread in the pandemic as enterprises find different ways to improve their business. The emerging financial crisis may result in the rise of bartering via technological tools, for instance new cloud-based barter platforms [7]. The magazine [7] uses the following wordplay- ”a barter way of life”- which offers a perfect illustration of the recent consumer and business attitude to bartering. In addition to clearing excess inventories, barter can be an important complement to trade credits [20]. The trade credit is a remarkable tool to boost market demand by attracting more customers [44, 45, 9].

In light of the foregoing considerations, this article focuses on commercial (retail) bartering with the following sequence of events. First, Firm X registers on an exchange platform and provides a general description of the product to be bartered. With the help of the broker, it finds Firm Y that looks for the offered product. Those two firms engage in a cashless transaction. They pay a commission of approximately $5 - 15\%$ of the traded value to the broker. A modern barter system may also consist in multilateral bartering. In addition, transactions may also involve barter currency whose use is usually strictly limited to the specific barter platform. Another noteworthy fact is that the barter price is the same as the market price which distinguishes such an approach from principles followed on the secondary market.

We examine barter exchange in the newsvendor problem with multiplicative demand exponentially dependent on the retail price. More precisely, we supplement the newsvendor problem considered by [19] with the price-dependent demand and conduct the optimization as a new task. The contribution to the previous research is as follows: the barter price as a decision variable is introduced into the model of [19] where the price was assumed to be exogenous, and multiplicative uncertainty is used instead of the additive one which was studied in [8]. Consequently, we consider the following two cases: with the co-movement of prices or with uncorrelated barter prices of exchanged products. Finally, we conduct the sensitivity analysis and illustrate the results by using the numerical example.

The study is motivated by the fact that there may exist differences between the solutions of Operations Research (OR) models with additive and multiplicative uncertainty. Both types of uncertainty have been widely studied. However, there is no general theory that links the form of demand uncertainty with specific products. The multiplicative stochastic demand can model the demand for high-fashion or new products, while the additive demand can model the demand for branded products [1]. For some products, both additive and multiplicative models have empirical support, e.g. in case of electricity demand [46]. Recently, [10] determined pricing and demand management of air tickets using a multiplicative demand model. The proposed demand model was tested using real flight data. [43] examined the inventory model for deteriorating items with fuzzy lead time, negative exponential demand, and partially backlogged shortages. It is also worth mentioning that sometimes there exist different behaviors between additive and multiplicative uncertainty i.e. these two demand uncertainty forms often lead to different pricing policies. The demand uncertainty effect is also considered in [12], where the additive/multiplicative dichotomy of this uncertainty is analyzed. DeYong [12] studies the dichotomy in the price-setting newsvendor problem in which, in the case of additive uncertainty, the optimal price is lower for the stochastic system than for the deterministic one, which is contrary to the results of the same problem with multiplicative uncertainty. Therefore, for the sake of generality and completeness, this article studies the newsvendor problem with barter exchange and multiplicative demand which complements the additive case considered in [8].

This article makes several contributions to the literature:

- expansion of the relatively small volume of bartering research in OR problems,
- introduction of the price-setting newsvendor problem with barter exchange and multiplicative uncertainty and development of the solution to this model which complements the results for additive uncertainty,
- finding solutions to the model in two cases: with and without the co-movement of barter prices,
- finding the sufficient conditions for the existence of a unique solution to the problem,
- presentation of the sensitivity analysis with respect to the rate of commission.

2. Related literature

This study is closely related to the newsvendor model with multiplicative uncertainty and to barter exchange in the supply chain management. This section reviews recent literature concerning these issues. The objective of the newsvendor problem is finding an optimal replenishment policy for a perishable product in the face of uncertain demand. A price-taking newsvendor problem was considered by numerous authors who contribute to inventory management [49, 38]. Many modifications of this basic model have been introduced which effect the complexity of the problem [39, 26]. One of the major generalizations of the classical newsvendor is introducing price as a decision variable. This modification has been frequently applied, but lately, it has become an increasingly popular subject matter studied by scientists in the field [12]. In the price-setting newsvendor problem, the price-dependent demand mainly includes additive or multiplicative uncertainty.

The price-setting newsvendor problem with multiplicative demand has been considered by [51, 52, 27, 41]. Ye and Sun [51] investigated the price-setting newsvendor problem with strategic customers under two demand cases: additive and multiplicative. For each case, they proved that neglecting the price-sensitivity of demand leads to sub-optimal solution. Yu et al. [52] studied the price-setting newsvendor model involving a manufacturer with random yield and demand uncertainty using stochastic comparisons. Kirshner and Shao [27] studied optimism and overconfidence of a newsvendor as weights on demand and profit. They used the prospect theory to show that optimism increases inventory. The newsvendor problem with isoelastic demand under the mean-variance criteria was examined in [41]. The multiplicative demand with negative exponential function was considered in [40] and [43]. In [40] the authors investigated the consignment contract with a vendor or retailer managed inventory, while in [43] the inventory model for deteriorating items with fuzzy lead time and partially backlogged shortages were examined.

Barter exchange is applied to maintain and balance trade volumes in order to maximize the participants' utility. Researchers attempt to specify the optimal allocation and efficient exchange algorithms for bartering in industries such as the medical or energy industry, education, tourism, materials processing industries, knowledge exchange. Authors considered the social, economic and logistic subject matters of commercial barter. The economic effect of barter is concentrated on the liquidity problem [50], a strategic response to economic crisis [31], reduction of taxes [42], market segmentation and price discrimination [18, 42] and negotiations [17]. Online barter platforms can overcome some restrictions on barter i.e. inefficiencies in facilitating exchange in comparison to money. Moreover, they may be helpful during a crisis such as a social and economic crisis connected with the Covid-19 outbreak [17]. Barter exchange can constitute a strategic answer to small and medium enterprises' growth, helping to improve sales and to enter new markets [6]. Several researchers attempt to answer how barter influences inventory and supply chain management [14, 36, 13]. However, there are few papers on modeling inventory decisions with the use of barter exchange [3, 19, 54].

Commercial bartering exists in business alongside personal or corporate bartering. In [19], the authors employed commercial barter to the newsvendor problem. The barter price, which is equal to the selling price, is specified in advance and optimal solutions in the case of the non-random and random barter supply were calculated. Zhang et al. [54] considered bartering in a wholesale-price pull contract, in which the retailer chooses the wholesale price and the manufacturer determines the order quantity. In their considerations, the demand is defined as a random variable with a given distribution function. Huang et al. [20] investigated a two-level supply chain of one manufacturer and one capital-constrained retailer with trade credit. In this agreement the retailer swaps unsold products for subsidiary products on a barter platform. They derived the retailer's optimal order quantity and the manufacturer's wholesale price under stochastic demand and exogenous retail price.

We extend the results of [19] and assume that the retail price is specified by the retailer and demand is multiplicative and exponentially dependent on price. Using multiplicative uncertainty, the article also complements the results of [8] where the focus was on the additive demand and the possibility of negative demand realizations.

3. Model and analysis

3.1. Basic information

We concentrate on the price-setting newsvendor model with bartering and multiplicative demand. In the considered problem, the customer demand for Product A depends exponentially on the selling price. The retailer sets order quantity Q and price p to satisfy the demand for Product A through the maximization of the retailer's expected profit. For simplicity, it is assumed that the unsatisfied demand is lost, there is no penalty cost and the salvage value of unsold Product A is 0. Additionally, some quantity of unsold Product A can be exchanged at almost full price for Product B on a barter platform. The retailer needs Product B for its employees. Product B could be, e.g. stationery and office supplies for everyday use, food products needed during coffee breaks. After this swap, the retailer buys the remaining needed portion of Product B on the market.

Market demand for Product A is stochastic and multiplicative given by

$$D(p,\varepsilon) = y(p)\varepsilon \tag{1}$$

where $y(p) = a\mathrm{e}^{-bp}$, $a, b > 0$, p is the selling price and $\varepsilon \in [\alpha, \beta]$, $0 < \alpha < \beta$, is a random variable with cdf F and continuously differentiable pdf f on $[\alpha, \beta]$, mean μ and finite variance σ^2.

Deterministic negative exponential demand $y(p) = a\mathrm{e}^{-bp}$ is a frequently overlooked alternative to the isoelastic demand defined as $y(p) = a_1 p^{-b_1}$ [40]. However, the negative exponential one has numerous attractive properties that make it very useful in many circumstances. Parameters a and b have a different meaning than the parameters of the isoelastic demand. Parameter a can be explained as a type of market size. It can be understood as the market saturation quantity when the commodity is free. Parameter b is such that $1/b$ turns out to be the price at which the demand becomes elastic. It is the unit elasticity price. The price elasticity $-bp$ starts at zero when the price is zero and becomes more elastic and proportional to price. When $p = 1/b$, the elasticity is exactly equal to unity. The higher the price is, the more elastic the demand becomes [4]. The list of model parameters and notation are presented in Table 1.

Table 1. List of notations and parameters

Symbol	Description
r	commission that the retailer pays to the barter platform given in the share of the bartered value, in practice $r \in [0.05, 0.15]$ [19]
c	wholesale price of Product A per unit
p_A	selling (barter) price of Product A per unit, $p_A \geq c$
p_B	barter price of Product B per unit
Q_A	order quantity of Product A
Q_B	quantity of Product B the retailer needs for its employees
Q_e	early demand on Product B if $p_B = 0$ specified by the retailer
w_B	value of Product B the retailer needs for its employees
z_A	price-sensitive stock factor of Product A
p_A^*, p_A^c	optimal selling price of Product A
z_A^*, z_A^c	optimal stock factor of Product A
Π, Π_c	retailer's profit
$\Lambda(z)$	$\Lambda(z) = \int_\alpha^z (z-u) f(u) du$, $\frac{d\Lambda(z)}{dz} = F(z)$, $z \in [\alpha, \beta]$

We describe the retailer's profit in the following cases.

1. If $0 \leq Q_A \leq D(p_A, \varepsilon)$, the retailer buys Product B on the market, and then its profit can be calculated from
$$\Pi(p_A, Q_A) = (p_A - c)Q_A - p_B Q_B \tag{2}$$
2. If $p_A D(p_A, \varepsilon) < p_A Q_A \leq p_A D(p_A, \varepsilon) + p_B Q_B$, $Q_A - D(p_A, \varepsilon)$ items of Product A are unsold. The retailer swaps $Q_A - D(p_A, \varepsilon)$ units of Product A for Product B of the same value, pays the commission $r p_A(Q_A - D(p_A, \varepsilon))$, and buys Product B of value $p_B Q_B - p_A(Q_A - D(p_A, \varepsilon))$ on the market. Therefore, the retailer's profit is given by
$$\Pi(p_A, Q_A) = ((1-r)p_A - c)Q_A + r p_A D(p_A, \varepsilon) - p_B Q_B \tag{3}$$
3. If $p_A Q_A - p_B Q_B > p_A D(p_A, \varepsilon)$, the retailer barters Product A for Product B of the same value at the cost of the commission $r p_B Q_B$. The remaining inventory is lost. Consequently, the retailer's profit is equal to
$$\Pi(p_A, Q_A) = -cQ_A + p_A D(p_A, \varepsilon) - r p_B Q_B \tag{4}$$

Thus,

$$\Pi(p_A, Q_A) = \begin{cases} (p_A - c)Q_A - p_0 Q_0 & \text{if } 0 \le Q_A \le D(p_A, \varepsilon) \\ ((1-r)p_A - c)Q_A + rp_A D(p_A, \varepsilon) - p_B Q_B & \text{if } p_A D(p_A, \varepsilon) < p_A Q_A \le \\ & p_A D(p_A, \varepsilon) + p_B Q_B \\ -cQ_A + p_A D(p_A, \varepsilon) - rp_B Q_B & \text{if } p_A Q_A - p_B Q_B > p_A D(p_A, \varepsilon) \end{cases} \tag{5}$$

Now, let us consider two possibilities. The first possibility assumes that there is the co-movement of prices of Product A and B, namely, those prices are highly positively correlated. It can happen for the prices of processed products and the purchase price of raw material, for food prices and oil price [33, 15], for crude oil and petroleum-product prices [37, 30], for energy prices and agricultural commodity prices [28], for coffee and cocoa prices [48]. In the second possibility, the price of Product B is uncorrelated with the price of Product A.

In the subsequent analysis instead of using Q_A we will use the stock factor $z_A = Q_A / y(p_A)$ [35, 41]. Since $Q_A \in [y(p_A)\alpha, y(p_A)\beta]$ (i.e. the order quantity must be within the range defined by the possible demand values), it follows that $z_A \in [\alpha, \beta]$.

The following lemma will be useful in the rest of the article. The proofs of theorems, propositions and lemmas are relegated to Appendix A.

Lemma 1. The following statements are true for any $z \in [\alpha, \beta]$:

$$\int_z^\beta (z-u)f(u)du = z - \mu - \Lambda(z) \tag{6}$$

$$\int_{z-x}^z (z-u)f(u)du = \Lambda(z) - \Lambda(z-x) - xF(z-x) \tag{7}$$

$$\int_\alpha^{z-x} (z-u)f(u)du = \Lambda(z-x) + xF(z-x) \tag{8}$$

3.2. Newsvendor problem with the co-movement of prices

Certain works highlight the fact that the co-movements of prices are a central and distinctive feature of commodity prices. These co-movements occur due to common factors influencing those prices, or due to herd behavior. Some studies concentrate on co-movements in the prices of similar commodities, others have found co-movements among different groups of prices ([34] and references therein).

In this section, we consider the model in which the price of Product A is equal to the price of Product B, $p_A = p_B$, which implies that those prices are perfectly positively correlated. It is possible for similar products from the same product category, like printed media products. Moreover, we assume that the quantity of Product B depends on the barter price of this product, namely the employees' demand for Product B is negatively exponentially dependent on price and determined by

$$Q_B = Q_e \mathrm{e}^{-bp_A} \tag{9}$$

where quantity Q_e is the early demand on Product B. The quoted assumptions let the model be mathematically tractable and allow precise solutions to the maximization problem to be offered. The more complicated models with other various assumptions on the barter price or barter quantity can be solved numerically, similarly to the model studied in Subsection 3.3. We need some additional assumptions.

1. $h(z, r, x)$ - mixture failure rate defined by

$$h(z, r, x) = \frac{rf(z) + (1-r)f(z-x)}{r\bar{F}(z) + (1-r)\bar{F}(z-x)} \tag{10}$$

   is a non-decreasing function of z, where $r \in [0, 1]$.

2. $0 \leq \frac{Q_e}{a} < \alpha$

Assumption 1 is needed to prove the unimodality of expected profit. If $r = 1$, then $h(z, r, x)$ reduces to failure rate function $h(z)$ defined by $h(z) = f(z)/\bar{F}(z)$ with $\bar{F}(z) = 1 - F(z)$ [24]. A lot of distributions used in OR models have an increasing failure rate and a twice differentiable F with a continuous second derivative, i.e. the normal, uniform, Power, Chi-squared, Logistic, Gamma, and Weibull distributions with specific parameters. It is well known that mixtures of decreasing failure rate distributions are always decreasing. However, the mixture failure rate of two IFR distributions does not have to be increasing, namely, IFR class of distributions is not closed under the operation of mixing. For example, the mixture of Gamma distributions with the same scale parameter can result either in the increasing mixture failure rate, or in the modified bathtube mixture failure rate. Similar shapes occur for the mixtures of the two IFR Weibull distributions or two truncated normal distribution for which mixture failure rate increase only for some specific parameters [16].

Assumption 2 corresponds to $z_A - \frac{Q_e}{a} > 0$ for any $z_A \in [\alpha, \beta]$, which is equivalent to $Q_A > Q_B$. This means that the order quantity of Product A is always greater than the quantity of Product B needed by the retailer. This assumption assures that the unique optimal solution exists.

Next, the newsvendor model with barter exchange and the classical newsvendor model with the consumption of Product B are analyzed.

3.2.1. Newsvendor with bartering of Product A and B

The model (5) is modified by introducing $p_B = p_A$, and substituting the stock factor $z_A = Q_A/(ae^{-bp_A})$ and Q_B given by (9) into (5). Then, the profit can be written as

$$\Pi(p_A, z_A) = \begin{cases} a\mathrm{e}^{-bp_A}((p_A - c)z_A - p_A\frac{Q_e}{a}) & \text{if } z_A \leq \varepsilon \leq \beta \\ a\mathrm{e}^{-bp_A}((p_A - c)z_A - rp_A(z_A - \varepsilon) - p_A\frac{Q_e}{a}) & \text{if } z_A - \frac{Q_e}{a} \leq \varepsilon < z_A \\ a\mathrm{e}^{-bp_A}((p_A - c)z_A - p_A(z_A - \varepsilon) + p_A(1-r)\frac{Q_e}{a} - p_A\frac{Q_e}{a}) & \text{if } \alpha \leq \varepsilon \leq z_A - \frac{Q_e}{a} \end{cases} \tag{11}$$

Therefore, by Lemma 1

$$\mathrm{E}\Pi(p_A, z_A) = a\mathrm{e}^{-bp_A}((p_A - c)z_A - p_A\frac{Q_e}{a} - rp_A\Lambda(z_A) - p_A(1-r)\Lambda(z_A - \frac{Q_e}{a})) \tag{12}$$

We consider the optimization problem

$$\max_{\substack{p_A \in (c, \infty), \\ z_A \in [\alpha, \beta]}} \mathrm{E}\Pi(p_A, z_A) \tag{13}$$

Now, we need the sequential optimization method [53] which first determines the optimal price for a given stock factor. The existence of the optimal price requires the concavity of the objective function with respect to price. Next, the objective function is expressed in terms of one variable and it is optimized. According to this method we obtain the results below.

Theorem 1. Under all of the above assumptions, the optimal solution to the problem (13) is as follows:

$$p_A^* = p_A^*(z_A^*)$$

where

$$p_A^*(z_A) = \frac{1}{b} + \frac{cz_A}{z_A - \frac{Q_e}{a} - r\Lambda(z_A) - (1-r)\Lambda(z_A - \frac{Q_e}{a})} \tag{14}$$

and z_A^* is the unique solution to

$$(bcz_A^* + z_A^* - \frac{Q_e}{a} - r\Lambda(z_A^*) - (1-r)\Lambda(z_A^* - \frac{Q_e}{a}))(1 - rF(z_A^*) - (1-r)F(z_A^* - \frac{Q_e}{a})) - bc(z_A^* - \frac{Q_e}{a} - r\Lambda(z_A^*) - (1-r)\Lambda(z_A^* - \frac{Q_e}{a})) = 0 \tag{15}$$

if

$$p_A^*(\beta)(1-r)\bar{F}(\beta - \frac{Q_e}{a})) - c < 0 \tag{16}$$

Then, $p_A^*(z_A) > c$ and the optimal expected profit is given by

$$\begin{aligned} \mathrm{E}\Pi^* &= \mathrm{E}\Pi(p_A^*(z_A^*), z_A^*) \\ &= \frac{a}{b}\mathrm{e}^{-1-\frac{bcz_A^*}{z_A^* - \frac{Q_e}{a} - r\Lambda(z_A^*) - (1-r)\Lambda(z_A^* - \frac{Q_e}{a})}} \left(z_A^* - \frac{Q_e}{a} - r\Lambda(z_A^*) - (1-r)\Lambda(z_A^* - \frac{Q_e}{a})\right) \end{aligned} \tag{17}$$

3.2.2. Classical newsvendor problem with Product B consumption

In order to compare the model with bartering with the classical newsvendor, this subsection analyzes the latter model and gives optimal solutions to this case. The retailer's consumption of Product B is incorporated into the classical newsvendor model without bartering. The retailer will buy Q_B items of Product B [19]. As a consequence, the retailer's expected profit is given by

$$\mathrm{E}\Pi_c(p_A, z_A) = a\mathrm{e}^{-bp_A}((p_A - c)z_A - p_A\frac{Q_e}{a} - p_A\Lambda(z_A)) \tag{18}$$

Now, the optimization problem is considered, namely

$$\max_{\substack{p_A \in (c,\infty), \\ z_A \in [\alpha,\beta]}} \mathrm{E}\Pi_c(p_A, z_A) \tag{19}$$

Note that the objective function of the newsvendor problem with consumption corresponds to the objective function of the newsvendor problem with bartering, assuming $r = 1$.

Now, we also employ the sequential optimization method. According to such an approach, the first step in the optimization is to calculate the price that maximizes (18) for any given $z_A \in [\alpha, \beta]$. We obtain the following theorem.

Theorem 2. Under all of the above assumptions, the optimal solution to the problem (19) is given by

$$p_A^c = p_A^c(z_A^c)$$

where

$$p_A^c(z_A) = \frac{1}{b} + \frac{cz_A}{z_A - \frac{Q_e}{a} - \Lambda(z_A)} \tag{20}$$

and z_A^c is the unique solution to

$$(bcz_A^c + z_A^c - \frac{Q_e}{a} - \Lambda(z_A^c))\bar{F}(z_A^c) - bc(z_A^c - \frac{Q_e}{a} - \Lambda(z_A^c)) = 0 \tag{21}$$

Then, $p_A^c(z) > c$ and the optimal expected profit is equal to

$$\mathrm{E}\Pi_c^* = \mathrm{E}\Pi_c(p_A^c, z_A^c) = \frac{a}{b}\mathrm{e}^{-1-\frac{bcz_A^c}{z_A^c - \frac{Q_e}{a} - \Lambda(z_A^c)}}(z_A^c - \frac{Q_e}{a} - \Lambda(z_A^c)) \tag{22}$$

3.3. Newsvendor problem without the co-movement of prices

We consider the problem $\max \mathrm{E}\Pi(p_A, z_A)$, where $\Pi(p_A, z_A)$ is a transformation of (5) by substituting the stock factor $z_A \in [\alpha, \beta]$, and value w_B which is the amount the retailer specifies to be dedicated to buying necessary products, e.g. office supplies [19]. The given value w_B corresponds to $p_B Q_B$. However, in this case, the price of Product B does not have to be constant, but the value of needed products w_B is constant and given in advance. Moreover, we assume that the price of Product B does not depend on the price of Product A. Then, the profit function can be written as

$$\Pi(p_A, z_A) = \begin{cases} a\mathrm{e}^{-bp_A}(p_A - c)z_A - w_B & \text{if } z_A \le \varepsilon \le \beta \\ a\mathrm{e}^{-bp_A}((p_A - c)z_A - rp_A(z_A - \varepsilon)) - w_B & \text{if } z_A - \frac{w_B}{ap}\mathrm{e}^{bp_A} \le \varepsilon < z_A \\ a\mathrm{e}^{-bp_A}((p_A - c)z_A - p(z_A - \varepsilon)) - rw_B & \text{if } \alpha \le \varepsilon < z_A - \frac{w_B}{ap_A}\mathrm{e}^{bp_A} \end{cases} \tag{23}$$

Therefore,

$$\begin{aligned} \mathrm{E}\Pi(p_A, z_A) &= \int_{z_A}^{\beta} (a\mathrm{e}^{-bp_A}(p_A - c)z_A - w_B) f(u)\mathrm{d}u \\ &+ \int_{z_A - \frac{w_B}{ap_A}\mathrm{e}^{bp_A}}^{z_A} (a\mathrm{e}^{-bp_A}((p_A - c)z_A - rp_A(z_A - u)) - w_B) f(u)\mathrm{d}u \\ &+ \int_{\alpha}^{z_A - \frac{w_B}{ap_A}\mathrm{e}^{bp_A}} (a\mathrm{e}^{-bp_A}((p_A - c)z_A - p_A(z_A - u)) + (1-r)w_B - w_B) f(u)\mathrm{d}u \\ &= a\mathrm{e}^{-bp_A}(p_A - c)z_A - w_B + (1-r)w_B F\left(z_A - \frac{w_B}{ap_A}\mathrm{e}^{bp_A}\right) \\ &- a\mathrm{e}^{-bp_A}\left(rp_A \int_{z_A - \frac{w_B}{ap_A}\mathrm{e}^{bp_A}}^{z_A} (z_A - u) f(u)\mathrm{d}u + p_A \int_{A}^{z_A - \frac{w_B}{ap_A}\mathrm{e}^{bp_A}} (z_A - u) f(u)\mathrm{d}u\right) \end{aligned}$$

With the help of Lemma 1, we obtain

$$\mathrm{E}\Pi(p_A, z_A) = a\mathrm{e}^{-bp_A}\left((p_A - c)z_A - rp_A\Lambda(z_A) - p_A(1-r)\Lambda\left(z_A - \frac{w_B}{ap_A}\mathrm{e}^{bp_A}\right)\right) - w_B \tag{24}$$

Note that $\mathrm{E}\Pi(p_A, z_A)$ attains only negative values for any $z_A \in [\alpha, \beta]$ if $\frac{w_B}{ap_A}\mathrm{e}^{bp_A} > \beta$. Therefore, we limit our considerations to the set $\{p_A : p_A \ge c \text{ and } \frac{w_B}{ap_A}\mathrm{e}^{bp_A} \le \beta\}$ which is non-empty if solutions p_1 and p_2 to equation $\frac{w_0}{ap}\mathrm{e}^{bp} = B$ exist. Therefore, from now on, we consider the optimization problem

$$\max_{\substack{p_A \in [\max\{c, p_1\}, p_2], \\ z_A \in [\alpha, \beta]}} \mathrm{E}\Pi(p_A, z_A) \tag{25}$$

The assumption that p_A and z_A belong to closed intervals allows us to use the Extreme Value Theorem and to state that (25) always has an optimal solution. Moreover, we prove the following proposition.

Proposition 1. The expected profit defined by (24) is concave on z_A for $z_A \in [\alpha, \beta]$ and a given p_A. The unique optimal solution to (25) exists if the function defined by (24) is concave on p_A.

3.4. Impact of barter on newsvendor decisions with numerical examples

In this subsection, we examine the impacts of barter on the newsvendor decisions and profit. Therefore, the sensitivity analysis is conducted with respect to the commission and the early demand for Product B. First, the following result is needed.

Lemma 2. In the case with the co-movement of prices, we have $E\Pi(p_A^*(z_A), z_A) > E\Pi_c(p_A^c(z_A), z_A)$ for any $z_A \in [\alpha, \beta]$.

Using the above lemma, the next proposition is proved.

Proposition 2. The following cases are true:

1. In the classical newsvendor model with consumption and in the newsvendor model with barter exchange and the co-movement of prices, the expected profit is a decreasing function of the early demand Q_e on Product B.
2. In the newsvendor model with barter exchange and the co-movement of prices, the optimal expected profit is a decreasing function of commission r.
3. In the case of the co-movement of prices, the following inequality holds $E\Pi^* > E\Pi_c^*$ for a given r and Q_e.

The numerical results presented in Table 3 and 4 verify Proposition 2. If ε is uniformly distributed on $[\alpha, \beta]$, $0 < \alpha < \beta$, then the mixture failure rate is equal to $h(z, r, x) = 1/(1 - z + (1 - r)x)$ which is increasing. Consequently, the desired assumption (1) is satisfied. We set parameter values as presented in Table 2.

Table 2. Parameter values, $\varepsilon \sim U[\alpha, \beta]$

Parameter	Values
α	0.2
β	2
r	$0.05, 0.075, 0.1, 0.125, 0.15$
a	100
b	1
c	1
Q_e	$1, 2, 3, 4, 5$
w_B	$0.1, 0.2, 0.3, 0.4, 0.5$

The optimal profit of the newsvendor with bartering compared to the classical newsvendor with consumption is related to the commission and early demand on Product B. With increasing commission, the cost of barter increases and the retailer should decrease the quantity of the bartered product. Therefore, the profit decreases. Given a certain barter commission, if the early demand on Product B increases then the retailer can barter more unsold products. The retailer can move more distressed inventory for bartering which induces ordering more products and decreases the profit. Finally, it can be stated that bartering is always beneficial for the retailer producing a higher profit than the profit of the classical newsvendor with consumption. The above sensitivity analysis is consistent with the results obtained for the model without pricing [19]. Similar conclusions hold for the newsvendor with uncorrelated barter prices.

The shapes of the expected profit functions in three dimensions for correlated and uncorrelated barter prices are presented in Figure 1 and Figure 2, respectively. The maximum profits and the optimal quantities regarding the figures are presented in bold text in Table 3. It should be noted, that the model has to be selected separately for specific products taking into account if their prices depend on each other or not. In fact, the models and the related figures cannot be efficiently compared because of different meaning of parameters Q_e and w_B - in the model with uncorrelated barter prices the value of bartered products is fixed and in the model with prices co-movement it changes with price of Product A.

4. Conclusions

Bartering is not only for individuals looking for baking items or grocery shopping. In barter exchanges for business, participating organizations strive to increase their annual business by 10% to 15% through swapping their services for the services of other businesses. Businesses are increasingly interested in joining barter exchanges, including doctors, lawyers, service companies and retailers. This phenomenon

Table 3. Impact of commission r on profit, retail price and stock factor, $w_B = 0.2$, $Q_e = 2$

Uncorrelated prices				Correlated prices			Classical newsvendor		
r	$E\Pi^*$	p_A^*	z_A^*	$E\Pi^*$	p_A^*	z_A^*	$E\Pi_c^*$	p_A^c	z_A^c
0.05	9.1452	2.3155	1.2309	9.0285	2.3367	1.2487			
0.075	9.1423	2.3154	1.2307	9.0220	2.3369	1.2482			
0.1	**9.1395**	**2.3152**	**1.2304**	**9.0155**	**2.3371**	**1.2478**			
0.125	9.1366	2.3151	1.2302	9.0089	2.3373	1.2474			
0.15	9.1338	2.3150	1.2299	9.0024	2.3375	1.2470			
1.00							8.7834	2.3449	1.2324

Table 4. Impact of value of Product B and the early demand on Product B on profit, price and stock factor, $r = 0.1$

Uncorrelated prices				Correlated prices				Classical newsvendor		
w_B	$E\Pi^*(w_B)$	p_A^*	z_A^*	Q_e	$E\Pi^*(Q_e)$	p_A^*	z_A^*	$E\Pi_c^*(Q_e)$	p_A^c	z_A^c
0.1	9.1884	2.3131	1.2258	1	9.1258	2.3240	1.2345	9.0093	2.3278	1.2267
0.2	9.1395	2.3152	1.2304	2	9.0155	2.3371	1.2478	8.7834	2.3449	1.2324
0.3	9.0906	2.3174	1.2351	3	8.9063	2.3502	1.2611	8.5597	2.3624	1.2381
0.4	9.0417	2.3196	1.2398	4	8.7983	2.3632	1.2743	8.3384	2.3802	1.2438
0.5	8.9928	2.3217	1.2445	5	8.6915	2.3762	1.2875	8.1193	2.3984	1.2495

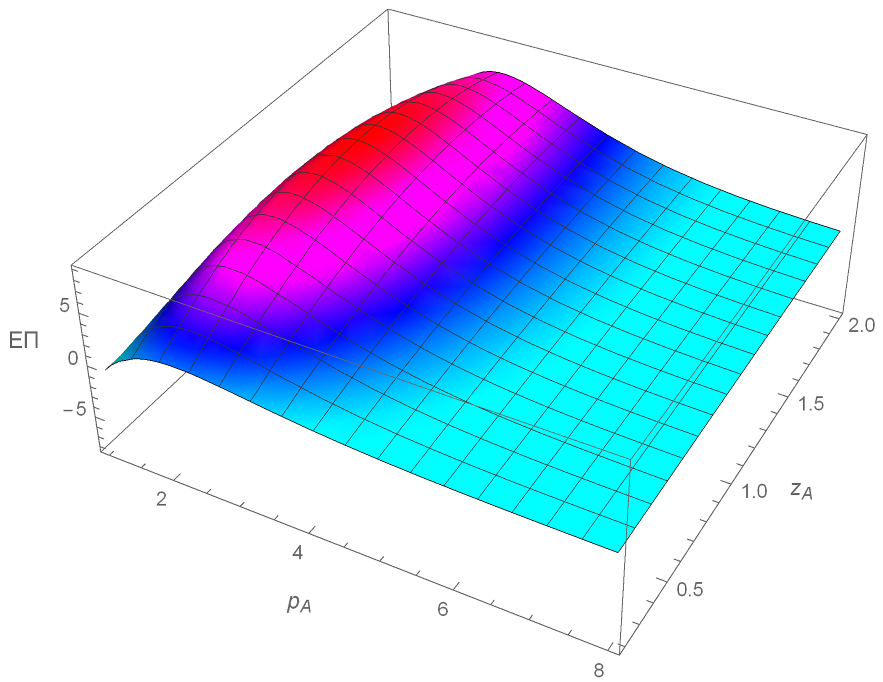

Figure 1. Shape of the expected profit in the case of the co-movement of prices, $(\alpha, \beta, a, b, c, Q_e, r) = (0.2, 2, 100, 1, 1, 2, 0.1)$

was noted by the US-based IRTA, a non-profit organization promoting and advancing modern trade and barter systems. Members can exchange their professional services for barter credit, which they can then use to buy the services of another member. In the major of cases barter constitutes multilateral trade. Barter exchanges have been called “new buying" during the coronavirus pandemic. Bartering is seeing

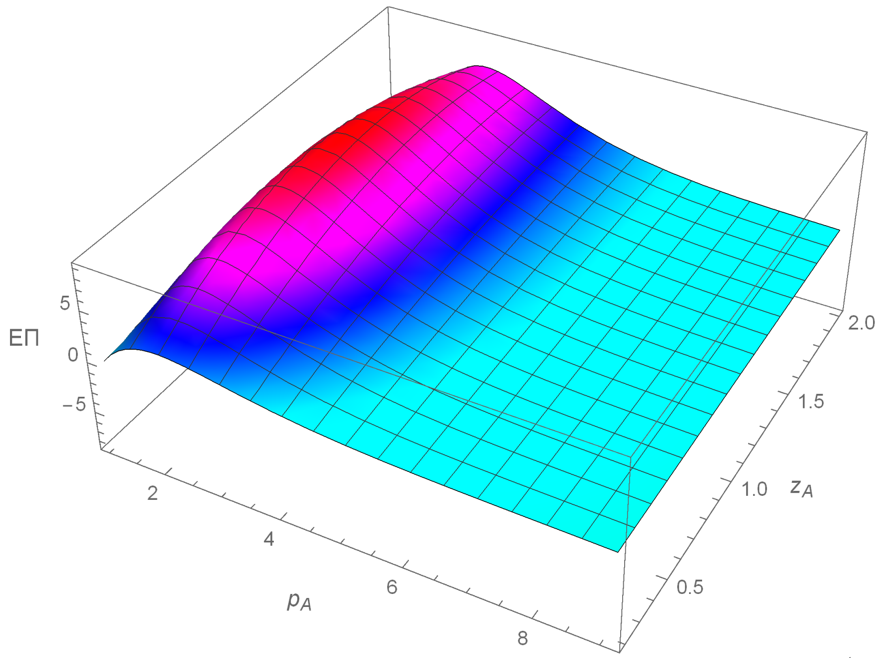

Figure 2. Shape of the expected profit in the case of uncorrelated barter prices, $(\alpha, \beta, a, b, c, w_B, r) = (0.2, 2, 100, 1, 1, 0.2, 0.1)$

more activity and gaining more interest than ever before because both cash and credit are tight [23]. Today's bartering is not a direct trade barter model in a traditional form. Business-to-business trading has been providing this alternative method of payment that conserves cash, attracts new customers and gives businesses a competitive advantage [47].

The objective of this study is to determine the general mathematical model adapted to multiplicative stochastic demand and commercial barter. We solve the examined problem with pricing and assuming that demand is exponentially price-dependent. The sequence of events in the model is such that first a retailer sells Product A on the market and needs a certain quantity of Product B for its own purposes. The retailer can exchange unsold Product A for B on a barter platform. Then, it can buy the rest of the needed portion of Product B on the market. The retailer chooses the order quantity and barter price of Product A which maximizes its expected profit. We consider the possibility of the co-movement of prices of Product A and B and without it. In the first case, we present exact formulas for an optimal order quantity and price. Moreover, we also consider the case where there is no barter exchange but Product B consumption still exists. Due to the mathematical complexity, other cases of the problem can be solved numerically. We examine the sensitivity analysis with respect to the commission and early demand for Product B and present numerical examples. We show that barter exchange is always beneficial for the retailer producing a higher profit than the profit of the classical newsvendor with consumption.

This article generalizes the newsvendor model with barter exchange studied in [19] by introducing pricing and multiplicative uncertainty. The newsvendor problem with bartering and additive demand was solved in [8]. However, in [8] the focus was on the possibility of negative demand realizations. Therefore, this study also complements the results of [8] with another type of uncertainty. It should be noted that there are no strict rules as to which kind of uncertainty should be used in specific cases. But, it was shown that it can be linked to specific products [12]. Finally, the novelty lies in the use of the mixture failure rate in the proofs of the main results. We consider the endogenous price in this study which suites the reality better than assuming fixed price. However, it appeared that general implications are similar to those obtained in the exogenous price model. The newsvendor problem with barter exchange and multiplicative demand can be more beneficial to the retailer compared to the model of the classical newsvendor. Therefore, our work may provide a guidance for the retailer on how to make the decisions on a barter platform.

The newsvendor model with bartering studied in this article is considered from the retailer's perspective. Future studies can investigate the supply chain models with two or more actors under various agreements. As a building block of coordinating contracts, the newsvendor model incorporates the fundamental technique for stochastic decision-making. Therefore, a barter exchange probably could be introduced to these contracts and increase the supply chain members' profits. Moreover, the natural extension may be the multi-period or multi-product inventory model with bartering for a retailer.

Acknowledgement: This work was supported by the National Science Centre, Poland under Grant 2019/35/D/HS4/00801.

References

[1] Agrawal, V., and Seshadri, S. Impact of uncertainty and risk aversion on price and order quantity in the newsvendor problem. *Manufacturing & Service Operations Management* 2, 4 (2000), 410–423.

[2] Anderlini, L., and Sabourian, H. Some notes on the economics of barter, money and credit. In *Barter, Exchange and Value: An Anthropological Approach*, C. Humphrey and S. Hugh-Jones, Eds. Cambridge University Press, 1992, pp. 75–101.

[3] Anderson, R., Ashlagi, I., Gamarnik, D., and Kanoria, Y. Efficient dynamic barter exchange. *Operations Research* 65, 6 (2017), 1446–1459.

[4] Antweiler, W. The virtues of negative exponential demand. *Werner's Blog - Opinion, Analysis, Commentary* (30 June 2018), accessed 7 June 2021.

[5] Bartik, A. W., Bertrand, M., Cullen, Z., Glaeser, E. L., Luca, M., and Stanton, C. The impact of COVID-19 on small business outcomes and expectations. *Proceedings of the National Academy of Sciences* 117, 30 (2020), 17656–17666.

[6] Battistella, L. M., and Hynes, N. Something new under the sun? A case study of modern Italian barter systems. *Journal of Applied Business and Economics* 21, 2 (2019), 27–50.

[7] Bearne, S. A barter way of life: Covid crisis fuels swapping of goods and skills. *The Guardian* (31 October 2020), accessed 25 May 2021.

[8] Bieniek, M. Bartering: price-setting newsvendor problem with barter exchange. *Sustainability* 13, 12 (2021), 1–22.

[9] Chaudhari, U., Shah, N., and Jani, M. Inventory modelling of deteriorating item and preservation technology with advance payment scheme under quadratic demand. In *Optimization and Inventory Management. Asset Analytics (Performance and Safety Management)*, N. Shah and M. Mittal, Eds. Springer, Singapore, 2020, pp. 69–79.

[10] Dalalah, D., Khasawneh, M., and Khan, S. Pricing and demand management of air tickets using a multiplicative newsvendor model. *Journal of Revenue and Pricing Management* 21, 5 (2022), 517–528.

[11] Dalton, G. Barter. *Journal of Economic Issues* 16, 1 (1982), 181–190.

[12] DeYong, G. D. The price-setting newsvendor: review and extensions. *International Journal of Production Research* 58, 6 (2020), 1776–1804.

[13] Efremova, O. I., and Gabdullin, L. V. Logistic barter features as a new concept of supply chain management. *Academy of Strategic Management Journal* 15, Special Issue 2 (2016), 19–25.

[14] Erridge, A., and Zhabykenov, D. The role of purchasing in countertrade. *European Journal of Purchasing and Supply Management* 4, 2-3 (1998), 97 – 107.

[15] Esmaeili, A., and Shokoohi, Z. Assessing the effect of oil price on world food prices: application of principal component analysis. *Energy Policy* 39, 2 (2011), 1022–1025.

[16] Finkelstein, M. Understanding the shape of the mixture failure rate (with engineering and demographic applications). *Applied Stochastic Models in Business and Industry* 25, 6 (2009), 643–663.

[17] Gunia, B. C., and Lewicki, R. J. Bartering as a blind spot: a call to action from Covid-19. *Negotiation Journal* 36, 4 (2020), 561–572.

[18] Guriev, S., and Kvassov, D. Barter for price discrimination. *International Journal of Industrial Organization* 22, 3 (2004), 329–350.

[19] Hua, G., Zhang, Y., Cheng, T., Wang, S., and Zhang, J. The newsvendor problem with barter exchange. *Omega* 92 (2020), 102149.

[20] Huang, Y., Pi, Z., and Fang, W. Trade credit with barter in a capital-constrained supply chain. *Sustainability* 13, 20 (2021), 11361.

[21] IRTA. The barter and trade industry (2021), accessed 15 March 2021.

[22] Jevons, W. S. *Money and the Mechanism of Exchange*, vol. 17, D. Appleton, New York, 1876.

[23] Jones, J. Could bartering become the new buying in a changed world? *Worklife* (27 August 2020), accessed 25 May 2021.

[24] Karlin, S., and Carr, C. R. Prices and optimal inventory policy. *Studies in Applied Probability and Management Science* 4, 1 (1962), 159–172.

[25] Kharif, O. The return of the barter economy, swapping eggs for toilet paper. *Bloomberg* (31 Mach 2020), accessed 25 May 2021.

[26] Khouja, M. The single-period (news-vendor) problem: literature review and suggestions for future research. *Omega* 27, 5 (1999), 537–553.

[27] Kirshner, S. N., and Shao, L. The overconfident and optimistic price-setting newsvendor. *European Journal of Operational Research* 277, 1 (2019), 166–173.

[28] Koirala, K. H., Mishra, A. K., D'Antoni, J. M., and Mehlhorn, J. E. Energy prices and agricultural commodity prices: testing correlation using copulas method. *Energy* 81 (2015), 430–436.

[29] Kregel, J. The economic problem: from barter to commodity money to electronic money. *Levy Economics Institute, Working Papers Series* (2021), 1–33.

[30] Liu, L., and Ma, G. Cross-correlation between crude oil and refined product prices. *Physica A: Statistical Mechanics and its Applications* 413 (2014), 284–293.

[31] Melis, C., and Guidici, E. Is barter a strategic response to the global capitalist crisis? In *Capitalism and the Social Relationship*, H. Kazeroony and A. Stachowicz-Stanusch, Eds. Palgrave Macmillan, London, 2014, pp. 265–277.

[32] Ngoc, T. V. Barter online network. *CRIS - Bulletin of the Centre for Research and Interdisciplinary Study* 2015, 2 (2015), 7–24.

[33] Pal, D., and Mitra, S. K. Interdependence between crude oil and world food prices: A detrended cross correlation analysis. *Physica A: Statistical Mechanics and its Applications* 492 (2018), 1032–1044.

[34] Patnaik, A. Price co-movements, commonalities and responsivness to monetary policy: empirical analysis under Indian conditions. *Asia-Pacific Sustainable Development Journal* 25, 2 (2018), 77–97.

[35] Petruzzi, N., and Dada, M. Pricing and newsvendor problem: a review with extensions. *Operations Research* 47, 2 (1999), 183–194.

[36] Plank, R. E., Reid, D. A., and Bates, F. Barter: An alternative to traditional methods of purchasing. *International Journal of Purchasing and Materials Management* 30, 1 (1994), 51–57.

[37] Polanco Martínez, J. M., Abadie, L. M., and Fernández-Macho, J. A multi-resolution and multivariate analysis of the dynamic relationships between crude oil and petroleum-product prices. *Applied Energy* 228 (2018), 1550–1560.

[38] Porteus, E. L. Stochastic inventory theory. In *Stochastic Models*, D. P. Heyman and M. J. Sobel, Eds., vol. 2 of *Handbooks in Operations Research and Management Science*. Elsevier, 1990, pp. 605–652.

[39] Qin, Y., Wang, R., Vakharia, A. J., Chen, Y., and Seref, M. M. The newsvendor problem: review and directions for future research. *European Journal of Operational Research* 213, 2 (2011), 361–374.

[40] Ru, J., and Wang, Y. Consignment contracting: Who should control inventory in the supply chain? *European Journal of Operational Research* 201, 3 (2010), 760–769.

[41] Rubio-Herrero, J., and Baykal-Gürsoy, M. Mean-variance analysis of the newsvendor problem with price-dependent, isoelastic demand. *European Journal of Operational Research* 283, 3 (2019), 942–953.

[42] Seabright, P., Ed. *The Vanishing Rouble: Barter Networks and Non-monetary Transactions in Post-Soviet Societies*. Cambridge University Press, 2000.

[43] Sen, N., and Saha, S. Inventory model for deteriorating items with negative exponential demand, probabilistic deterioration and fuzzy lead time under partial back logging. *Operations Research and Decisions* 30, 3 (2020), 97–112.

[44] Shah, N. H., Jani, M. Y., and Chaudhari, U. Retailer's optimal policies for price-credit dependent trapezoidal demand under two-level trade credit. *International Journal of Operations and Quantitative Management* 23, 2 (2017), 115–130.

[45] Shah, N. H., Jani, M. Y., and Chaudhari, U. Optimal ordering policy for deteriorating items under down-stream trade credit dependent quadratic demand with full up-stream trade credit and partial down-stream trade credit. *International Journal of Mathematics in Operational Research* 12, 3 (2018), 378–396.

[46] Shi, R., Zhang, J., and Ru, J. Impacts of power structure on supply chains with uncertain demand. *Production and Operations Management* 22, 5 (2013), 1232–1249.

[47] Strabley, J. Back to barter: A solution for struggling paint and coatings companies. *Paint and Coatings Industry* (13 July 2020), accessed 25 May 2021.

[48] Traore, F., and Badolo, F. On the co-movement between coffee and cocoa prices in international markets. *Applied Economics* 48, 40 (2016), 3877–3886.

[49] Whitin, T. M. Inventory control and price theory. *Management Science* 2, 1 (1955), 61–68.

[50] Williams, C. C. The new barter economy: an appraisal of local exchange and trading systems (LETS). *Journal of Public Policy* 16, 1 (1996), 85–101.

[51] Ye, T., and Sun, H. Price-setting newsvendor with strategic consumers. *Omega* 63 (2016), 103 – 110.

[52] Yu, L., Pei, J., Liu, X., Fan, W., and Pardalos, P. M. The effect of yield rate in a general price-setting newsvendor model with a yield-dependent secondary market. *International Transactions in Operational Research* 26, 6 (2019), 2337–2361.

[53] Zabel, E. Monopoly and uncertainty. *The Review of Economic Studies* 37, 2 (1970), 205–219.

[54] Zhang, B., Chen, M., and Wei, L. Impacts of barter exchange and decision biases in a two-level supply chain under pull contract. *International Transactions in Operational Research* 29, 3 (2022), 1868–1896.

A. Appendix

Proof of Lemma 1. Using the formula for the function of $\Lambda(z)$ and the standard algebraic computations we obtain the desired equations. □

Proof of Theorem 1. By solving the first order condition

$$\frac{\delta \mathrm{E}\Pi(p_A, z_A)}{\delta p_A} = a\mathrm{e}^{-bp_A}(-bg(z_A)p_A + bcz_A + g(z_A)) = 0$$

where

$$g(z_A) := z_A - \frac{Q_e}{a} - r\Lambda(z_A) - (1-r)\Lambda(z_A - \frac{Q_e}{a})$$

we obtain the optimal price function given by (14). Moreover, it should be noted that $\delta \mathrm{E}\Pi(p_A, z_A)/\delta p_A > 0$ for all $p_A < p_A^*(z_A)$, $\delta \mathrm{E}\Pi(p_A, z_A)/\delta p_A < 0$ for all $p_A > p_A^*(z_A)$ if $g(z_A) > 0$, which assures that $p_A^*(z_A)$ is a unique maximum. It can be seen that $g'(z_A) = r\bar{F}(z_A) + (1-r)\bar{F}(z_A - \frac{Q_e}{a}) > 0$ which means that $g(z_A)$ is increasing and it is enough to assume that $g(\alpha) = \alpha - \frac{Q_e}{a} > 0$ which is assured by assumption (2). Moreover, since $z_A - \Lambda(z_A)$ is always positive and increasing function of $z_A \in [\alpha, \beta]$ and $g(z_A)$ can be written as $g(z_A) = (1-r)(z_A - \frac{Q_e}{a} - \Lambda(z_A - \frac{Q_e}{a})) + r(z_A - \Lambda(z_A)) - r\frac{Q_e}{a}$, then $p_A^*(z_A) > c + \frac{1}{b} + c\frac{\Lambda(z_A) + r\frac{Q_e}{a}}{z_A - \Lambda(z_A) - r\frac{Q_e}{a}} > c$.

Now, we examine the quantity z_A^* which maximizes $\mathrm{E}\Pi(p_A^*(z_A), z_A)$ with $p_A^*(z_A)$ given by (14). We need the first derivative of expected profit with respect to z_A. It is given by

$$\frac{\mathrm{d}\mathrm{E}\Pi(p_A^*(z_A), z_A)}{\mathrm{d}z_A} = \frac{a\mathrm{e}^{-bp_A^*(z_A)}}{b(z_A - \frac{Q_e}{a} - r\Lambda(z_A) - (1-r)\Lambda(z_A - \frac{Q_e}{a}))} W(z_A) \tag{26}$$

where

$$\begin{aligned} W(z_A) &= \left(bcz_A + z_A - \frac{Q_e}{a} - r\Lambda(z_A) - (1-r)\Lambda(z_A - \frac{Q_e}{a})\right) \\ &\cdot \left(1 - rF(z_A) - (1-r)F(z_A - \frac{Q_e}{a})) - bc(z_A - \frac{Q_e}{a} - r\Lambda(z_A) - (1-r)\Lambda(z_A - \frac{Q_e}{a})\right) \end{aligned}$$

Because the first factor in (26) is always positive, the first order condition requires that the optimal z_A^* satisfies $W(z_A^*) = 0$, which gives (15). Quantity z_A^* always exists in interval (α, β), since $W(z)$ is continuous, $W(\alpha) = \alpha - \frac{Q_e}{a} + bc\frac{Q_e}{a} > 0$ by assumption (2) and $W(\beta) < 0$ by assumption (16). For simplicity let $h := h(z, r, \frac{Q_e}{a})$. To verify the uniqueness of z_A^* let us derive

$$\begin{aligned} W'(z_A) &= (1 - rF(z_A) - (1-r)F(z_A - \frac{Q_e}{a})) \\ &\left[1 - rF(z_A) - (1-r)F(z_A - \frac{Q_e}{a}) - h(bcz_A + z_A - \frac{Q_e}{a} - r\Lambda(z_A) - (1-r)\Lambda(z_A - \frac{Q_e}{a}))\right] \end{aligned}$$

Then

$$\begin{aligned} W''(z_A)|_{W'(z_A)=0} &= -hW'(z_A) + (1 - rF(z_A) - (1-r)F(z_A - \frac{Q_e}{a})) \\ &\left[-rf(z_A) - (1-r)f(z_A - \frac{Q_e}{a}) - h'(bcz_A + z_A - \frac{Q_e}{a} - r\Lambda(z_A) - (1-r)\Lambda(z_A - \frac{Q_e}{a})) \right. \\ &\left. -h(bc + 1 - rF(z_A) - (1-r)F(z_A - \frac{Q_e}{a}))\right] \end{aligned}$$

Now, we can observe that if $h' > 0$ then $W''(z_A) < 0$ at $W'(z_A) = 0$, which implies that $W(z_A)$ is a unimodal function. It also guarantees the uniqueness of $z_A^* \in [\alpha, \beta]$. The proof is complete. □

Proof of Theorem 2. The mathematical proof is similar to the proof of Theorem 1 with $r = 1$. □

Proof of Proposition 1. It can be seen that for a given p_A, the first derivative is equal to

$$\frac{\delta \mathrm{E}\Pi(p_A, z_A)}{\delta z_A} = ae^{-bp_A}\left(p_A - c - rp_A F(z_A) - p_A(1-r)F\left(z_A - \frac{w_B}{ap_A}\mathrm{e}^{bp_A}\right)\right)$$

and therefore $\frac{\delta \mathrm{E}\Pi(p_A,z_A)}{\delta z_A}|_{z_A=\alpha} = ae^{-bp_A}(p_A - c) > 0$ and the second derivative given by

$$\frac{\delta^2 \mathrm{E}\Pi(p_A, z_A)}{\delta z_A^2} = -ae^{-bp_A}\left(rp_A f(z_A) + p_A(1-r)f\left(z_A - \frac{w_B}{ap_A}\mathrm{e}^{bp_A}\right)\right)$$

is negative, which implies the statement of the lemma. □

Proof of Lemma 2. Using (17) and (22) we obtain

$$\begin{aligned}
&\frac{b\mathrm{e}}{a}(\mathrm{E}\Pi(p_A^*(z_A), z_A) - \mathrm{E}\Pi_c(p_A^c(z_A), z_A))\\
&= \mathrm{e}^{\frac{-bcz_A}{z_A - \frac{Q_e}{a} - r\Lambda(z_A) - (1-r)\Lambda(z_A - \frac{Q_e}{a})}}\left(z_A - \frac{Q_e}{a} - \Lambda(z_A) - (1-r)(\Lambda(z_A) - \Lambda(z_A - \frac{Q_e}{a}))\right)\\
&- \mathrm{e}^{\frac{-bcz_A}{z_A - \frac{Q_e}{a} - \Lambda(z_A)}}\left(z_A - \frac{Q_e}{a} - \Lambda(z_A)\right) > \mathrm{e}^{\frac{-bcz_A}{z_A - \frac{Q_e}{a} - \Lambda(z_A)}}(1-r)(\Lambda(z_A) - \Lambda(z_A - \frac{Q_e}{a})) > 0
\end{aligned}$$

The proof is complete. □

Proof of Proposition 2. By the chain rule

$$\frac{\mathrm{d}\mathrm{E}\Pi^*}{\mathrm{d}Q_e} = \frac{\delta \mathrm{E}\Pi^*}{\delta Q_e} + \frac{\delta \mathrm{E}\Pi^*}{\delta z_A^*}\frac{\mathrm{d}z_A^*}{\mathrm{d}Q_e},$$

which gives $\frac{\mathrm{d}\mathrm{E}\Pi^*}{\mathrm{d}Q_e} = \frac{\delta \mathrm{E}\Pi^*(Q_e)}{\delta Q_e}$. By (17), (2) and the fact that $\Lambda(z)$ is increasing,

$$\begin{aligned}
&\operatorname{sgn}\left\{\frac{\mathrm{d}\mathrm{E}\Pi^*}{\mathrm{d}Q_e}\right\}\\
&= \operatorname{sgn}\left\{((1-r)F(z_A^* - \frac{Q_e}{a}) - 1)\frac{bcz_A^* + z_A^* - \frac{Q_e}{a} - \Lambda(z_A^*) - (1-r)(\Lambda(z_A^*) - \Lambda(z_A^* - \frac{Q_e}{a}))}{z_A^* - \frac{Q_e}{a} - \Lambda(z_A^*) - (1-r)(\Lambda(z_A^*) - \Lambda(z_A^* - \frac{Q_e}{a}))}\right\} < 0
\end{aligned}$$

which implies that $\mathrm{E}\Pi^*$ decreases with Q_e. Similarly,

$$\operatorname{sgn}\left\{\frac{\mathrm{d}\mathrm{E}\Pi_c^*}{\mathrm{d}Q_e}\right\} = \operatorname{sgn}\{-\frac{bcz_A^c}{z_A^c - \frac{Q_e}{a} - \Lambda(z_A^c)} - 1\} < 0$$

implies that $\mathrm{E}\Pi_c^*(Q_e)$ also decreases with Q_e. Moreover,

$$\begin{aligned}
&\operatorname{sgn}\left\{\frac{\mathrm{d}\mathrm{E}\Pi^*}{\mathrm{d}r}\right\}\\
&= \operatorname{sgn}\{(\Lambda(z_A^* - \frac{Q_e}{a}) - \Lambda(z_A^*))(bcz_A^* + z_A^* - \frac{Q_e}{a} - r\Lambda(z_A^*) - (1-r)\Lambda(z_A^* - \frac{Q_e}{a}))\} < 0
\end{aligned}$$

which implies that $\mathrm{E}\Pi^*(r)$ decreases with r. By Lemma 2, $\mathrm{E}\Pi^* > \mathrm{E}\Pi(p_A^*(z_A^c), z_A^c) > \mathrm{E}\Pi_c(p_A^c(z_A^c), z_A^c) = \mathrm{E}\Pi_c^*$, which ends the proof. □