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A UNIQUE WEAK SOLUTION FOR A KIND OF COUPLED SYSTEM OF FRACTIONAL SCHRÖDINGER EQUATIONS

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Abstract. In this paper, we prove the existence of a unique weak solution for a class of fractional systems of Schrödinger equations by using the Minty–Browder theorem in the Cartesian space. To this aim, we need to impose some growth conditions to control the source functions with respect to dependent variables.

Keywords: fractional Laplacian, uniqueness, weak solution, nonlinear system.

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1. INTRODUCTION AND PRELIMINARIES

Fractional differential equations (FDEs) are used in the study of fluid flow, diffusive transport akin to diffusion, rheology, probability, electrical networks, etc. [1, 8, 14, 15, 17–23]. There is a wide range of works deal with fractional equations with fractional Laplacian, for example, readers are encouraged to study [5, 7, 10, 16, 25, 28]. Recently, the existence of solutions for Schrödinger–Hardy systems, p-fractional Hardy–Schrödinger–Kirchhoff systems as well as a class of systems involving fractional (p, q)-Laplacian operators (see [11–13]) are studied.

Here, we consider the following fractional Schrödinger coupled system

$$\begin{cases} (-\Delta^s)u + V_1(x)u + f(x,u,v) = \lambda u \quad \text{for all } x \in \mathbb{R}^N, \\ (-\Delta^s)v + V_2(x)v + g(x,u,v) = \lambda v \quad \text{for all } x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $N \geq 2$, $s \in (0,1)$, $(-\Delta^s)$ denotes the fractional Laplacian and $f, g \in C(\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$. The fractional Laplacian $(-\Delta^s)$ with $s \in (0,1)$ of a function $\phi \in \varphi$ is defined by (see [2, 27])

$$\Lambda((-\Delta^s)\phi)(k) = |k|^{2s}\Lambda(\phi)(k) \quad \text{for all } s \in (0,1),$$

where φ denotes the Schwartz space consisting of rapidly decreasing C^∞-functions in $\mathbb{R}^N$ and Λ is the Fourier transform, i.e.

$$\Lambda(\phi)(k) = \frac{1}{(2\pi)^{N/2}} \int\limits_{\mathbb{R}^N} \exp\{-2\pi ikx\}\phi(x)dx.$$

Another definition of the fractional Laplacian of a smooth enough function ϕ can be given by the following singular integral

$$(-\Delta^s)\phi(x) = c_{N,S}\, P.V. \int\limits_{\mathbb{R}^N} \frac{\phi(x)-\phi(y)}{|x-y|^{N+2s}}dy,$$

where $P.V.$ is the principal value and

$$c_{N,S} = \frac{4^s\Gamma(N/2+s)}{\pi^{N/2}|\Gamma(-s)|}.$$

Recently, much attention have been given to the investigation of fractional local problems such as some new contributions on the study of the existence of positive solutions to the critical fractional Laplacian elliptic Dirichlet equations in a bounded domain [3].

An important class of these types of problems is the wave solutions of fractional Schrödinger equations. We refer the reader to [9, 24, 26].

Xu *et al.* proved in [27] the existence of a unique nontrivial solution for the following problem

$$(-\Delta^s)u + V(x)u + f(x,u) = \lambda u \quad \text{for all } x \in \mathbb{R}^N, \tag{1.2}$$

where $N \geq 2$, $s \in (0,1)$, $\lambda \in \mathbb{R}$ and $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$. With this motivation, we extend the above Sturm–Liouville problem to a coupled system with two different potential known energy functions and two different unknown wave functions. To this aim, we restrict ourselves to some new suppositions by defining some new convenient spaces.

In the last part of this section, we present some definitions and notations that we need throughout the paper.

Here, we assume that the following conditions hold:

(i) $V \in C(\mathbb{R}^N, \mathbb{R})$, $\mathcal{V} = \inf_{x\in\mathbb{R}^N} V(x) > 0$,
(ii) $\exists r_0 > 0\, \forall M > 0 : \text{meas}\{x \in B(y; r_0) : V(x) \leq M\} \to 0$ $(|y| \to \infty)$.

Definition 1.1. The space $H^s(\mathbb{R}^N)$ is defined by

$$H^s(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) : \int\limits_{\mathbb{R}^N} (|\xi|^{2s}\hat{u}^2 + \hat{u}^2)d\xi < \infty \right\},$$

where $\hat{u} = \Lambda(u)$ with respect to the norm

$$\|u\|_{H^s(\mathbb{R}^N)} = \left(\int\limits_{\mathbb{R}^N} (|\xi|^{2s}\hat{u}^2 + \hat{u}^2)d\xi \right)^{\frac{1}{2}}.$$

Due to the appearance of potential energies $V_1(x)$ and $V_2(x)$ in the system of equations (1.1), we introduce the subspace

$$E = \left\{ u \in H^s(\mathbb{R}^N) : \int\limits_{\mathbb{R}^N} V(x)u^2 dx < \infty \right\}, \tag{1.3}$$

and the norm

$$\|u\|_E = \left(\int\limits_{\mathbb{R}^N} (|\xi|^{2s}\hat{u}^2 + \hat{u}^2)d\xi + \int\limits_{\mathbb{R}^N} V(x)u^2 dx \right)^{\frac{1}{2}}.$$

Notice that E is an inner product space by introducing the Sobolev inner product $\prec \cdot, \cdot \succ_E$ which is defined by

$$\prec u, v \succ_E = \int\limits_{\mathbb{R}^N} \Big((-\Delta)^{\frac{s}{2}} u(x)(-\Delta)^{\frac{s}{2}} v(x) + V(x)u(x)v(x) \Big) dx,$$

for all $u, v \in E$.

By Plancherel's theorem and condition (i), it is easily seen that $\|\cdot\|_E$ is equivalent to

$$\|u\| = \left(\int\limits_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}} u|^2 + V(x)u^2) dx \right)^{\frac{1}{2}}. \tag{1.4}$$

Let X be the Cartesian product $X = E \times E$. This is a Hilbert space with the corresponding product norm $\|(u,v)\|_X = (\|u\|^2 + \|v\|^2)^{\frac{1}{2}}$.

Lemma 1.2 ([6]). *The space E defined by* (1.3) *is continuously embedded into $L^p(\mathbb{R}^N)$ for $p \in [2, 2_s^*]$ and compactly embedded into $L^p_{loc}(\mathbb{R}^N)$ for $p \in [2, 2_s^*)$, where $2_s^* = \frac{2N}{N-2s}$, that is, there exists a positive constant c_p such that*

$$\|u\|_p \le c_p \parallel u \parallel \quad \textit{for all } p \in [2, 2_s^*]. \tag{1.5}$$

Remark 1.3. One can extend the preceding lemma for the general space X as

$$\|u\|_p^2 + \|v\|_p^2 \le S(\parallel u \parallel^2 + \parallel v \parallel^2) = S\|(u,v)\|_X^2, \tag{1.6}$$

where S is the maximum of Sobolev constants c_p for u and v.

Lemma 1.4 ([27]). *The space E defined by* (1.3) *is compactly embedded into $L^p(\mathbb{R}^N)$ for $p \in [2, 2_s^*)$.*

Remark 1.5. According to Lemma 1.4, one can show that X is compactly embedded into $\big(L^p(\mathbb{R}^N)\big)^2$ for $p \in [2, 2_s^*)$.

Definition 1.6. We say that $(u, v) \in X$ is a weak solution of the system of equations (1.1) if

$$\begin{aligned}&\int\limits_{\mathbb{R}^N}\Big((-\triangle)^{\frac{s}{2}}u(-\triangle)^{\frac{s}{2}}\phi_1 + V_1(x)u\phi_1\Big)dx + \int\limits_{\mathbb{R}^N} f(x,u,v)\phi_1 dx\\ &+\int\limits_{\mathbb{R}^N}\Big((-\triangle)^{\frac{s}{2}}v(-\triangle)^{\frac{s}{2}}\phi_2 + V_2(x)v\phi_2\Big)dx + \int\limits_{\mathbb{R}^N} g(x,u,v)\phi_2 dx\\ &-\lambda\Bigg(\int\limits_{\mathbb{R}^N} u\phi_1 dx + \int\limits_{\mathbb{R}^N} v\phi_2 dx\Bigg) = 0\end{aligned}$$

for all $\phi = (\phi_1, \phi_2) \in X$.

The next result, that is due to Minity and Browder, is useful for reaching our purpose.

Theorem 1.7 ([4]). *Let E be a reflexive Banach space and $A : X \to X^*$ be a continuous nonlinear map such that*

(i) $\prec Au_1 - Au_2, u_1 - u_2 \succ > 0$ *for all* $u_1, u_2 \in E,\ u_1 \neq u_2$,
(ii) *A is coercive.*

Then for every $\mathcal{F}$ in E^*, there exists unique $u \in E$ such that $Au = \mathcal{F}$.

2. THE MAIN RESULT

Here, we prove the existence of a unique nontrivial weak solution of problem (1.1).

Theorem 2.1. *Assume that the following conditions hold:*

(H_1) *There exist $c_1, c_2, d_1, d_2 > 0$ and $P \in [2, 2_s^*)$ such that for all $(x, s, t) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}$ we have*

$$|f(x,s,t)| \leq \alpha_1(x) + c_1|s|^{p-1} + d_1|t|^{p-1}$$

and

$$|g(x,s,t)| \leq \alpha_2(x) + c_2|s|^{p-1} + d_2|t|^{p-1},$$

where $\alpha_i \in L^q(\mathbb{R}^N)$, with $i = 1, 2$, and $q \in (\frac{2N}{N+2s}, 2]$. Further, $f(x, 0, v)$, $g(x, u, 0) \in L^q(\mathbb{R}^N \times \mathbb{R})$.

(H_2) *For all $x \in \mathbb{R}^N$ and $s_1, s_2, t_1, t_2 \in \mathbb{R}$ such that $s_1 \neq s_2, t_1 \neq t_2$, we have*

$$\frac{f(x,s_1,t_1) - f(x,s_2,t_2)}{s_1 - s_2} \geq \mu^*$$

and

$$\frac{g(x,s_1,t_1) - g(x,s_2,t_2)}{t_1 - t_2} \geq \mu^*.$$

Then problem (1.1) has a unique nontrivial weak solution $(u, v) \in X$ for $\lambda = \mu^*$.

Proof. Take the operator $A : X \to X^*$ as follows:

$$\begin{aligned}\prec A(u,v),(\phi_1,\phi_2)\succ =& \int\limits_{\mathbb{R}^N} \left((-\triangle)^{\frac{s}{2}}u(-\triangle)^{\frac{s}{2}}\phi_1 + V_1(x)u\phi_1\right)dx\\ &+ \int\limits_{\mathbb{R}^N} \left((-\triangle)^{\frac{s}{2}}v(-\triangle)^{\frac{s}{2}}\phi_2 + V_2(x)v\phi_2\right)dx\\ &+ \int\limits_{\mathbb{R}^N} f(x,u,v)\phi_1 dx + \int\limits_{\mathbb{R}^N} g(x,u,v)\phi_2 dx\\ &- \lambda\Bigg(\int\limits_{\mathbb{R}^N} u\phi_1 dx + \int\limits_{\mathbb{R}^N} v\phi_2 dx\Bigg),\end{aligned}$$

for all $u, v, \phi_1, \phi_2 \in E$. Notice that

$$\begin{aligned}\left|\int\limits_{\mathbb{R}^N} f(x,u,v)\phi_1 dx\right| &\le \int\limits_{\mathbb{R}^N} (\alpha_1(x) + c_1|u|^{p-1} + d_1|v|^{p-1})|\phi_1|dx\\ &\le \|\alpha_1\|_q\|\phi_1\|_p + c_1\|\phi_1\|_p\|u\|_p^{p-1} + d_1\|\phi_1\|_p\|v\|_p^{p-1}\\ &< \infty\end{aligned}$$

for all $u, v, \phi_1, \phi_2 \in E$. In a similar way, we get

$$\left|\int\limits_{\mathbb{R}^N} g(x,u,v)\phi_2 dx\right| < \infty.$$

On the other hand,

$$\left|\int\limits_{\mathbb{R}^N} u\phi_1 dx\right| \le \|u\|_2\|\phi_1\|_2 < \infty,$$

and

$$\left|\int\limits_{\mathbb{R}^N} v\phi_2 dx\right| \le \|v\|_2\|\phi_2\|_2 < \infty.$$

Thus, $\prec A(u,v),\phi \succ \in X^*$, that is, A is well-defined. It is sufficient to investigate that A satisfies Theorem 1.7. We have

$$\begin{aligned}
&\prec A(u_1,v_1)-A(u_2,v_2),(u_1-u_2,v_1-v_2)\succ\\
&=\prec A(u_1,v_1),(u_1-u_2,v_1-v_2)\succ - \prec A(u_2,v_2),(u_1-u_2,v_1-v_2)\succ\\
&=\int\limits_{\mathbb{R}^N}((-\triangle)^{\frac{s}{2}}u_1(-\triangle)^{\frac{s}{2}}(u_1-u_2)+V_1(x)u_1(u_1-u_2))dx\\
&\quad+\int\limits_{\mathbb{R}^N}(((-\triangle)^{\frac{s}{2}}v_1(-\triangle)^{\frac{s}{2}}(v_1-v_2))+V_2(x)v_1(v_1-v_2))dx\\
&\quad+\int\limits_{\mathbb{R}^N}f(x,u_1,v_1)(u_1-u_2)dx+\int\limits_{\mathbb{R}^N}g(x,u_1,v_1)(v_1-v_2)dx\\
&\quad-\mu^*\Bigg(\int\limits_{\mathbb{R}^N}(u_1(u_1-u_2)+v_1(v_1-v_2))dx\Bigg)\\
&\quad-\int\limits_{\mathbb{R}^N}(((-\triangle)^{\frac{s}{2}}u_2(-\triangle)^{\frac{s}{2}}(u_1-u_2))+V_1(x)u_2(u_1-u_2))dx\\
&\quad-\int\limits_{\mathbb{R}^N}(((-\triangle)^{\frac{s}{2}}v_2(-\triangle)^{\frac{s}{2}}(v_1-v_2))+V_2(x)v_2(v_1-v_2))dx\\
&\quad-\int\limits_{\mathbb{R}^N}f(x,u_2,v_2)(u_1-u_2)dx-\int\limits_{\mathbb{R}^N}g(x,u_2,v_2)(v_1-v_2)dx\\
&\quad+\mu^*\Bigg(\int\limits_{\mathbb{R}^N}(u_2(u_1-u_2)+v_2(v_1-v_2))dx\Bigg)\\
&=\int\limits_{\mathbb{R}^N}(|(-\triangle)^{\frac{s}{2}}(u_1-u_2)|^2+V_1(x)|u_1-u_2|^2)dx\\
&\quad+\int\limits_{\mathbb{R}^N}(|(-\triangle)^{\frac{s}{2}}(v_1-v_2)|^2+V_2(x)|v_1-v_2|^2)dx\\
&\quad+\int\limits_{\mathbb{R}^N}(f(x,u_1,v_1)-f(x,u_2,v_2))(u_1-u_2)dx\\
&\quad+\int\limits_{\mathbb{R}^N}(g(x,u_1,v_1)-g(x,u_2,v_2))(v_1-v_2)dx\\
&\quad-\mu^*\Bigg(\int\limits_{\mathbb{R}^N}(|u_1-u_2|^2+|v_1-v_2|^2)dx\Bigg).
\end{aligned}$$

By (H_2), since $u_1 \neq u_2$ and $v_1 \neq v_2$, this implies that

$$\prec A(u_1, v_1) - A(u_2, v_2), (u_1 - u_2, v_1 - v_2) \succ > 0.$$

To complete the proof, it suffices to show that A is coercive. By definition

$$\begin{aligned} < A(u,v),(u,v)> = & \int\limits_{\mathbb{R}^N} \left(|(-\triangle)^{\frac{s}{2}} u|^2 + V_1(x) u^2\right) dx \\ & + \int\limits_{\mathbb{R}^N} \left(|(-\triangle)^{\frac{s}{2}} v|^2 + V_2(x) v^2\right) dx + \int\limits_{\mathbb{R}^N} f(x,u,v) u dx \\ & + \int\limits_{\mathbb{R}^N} g(x,u,v) v dx - \mu^* \Bigg(\int\limits_{\mathbb{R}^N} (|u|^2 + |v|^2) dx \Bigg). \end{aligned}$$

Setting $s_1 = s, s_2 = 0$ and $t_1 = t, t_2 = 0$ in condition (H_2), we get

$$sf(x,s,t) \geq sf(x,0,t) + \mu^* |s|^2, \quad tg(x,s,t) \geq tg(x,s,0) + \mu^* |t|^2.$$

Hence

$$\begin{aligned} \prec A(u,v),(u,v) \succ \geq & \|u\|^2 + \|v\|^2 + \int\limits_{\mathbb{R}^N} (uf(x,0,v) + \mu^* |u|^2) dx \\ & + \int\limits_{\mathbb{R}^N} (vg(x,u,0) + \mu^* |v|^2) dx - \mu^* \Bigg(\int\limits_{\mathbb{R}^N} (|u|^2 + |v|^2) dx \Bigg). \end{aligned}$$

By the Hölder inequality, we obtain

$$\begin{aligned} \prec A(u,v),(u,v) \succ & \geq \|(u,v)\|_X^2 - \|u\|_p \Bigg(\int\limits_{\mathbb{R}^N} |f(x,0,v)|^q dx \Bigg)^{\frac{1}{q}} \\ & \quad - \|v\|_p \Bigg(\int\limits_{\mathbb{R}^N} |g(x,u,0)|^q dx \Bigg)^{\frac{1}{q}} \\ & \geq \|(u,v)\|_X^2 - c_p \|u\| \Bigg(\int\limits_{\mathbb{R}^N} |f(x,0,v)|^q dx \Bigg)^{\frac{1}{q}} \\ & \quad - c'_p \|v\| \Bigg(\int\limits_{\mathbb{R}^N} |g(x,u,0)|^q dx \Bigg)^{\frac{1}{q}}, \end{aligned}$$

where c_p and c'_p are Sobolev constants corresponding to u and v, respectively. But we know that the other norm, which is equivalent to the product norm, is

$$\|(u,v)\|_X = \max\{\|u\|, \|v\|\}.$$

So inequality (1.6) implies

$$\begin{aligned}&\prec A(u,v),(u,v)\succ\\&\geq \|(u,v)\|_X^2 - S\|(u,v)\|_X\left(\left(\int\limits_{\mathbb{R}^N}|f(x,0,v)|^q dx\right)^{\frac{1}{q}}+\left(\int\limits_{\mathbb{R}^N}|g(x,u,0)|^q dx\right)^{\frac{1}{q}}\right),\end{aligned}$$

where S is the maximum of Sobolev constants c_p and c'_p. Therefore,

$$\begin{aligned}&\lim_{\|(u,v)\|\to\infty}\frac{\prec A(u,v),(u,v)\succ}{\|(u,v)\|}\\&\geq \|(u,v)\|_X - S\left(\left(\int\limits_{\mathbb{R}^N}|f(x,0,v)|^q dx\right)^{\frac{1}{q}}+\left(\int\limits_{\mathbb{R}^N}|g(x,u,0)|^q dx\right)^{\frac{1}{q}}\right).\end{aligned}$$

It is clear that the right-hand side of the preceding inequality tends to infinity and so does the left-hand side. Then we get the desired result and the proof is complete. ☐

REFERENCES

[1] F. Abdolrazaghi, A. Razani, *On the weak solutions of an overdetermined system of nonlinear fractional partial integro-differential equations*, Miskolc Math. Notes **20** (2019), 3–16.

[2] G. Autuori, P. Pucci, *Elliptic problems involving the fractional Laplacian in $\mathbb{R}^N$*, J. Differential Equations **255** (2013), 2340–2362.

[3] B. Barrios, E. Colorado, A. De Pablo, U. Sánchez, *On some critical problems for the fractional Laplacian operator*, J. Differential Equations **252** (2012), 6133–6162.

[4] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer Science & Business Media, 2010.

[5] L. Caffarelli, L. Silvestre, *An extension problem related to the fractional Laplacian*, Comm. Partial Differential Equations **32** (2007), 1245–1260.

[6] X. Chang, *Ground state solutions of asymptotically linear fractional Schrödinger equations*, J. Math. Phys. **54** (2013), 061504.

[7] X. Chang, Z. Wang, *Ground state of scalar field equations involving a fractional Laplacian with general nonlinearity*, Nonlinearity **26** (2013), 479–494.

[8] K. Diethelm, *The Analysis of Fractional Differential Equations: An application-oriented exposition using differential operators of Caputo type*, Springer Science & Business Media, 2010.

[9] S. Dipierro, G. Palatucci, E. Valdinoci, *Existence and symmetry results for a Schrödinger type problem involving the fractional Laplacian*, arXiv:1202.0576, (2012).

[10] P. Felmer, A. Quaas, J. Tan, *Positive solutions of the nonlinear Schrödinger equation with the fractional Laplacian*, Proc. Roy. Soc. Edinburgh Sect. A **142** (2012), 1237–1262.

[11] A. Fiscella, P. Pucci, S. Saldi, *Existence of entire solutions for Schrödinger–Hardy systems involving two fractional operators*, Nonlinear Anal. **158** (2017), 109–131.

[12] A. Fiscella, P. Pucci, B. Zhang, *p-fractional Hardy–Schrödinger–Kirchhoff systems with critical nonlinearities*, Adv. Nonlinear Anal. **8** (2019) 1, 1111–1131.

[13] Y. Fu, H. Li, P. Pucci, *Existence of nonnegative solutions for a class of systems involving fractional* (p,q)*-Laplacian operators*, Chin. Ann. Math. Ser. B **39** (2018), 357–372.

[14] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier Science Limited, 2006.

[15] N. Nyamoradi, A. Razani, *Existence of solutions for a new p-Laplacian fractional boundary value problem with impulsive effects*, Journal of New Researches in Mathematics **5** (2019), 117–128.

[16] I. Podlubny, *The Laplace transform method for linear differential equations of the fractional order*, arXiv:funct-an/9710005, (1997).

[17] I. Podlubny, *Fractional Differential Equations*, Mathematics in Science and Engineering, vol. 198, Academic Press, San Diego, California, USA, 1999.

[18] A. Razani, *Weak and strong detonation profiles for a qualitative model*, J. Math. Anal. Appl. **276** (2002), 868–881.

[19] A. Razani, *Weak Chapman–Jouguet detonation profile for a qualitative model*, Bull. Aust. Math. Soc. **66** (2002), 393–403.

[20] A. Razani, *Existence of Chapman–Jouguet detonation for a viscous combustion model* J. Math. Anal. Appl. **293** (2004), 551–563.

[21] A. Razani, *Shock waves in gas dynamics*, Surv. Math. Appl. **2** (2007), 59–89.

[22] A. Razani, *Chapman–Jouguet travelling wave for a two-steps reaction scheme*, Ital. J. Pure Appl. Math. **39** (2018), 544–553.

[23] A. Razani, *Subsonic detonation waves in porous media*, Phys. Scr. **94** (2019), no. 085209.

[24] S. Secchi, *On fractional Schrödinger equations in* R^N *without the Ambrosetti–Rabinowitz condition*, arXiv:1210.0755 (2012).

[25] R. Servadei, E. Valdinoci, *The Brezis–Nirenberg result for the fractional Laplacian*, Trans. Amer. Math. Soc. **367** (2015), 67–102.

[26] J. Tan, Y. Wang, J. Yang, *Nonlinear fractional field equations*, Nonlinear Anal. **75** (2012), 2098–2110.

[27] J. Xu, Z. Wei, W. Dong, *Existence of weak solutions for a fractional Schrödinger equation*, Commun. Nonlinear Sci. Numer. Simul. **22** (2015), 1215–1222.

[28] Q. Yang, F. Liu, I. Turner, *Numerical methods for fractional partial differential equations with Riesz space fractional derivatives*, Appl. Math. Model. **34** (2010), 200–218.

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