

## IMPROVED ITERATIVE OSCILLATION TESTS FOR FIRST-ORDER DEVIATING DIFFERENTIAL EQUATIONS

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**Abstract.** In this paper, improved oscillation conditions are established for the oscillation of all solutions of differential equations with non-monotone deviating arguments and nonnegative coefficients. They lead to a procedure that checks for oscillations by iteratively computing  $\limsup$  and  $\liminf$  on terms recursively defined on the equation's coefficients and deviating argument. This procedure significantly improves all known oscillation criteria. The results and the improvement achieved over the other known conditions are illustrated by two examples, numerically solved in MATLAB.

**Keywords:** differential equation, non-monotone argument, oscillatory solution, nonoscillatory solution.

**Mathematics Subject Classification:** 34K06, 34K11.

### 1. INTRODUCTION

Consider the differential equation with a variable deviating argument of either delay

$$x'(t) + p(t)x(\tau(t)) = 0, \quad t \geq t_0, \quad (E)$$

or advanced type

$$x'(t) - q(t)x(\sigma(t)) = 0, \quad t \geq t_0, \quad (E')$$

where  $p, q$  are functions of nonnegative real numbers, and  $\tau, \sigma$  are functions of positive real numbers such that

$$\tau(t) < t, \quad t \geq t_0 \quad \text{and} \quad \lim_{t \rightarrow \infty} \tau(t) = \infty \quad (1.1)$$

and

$$\sigma(t) > t, \quad t \geq t_0, \quad (1.2)$$

respectively.

As is customary, a solution of  $(E)$  or  $(E')$  is called *oscillatory*, if it is neither eventually positive nor eventually negative. If there exists an eventually positive or an eventually negative solution, the equation is *nonoscillatory*. An equation is *oscillatory* if all its solutions oscillate.

The problem of establishing sufficient conditions for the oscillation of all solutions of equations  $(E)$  and  $(E')$  has been the subject of many investigations. The reader is referred to [1–7, 9–20, 22–29] and the references cited therein. Most of these papers concern the special case where the arguments are nondecreasing. A few papers studied the general case where the arguments are not necessarily monotone, see, for example, [1–6, 14, 23, 26] and the references therein.

The motivation for considering equations in the form of  $(E)$  or  $(E')$  with non-monotone arguments is justified not only by its pure mathematical interest, but also because such equations describe in a more realistic way a wide class of natural phenomena as natural disturbances (e.g. noise in communication systems) affecting parameters of the equation cause non-monotone deviations in the argument of the solutions. Therefore, an interesting question arises whether is it possible to obtain new oscillation criteria in the case where the argument  $\tau(t)$  or  $\sigma(t)$  is not necessarily monotone. In the present work, we achieve this goal by establishing criteria which, up to our knowledge, essentially improve all other known results in the literature.

The paper is organized as follows. First, we present, separately for a delay and advanced case, some of the related results which motivate the contents of this paper. Next, we establish new sufficient conditions of  $\limsup$  and  $\liminf$  type, for the oscillation of all solutions of  $(E)$  and  $(E')$ . We base our technique on the proper use of a recursive procedure leading to new inequalities which may replace former ones. To verify the significance of the obtained results, we provide two examples along with various comparisons among new and known criteria.

Throughout, we are going to use the following notation:

$$\alpha := \liminf_{t \rightarrow \infty} \int_{\tau(t)}^t p(s) ds,$$

$$\beta := \liminf_{t \rightarrow \infty} \int_t^{\sigma(t)} q(s) ds,$$

$$D(\omega) := \begin{cases} 0, & \text{if } \omega > 1/e, \\ \frac{1 - \omega - \sqrt{1 - 2\omega - \omega^2}}{2}, & \text{if } \omega \in [0, 1/e], \end{cases}$$

$$LD := \limsup_{t \rightarrow \infty} \int_{\tau(t)}^t p(s) ds, \quad \text{where } \tau(t) \text{ is nondecreasing,}$$

and

$$LA := \limsup_{t \rightarrow \infty} \int_t^{\sigma(t)} q(s) ds, \quad \text{where } \sigma(t) \text{ is nondecreasing.}$$

1.1. DELAY DIFFERENTIAL EQUATIONS (CHRONOLOGICAL REVIEW)

The first systematic study for the oscillation of all solutions to equation (E) was made by Myshkis in 1950 [25] when he proved that every solution of (E) oscillates if

$$\limsup_{t \rightarrow \infty} [t - \tau(t)] < \infty \quad \text{and} \quad \liminf_{t \rightarrow \infty} [t - \tau(t)] \liminf_{t \rightarrow \infty} p(t) > \frac{1}{e}. \quad (1.3)$$

In 1972, Ladas, Lakshmikantham and Papadakis [19] proved that, if

$$LD > 1, \quad (1.4)$$

then all solutions of (E) oscillate.

In 1982, Koplatadze and Chanturiya [13] improved (1.3) to

$$\alpha > \frac{1}{e}. \quad (1.5)$$

Concerning the constant  $1/e$  in (1.5), it is to be pointed out that if the inequality

$$\int_{\tau(t)}^t p(s) ds \leq \frac{1}{e}$$

holds eventually, then, according to a result in [13], (E) has a nonoscillatory solution.

Obviously, when the limit

$$\lim_{t \rightarrow \infty} \int_{\tau(t)}^t p(s) ds$$

does not exist, a gap appears between the conditions (1.4) and (1.5). How to fill this gap is an interesting problem which has attracted the attention of several authors. For example, in 2000, Jaroš and Stavroulakis [10] proved that, if

$$LD > \frac{1 + \ln \lambda_0}{\lambda_0} - D(\alpha), \quad (1.6)$$

where  $\lambda_0$  is the smaller root of the transcendental equation  $\lambda = e^{\alpha\lambda}$ , then all solutions of (E) oscillate.

Now we come to the case considered in the present work, i.e., that the argument  $\tau(t)$  is not necessarily monotone. Set

$$h(t) := \sup_{s \leq t} \tau(s), \quad t \geq t_0. \quad (1.7)$$

Clearly, the function  $h(t)$  is nondecreasing and  $\tau(t) \leq h(t) < t$  for all  $t \geq t_0$ .

Essential progress was made by Koplatadze and Kvinikadze [14] in 1994 who proved that if

$$\limsup_{t \rightarrow \infty} \int_{h(t)}^t p(s) \exp \left( \int_{h(s)}^{h(t)} p(u) \psi_j(u) du \right) ds > 1 - D(\alpha), \quad (1.8)$$

where

$$\psi_1(t) = 0, \quad \psi_j(t) = \exp \left( \int_{\tau(t)}^t p(u) \psi_{j-1}(u) du \right), \quad j \geq 2,$$

then all solutions of (E) oscillate.

In 2011, Braverman and Karpuz [2] proved that if

$$\limsup_{t \rightarrow \infty} \int_{h(t)}^t p(s) \exp \left( \int_{\tau(s)}^{h(t)} p(u) du \right) ds > 1, \quad (1.9)$$

then all solutions of (E) oscillate, while Stavroulakis [26] in 2014 improved (1.9) to

$$\limsup_{t \rightarrow \infty} \int_{h(t)}^t p(s) \exp \left( \int_{\tau(s)}^{h(t)} p(u) du \right) ds > 1 - D(\alpha). \quad (1.10)$$

In 2016, Morshedy and Attia [23] proved that, if

$$\limsup_{t \rightarrow \infty} \left[ \int_{g(t)}^t p_n(s) ds + D(\alpha) \exp \left( \int_{g(t)}^t \sum_{j=0}^{n-1} p_j(s) ds \right) \right] > 1, \quad (1.11)$$

where

$$p_0(t) = p(t) \quad \text{and} \quad p_n(t) = p_{n-1}(t) \int_{g(t)}^t p_{n-1}(s) \exp \left( \int_{g(s)}^t p_{n-1}(u) du \right) ds, \quad n \geq 1,$$

then all solutions of (E) oscillate. Here,  $g(t)$  is a nondecreasing continuous function such that  $\tau(t) \leq g(t) \leq t$ ,  $t_1 \geq t_0$ . Clearly,  $g(t)$  is more general than  $h(t)$  defined by (1.7).

In 2016 (2017), Chatzarakis [3] ([4]) proved that if for some  $j \in \mathbb{N}$

$$\limsup_{t \rightarrow \infty} \int_{h(t)}^t p(s) \exp \left( \int_{\tau(s)}^{h(t)} p_j(u) du \right) ds > 1 \quad (1.12)$$

or

$$\limsup_{t \rightarrow \infty} \int_{h(t)}^t p(s) \exp \left( \int_{\tau(s)}^{h(t)} p_j(u) du \right) ds > 1 - D(\alpha), \tag{1.13}$$

where

$$p_j(t) = p(t) \left[ 1 + \int_{\tau(t)}^t p(s) \exp \left( \int_{\tau(s)}^{h(t)} p_{j-1}(u) du \right) ds \right],$$

with  $p_0(t) = p(t)$ , then all solutions of (E) oscillate.

Lately, Chatzarakis and Li [5] improved (1.12) and (1.13) to

$$\limsup_{t \rightarrow \infty} \int_{h(t)}^t p(s) \exp \left( \int_{\tau(s)}^{h(t)} p(u) \exp \left( \int_{\tau(u)}^u P_j(\xi) d\xi \right) du \right) ds > 1, \tag{1.14}$$

and

$$\limsup_{t \rightarrow \infty} \int_{h(t)}^t p(s) \exp \left( \int_{\tau(s)}^{h(t)} p(u) \exp \left( \int_{\tau(u)}^u P_j(\xi) d\xi \right) du \right) ds > 1 - D(\alpha), \tag{1.15}$$

respectively, where

$$P_j(t) = p(t) \left[ 1 + \int_{\tau(t)}^t p(s) \exp \left( \int_{\tau(s)}^t p(u) \exp \left( \int_{\tau(u)}^u P_{j-1}(\xi) d\xi \right) du \right) ds \right],$$

with  $P_0(t) = \lambda_0 p(t)$  and  $\lambda_0$  is the smaller root of the transcendental equation  $\lambda = e^{\alpha\lambda}$ .

In the same paper, the authors proved that if for some  $j \in \mathbb{N}$

$$\limsup_{t \rightarrow \infty} \int_{h(t)}^t p(s) \exp \left( \int_{\tau(s)}^t p(u) \exp \left( \int_{\tau(u)}^u P_j(\xi) d\xi \right) du \right) ds > \frac{1}{D(\alpha)}, \tag{1.16}$$

or

$$\limsup_{t \rightarrow \infty} \int_{h(t)}^t p(s) \exp \left( \int_{\tau(s)}^{h(s)} p(u) \exp \left( \int_{\tau(u)}^u P_j(\xi) d\xi \right) du \right) ds > \frac{1 + \ln \lambda_0}{\lambda_0} - D(\alpha), \tag{1.17}$$

or

$$\liminf_{t \rightarrow \infty} \int_{h(t)}^t p(s) \exp \left( \int_{\tau(s)}^{h(s)} p(u) \exp \left( \int_{\tau(u)}^u P_j(\xi) d\xi \right) du \right) ds > \frac{1}{e}, \tag{1.18}$$

then all solutions of (E) oscillate.

1.2. ADVANCED DIFFERENTIAL EQUATIONS (CHRONOLOGICAL REVIEW)

By [21, Theorem 2.4.3], if

$$LA > 1, \tag{1.19}$$

then all solutions of  $(E')$  oscillate.

In 1983, Fukagai and Kusano [9], proved that if

$$\beta > \frac{1}{e}, \tag{1.20}$$

then all solutions of  $(E')$  oscillate, while if

$$\int_t^{\sigma(t)} q(s)ds \leq \frac{1}{e} \quad \text{for all sufficiently large } t,$$

then Eq.  $(E')$  has a nonoscillatory solution.

Assume that the argument  $\sigma(t)$  is not necessarily monotone. Set

$$\rho(t) = \inf_{s \geq t} \sigma(s), \quad t \geq t_0. \tag{1.21}$$

Clearly, the function  $\rho(t)$  is nondecreasing and  $\sigma(t) \geq \rho(t) > t$  for all  $t \geq t_0$ .

In 2015, Chatzarakis and Ocalan [6], proved that if

$$\limsup_{t \rightarrow \infty} \int_t^{\rho(t)} q(s) \exp \left( \int_{\rho(t)}^{\sigma(s)} q(u)du \right) ds > 1, \tag{1.22}$$

or

$$\liminf_{t \rightarrow \infty} \int_t^{\rho(t)} q(s) \exp \left( \int_{\rho(t)}^{\sigma(s)} q(u)du \right) ds > \frac{1}{e}, \tag{1.23}$$

then all solutions of  $(E')$  oscillate.

In 2016 (2017), Chatzarakis [3] ([4]) proved that, if for some  $j \in \mathbb{N}$

$$\limsup_{t \rightarrow \infty} \int_t^{\rho(t)} q(s) \exp \left( \int_{\rho(t)}^{\sigma(s)} q_j(u)du \right) ds > 1, \tag{1.24}$$

or

$$\limsup_{t \rightarrow \infty} \int_t^{\rho(t)} q(s) \exp \left( \int_{\rho(t)}^{\sigma(s)} q_j(u)du \right) ds > 1 - D(\beta), \tag{1.25}$$

where

$$q_j(t) = q(t) \left[ 1 + \int_t^{\sigma(t)} q(s) \exp \left( \int_{\rho(t)}^{\sigma(s)} q_{j-1}(u)du \right) ds \right], \quad j \geq 1$$

with  $q_0(t) = q(t)$ , then all solutions of  $(E')$  oscillate.

Lately, Chatzarakis and Li [5], improved (1.24) and (1.25) to

$$\limsup_{t \rightarrow \infty} \int_t^{\rho(t)} q(s) \exp \left( \int_{\rho(t)}^{\sigma(s)} q(u) \exp \left( \int_u^{\sigma(u)} Q_j(\xi) d\xi \right) du \right) ds > 1 \tag{1.26}$$

and

$$\limsup_{t \rightarrow \infty} \int_t^{\rho(t)} q(s) \exp \left( \int_{\rho(t)}^{\sigma(s)} q(u) \exp \left( \int_u^{\sigma(u)} Q_j(\xi) d\xi \right) du \right) ds > 1 - D(\beta), \tag{1.27}$$

respectively, where

$$Q_j(t) = q(t) \left[ 1 + \int_t^{\sigma(t)} q(s) \exp \left( \int_t^{\sigma(s)} q(u) \exp \left( \int_u^{\sigma(u)} P_{j-1}(\xi) d\xi \right) du \right) ds \right],$$

with  $Q_0(t) = \lambda_0 q(t)$  and  $\lambda_0$  is the smaller root of the transcendental equation  $\lambda = e^{\beta\lambda}$ .

In the same paper, the authors proved that, if for some  $j \in \mathbb{N}$

$$\limsup_{t \rightarrow \infty} \int_t^{\rho(t)} q(s) \exp \left( \int_t^{\sigma(s)} q(u) \exp \left( \int_u^{\sigma(u)} Q_j(\xi) d\xi \right) du \right) ds > \frac{1}{D(\beta)}, \tag{1.28}$$

or

$$\limsup_{t \rightarrow \infty} \int_t^{\rho(t)} q(s) \exp \left( \int_{\rho(s)}^{\sigma(s)} q(u) \exp \left( \int_u^{\sigma(u)} Q_j(\xi) d\xi \right) du \right) ds > \frac{1 + \ln \lambda_0}{\lambda_0} - D(\beta), \tag{1.29}$$

or

$$\liminf_{t \rightarrow \infty} \int_t^{\rho(t)} q(s) \exp \left( \int_{\rho(s)}^{\sigma(s)} q(u) \exp \left( \int_u^{\sigma(u)} Q_j(\xi) d\xi \right) du \right) ds > \frac{1}{e}, \tag{1.30}$$

then all solutions of  $(E')$  are oscillatory.

## 2. MAIN RESULTS

### 2.1. DELAY DIFFERENTIAL EQUATIONS

We further study  $(E)$  and derive new sufficient oscillation conditions, involving  $\limsup$  and  $\liminf$ , which improve all the previous results. The proofs of our main results are essentially based on the following lemmas.

**Lemma 2.1** ([8, Lemma 2.1.1]). *In addition to hypothesis (1.1), assume that  $h(t)$  is defined by (1.7). Then*

$$\alpha := \liminf_{t \rightarrow \infty} \int_{\tau(t)}^t p(s) ds = \liminf_{t \rightarrow \infty} \int_{h(t)}^t p(s) ds. \tag{2.1}$$

**Lemma 2.2** ([8, Lemma 2.1.3]). *In addition to hypothesis (1.1), assume that  $h(t)$  is defined by (1.7),  $\alpha \in (0, 1/e]$  and  $x(t)$  is an eventually positive solution of (E). Then*

$$\liminf_{t \rightarrow \infty} \frac{x(t)}{x(h(t))} \geq D(\alpha). \tag{2.2}$$

The next lemma provides a lower estimate for the ratio  $x(h(t))/x(t)$  in terms of the smaller root of the transcendental equation  $\lambda = e^{\alpha\lambda}$ .

**Lemma 2.3** ([17, Lemma 1]). *Assume that  $\alpha \in (0, 1/e]$  and let  $x$  be a positive solution of (E). Then*

$$\liminf_{t \rightarrow \infty} \frac{x(h(t))}{x(t)} \geq \lambda_0, \tag{2.3}$$

where  $\lambda_0$  is the smaller root of the transcendental equation  $\lambda = e^{\alpha\lambda}$ .

**Theorem 2.4.** *Assume that  $h(t)$  is defined by (1.7) and for some  $\ell \in \mathbb{N}$*

$$\limsup_{t \rightarrow \infty} \int_{h(t)}^t p(s) \exp \left( \int_{\tau(s)}^{h(t)} p(u) \exp \left( \int_{\tau(u)}^u R_\ell(\xi) d\xi \right) du \right) ds > 1, \tag{2.4}$$

where

$$R_\ell(t) = p(t) \left[ 1 + \int_{\tau(t)}^t p(s) \exp \left( \int_{\tau(s)}^t p(u) \exp \left( \int_{\tau(u)}^u R_{\ell-1}(\xi) d\xi \right) du \right) ds \right] \tag{2.5}$$

with  $R_0(t) = p(t) \left[ 1 + \lambda_0 \int_{\tau(t)}^t p(s) ds \right]$ , and  $\lambda_0$  is the smaller root of the transcendental equation  $\lambda = e^{\alpha\lambda}$ . Then all solutions of (E) are oscillatory.

*Proof.* Assume, for the sake of contradiction, that there exists a nonoscillatory solution  $x(t)$  of (E). Since  $-x(t)$  is also a solution of (E), we can confine our discussion only to the case where the solution  $x(t)$  is eventually positive. Then there exists  $t_1 > t_0$  such that  $x(t), x(\tau(t)) > 0$ , for all  $t \geq t_1$ . Thus, from (E) we have

$$x'(t) = -p(t)x(\tau(t)) \leq 0, \quad \text{for all } t \geq t_1,$$

which means that  $x(t)$  is an eventually nonincreasing positive function. Taking this into account and the fact  $\tau(t) \leq h(t)$ , (E) implies

$$x'(t) + p(t)x(h(t)) \leq 0, \quad t \geq t_1. \tag{2.6}$$



Observe that (2.3) implies that for each  $\epsilon > 0$  there exists a  $t_\epsilon$  such that

$$\frac{x(h(t))}{x(t)} > \lambda_0 - \epsilon \quad \text{for all } t \geq t_\epsilon \geq t_1. \quad (2.7)$$

Integrating (E) from  $\tau(t)$  to  $t$ , we have

$$x(t) - x(\tau(t)) + \int_{\tau(t)}^t p(s)x(\tau(s)) ds = 0, \quad (2.8)$$

which, in view of  $\tau(s) \leq h(s)$ , gives

$$x(t) - x(\tau(t)) + \int_{\tau(t)}^t p(s)x(h(s)) ds \leq 0. \quad (2.9)$$

Combining (2.9) and (2.7), we obtain

$$x(t) - x(\tau(t)) + (\lambda_0 - \epsilon) \int_{\tau(t)}^t p(s)x(s) ds \leq 0,$$

or

$$x(t) - x(\tau(t)) + (\lambda_0 - \epsilon)x(t) \int_{\tau(t)}^t p(s) ds \leq 0. \quad (2.10)$$

Multiplying the last inequality by  $p(t)$ , we find

$$p(t)x(t) - p(t)x(\tau(t)) + (\lambda_0 - \epsilon)p(t)x(t) \int_{\tau(t)}^t p(s) ds \leq 0,$$

which, in view of (E), becomes

$$x'(t) + p(t)x(t) + (\lambda_0 - \epsilon)p(t)x(t) \int_{\tau(t)}^t p(s) ds \leq 0,$$

or

$$x'(t) + p(t) \left[ 1 + (\lambda_0 - \epsilon) \int_{\tau(t)}^t p(s) ds \right] x(t) \leq 0.$$

Thus,

$$x'(t) + R_0(t, \epsilon)x(t) \leq 0, \quad (2.11)$$

where

$$R_0(t, \epsilon) = p(t) \left[ 1 + (\lambda_0 - \epsilon) \int_{\tau(t)}^t p(s) ds \right].$$

Applying the Grönwall inequality in (2.11), we obtain

$$x(s) \geq x(t) \exp \left( \int_s^t R_0(\xi, \epsilon) d\xi \right), \quad t \geq s. \quad (2.12)$$

Now we divide (E) by  $x(t) > 0$  and integrate on  $[s, t]$ , so

$$- \int_s^t \frac{x'(u)}{x(u)} du = \int_s^t p(u) \frac{x(\tau(u))}{x(u)} du,$$

or

$$\ln \frac{x(s)}{x(t)} = \int_s^t p(u) \frac{x(\tau(u))}{x(u)} du. \quad (2.13)$$

Since  $\tau(u) < u$ , the last inequality gives

$$\begin{aligned} \ln \frac{x(s)}{x(t)} &= \int_s^t p(u) \frac{x(\tau(u))}{x(u)} du \\ &\geq \int_s^t p(u) \frac{x(u)}{x(u)} \exp \left( \int_{\tau(u)}^u R_0(\xi, \epsilon) d\xi \right) du = \int_s^t p(u) \exp \left( \int_{\tau(u)}^u R_0(\xi, \epsilon) d\xi \right) du \end{aligned}$$

or

$$x(s) \geq x(t) \exp \left( \int_s^t p(u) \exp \left( \int_{\tau(u)}^u R_0(\xi, \epsilon) d\xi \right) du \right). \quad (2.14)$$

Setting  $s = \tau(s)$  in (2.14), we take

$$x(\tau(s)) \geq x(t) \exp \left( \int_{\tau(s)}^t p(u) \exp \left( \int_{\tau(u)}^u R_0(\xi, \epsilon) d\xi \right) du \right). \quad (2.15)$$

Combining (2.8) and (2.15), we obtain

$$x(t) - x(\tau(t)) + x(t) \int_{\tau(t)}^t p(s) \exp \left( \int_{\tau(s)}^t p(u) \exp \left( \int_{\tau(u)}^u R_0(\xi, \epsilon) d\xi \right) du \right) ds \leq 0.$$

Multiplying the last inequality by  $p(t)$ , we find

$$p(t)x(t) - p(t)x(\tau(t)) + p(t)x(t) \int_{\tau(t)}^t p(s) \exp \left( \int_{\tau(s)}^t p(u) \exp \left( \int_{\tau(u)}^u R_0(\xi, \epsilon) d\xi \right) du \right) ds \leq 0,$$

which, in view of (E), becomes

$$x'(t) + p(t)x(t) + p(t)x(t) \int_{\tau(t)}^t p(s) \exp \left( \int_{\tau(s)}^t p(u) \exp \left( \int_{\tau(u)}^u R_0(\xi, \epsilon) d\xi \right) du \right) ds \leq 0.$$

Hence, for sufficiently large  $t$ ,

$$x'(t) + p(t) \left[ 1 + \int_{\tau(t)}^t p(s) \exp \left( \int_{\tau(s)}^t p(u) \exp \left( \int_{\tau(u)}^u R_0(\xi, \epsilon) d\xi \right) du \right) ds \right] x(t) \leq 0,$$

or

$$x'(t) + R_1(t, \epsilon)x(t) \leq 0, \tag{2.16}$$

where

$$R_1(t, \epsilon) = p(t) \left[ 1 + \int_{\tau(t)}^t p(s) \exp \left( \int_{\tau(s)}^t p(u) \exp \left( \int_{\tau(u)}^u R_0(\xi, \epsilon) d\xi \right) du \right) ds \right].$$

Clearly (2.16) resembles (2.11) with  $R_0$  replaced by  $R_1$ , so an integration of (2.16) on  $[s, t]$  leads to

$$x(s) \geq x(t) \exp \left( \int_s^t R_1(\xi, \epsilon) d\xi \right). \tag{2.17}$$

Taking the steps starting from (2.12) to (2.15) we may see that  $x$  satisfies the inequality

$$x(\tau(s)) \geq x(t) \exp \left( \int_{\tau(s)}^t p(u) \exp \left( \int_{\tau(u)}^u R_1(\xi, \epsilon) d\xi \right) du \right). \tag{2.18}$$

Combining now (2.8) and (2.18), we obtain

$$x(t) - x(\tau(t)) + x(t) \int_{\tau(t)}^t p(s) \exp \left( \int_{\tau(s)}^t p(u) \exp \left( \int_{\tau(u)}^u R_1(\xi, \epsilon) d\xi \right) du \right) ds \leq 0.$$

Multiplying the last inequality by  $p(t)$ , as before, we find

$$x'(t) + p(t) \left[ 1 + \int_{\tau(t)}^t p(s) \exp \left( \int_{\tau(s)}^t p(u) \exp \left( \int_{\tau(u)}^u R_1(\xi, \epsilon) d\xi \right) du \right) ds \right] x(t) \leq 0.$$

Therefore, for sufficiently large  $t$ , we have

$$x'(t) + R_2(t, \epsilon)x(t) \leq 0,$$

where

$$R_2(t, \epsilon) = p(t) \left[ 1 + \int_{\tau(t)}^t p(s) \exp \left( \int_{\tau(s)}^t p(u) \exp \left( \int_{\tau(u)}^u R_1(\xi, \epsilon) d\xi \right) du \right) ds \right].$$

It becomes apparent, now, that by repeating the above steps, we can build inequalities on  $x'(t)$  with progressively higher indices  $R_\ell(t)$ ,  $\ell \in \mathbb{N}$ . In general, for sufficiently large  $t$ , the positive solution  $x(t)$  satisfies the inequality

$$x'(t) + R_\ell(t, \epsilon)x(t) \leq 0, \quad \ell \in \mathbb{N}, \tag{2.19}$$

where

$$R_\ell(t, \epsilon) = p(t) \left[ 1 + \int_{\tau(t)}^t p(s) \exp \left( \int_{\tau(s)}^t p(u) \exp \left( \int_{\tau(u)}^u R_{\ell-1}(\xi, \epsilon) d\xi \right) du \right) ds \right].$$

In order to take our final step, we recall that

$$h(t) := \sup_{s \leq t} \tau(s)$$

and note that  $h$  is a nondecreasing function. Moreover, since  $\tau(s) \leq h(s) \leq h(t)$  we have

$$x(\tau(s)) \geq x(h(t)) \exp \left( \int_{\tau(s)}^{h(t)} p(u) \exp \left( \int_{\tau(u)}^u R_\ell(\xi, \epsilon) d\xi \right) du \right).$$

Hence

$$x(t) - x(h(t)) + x(h(t)) \int_{h(t)}^t p(s) \exp \left( \int_{\tau(s)}^{h(t)} p(u) \exp \left( \int_{\tau(u)}^u R_\ell(\xi, \epsilon) d\xi \right) du \right) ds \leq 0. \tag{2.20}$$

The inequality is valid if we omit  $x(t) > 0$  in the left-hand side. Therefore

$$x(h(t)) \left[ \int_{h(t)}^t p(s) \exp \left( \int_{\tau(s)}^{h(t)} p(u) \exp \left( \int_{\tau(u)}^u R_\ell(\xi, \epsilon) d\xi \right) du \right) ds - 1 \right] < 0,$$

which means that

$$\limsup_{t \rightarrow \infty} \int_{h(t)}^t p(s) \exp \left( \int_{\tau(s)}^{h(t)} p(u) \exp \left( \int_{\tau(u)}^u R_\ell(\xi, \epsilon) d\xi \right) du \right) ds \leq 1.$$

Since  $\epsilon$  may be taken arbitrarily small, this inequality contradicts (2.4).

The proof of the theorem is complete. □

**Theorem 2.5.** *Assume that  $h(t)$  is defined by (1.7) and  $\alpha \in (0, 1/e]$ . If for some  $\ell \in \mathbb{N}$*

$$\limsup_{t \rightarrow \infty} \int_{h(t)}^t p(s) \exp \left( \int_{\tau(s)}^{h(t)} p(u) \exp \left( \int_{\tau(u)}^u R_\ell(\xi) d\xi \right) du \right) ds > 1 - D(\alpha), \tag{2.21}$$

where  $R_\ell$  is defined by (2.5), then all solutions of (E) are oscillatory.

*Proof.* Let  $x$  be an eventually positive solution of (E). Then, as in the proof of Theorem 2.4, we obtain (2.20), i.e, for sufficiently large  $t$  we have

$$x(t) - x(h(t)) + x(h(t)) \int_{h(t)}^t p(s) \exp \left( \int_{\tau(s)}^{h(t)} p(u) \exp \left( \int_{\tau(u)}^u R_\ell(\xi, \epsilon) d\xi \right) du \right) ds \leq 0.$$

Thus,

$$\int_{h(t)}^t p(s) \exp \left( \int_{\tau(s)}^{h(t)} p(u) \exp \left( \int_{\tau(u)}^u R_\ell(\xi, \epsilon) d\xi \right) du \right) ds \leq 1 - \frac{x(t)}{x(h(t))},$$

which gives

$$\limsup_{t \rightarrow \infty} \int_{h(t)}^t p(s) \exp \left( \int_{\tau(s)}^{h(t)} p(u) \exp \left( \int_{\tau(u)}^u R_\ell(\xi, \epsilon) d\xi \right) du \right) ds \leq 1 - \liminf_{t \rightarrow \infty} \frac{x(t)}{x(h(t))}. \tag{2.22}$$

By combining Lemmas 2.1 and 2.2, it becomes obvious that inequality (2.2) is fulfilled. So, (2.22) leads to

$$\limsup_{t \rightarrow \infty} \int_{h(t)}^t p(s) \exp \left( \int_{\tau(s)}^{h(t)} p(u) \exp \left( \int_{\tau(u)}^u R_\ell(\xi, \epsilon) d\xi \right) du \right) ds \leq 1 - D(\alpha).$$

Since  $\epsilon$  may be taken arbitrarily small, this inequality contradicts (2.21).

The proof of the theorem is complete. □

**Remark 2.6.** It is clear that the left-hand sides of both conditions (2.4) and (2.21) are identical, also the right-hand side of condition (2.21) reduces to (2.4) in case that  $\alpha = 0$ . So it seems that Theorem 2.5 is the same as Theorem 2.4 when  $\alpha = 0$ . However, one may notice that condition  $\alpha \in (0, 1/e]$  is required in Theorem 2.5 but not in Theorem 2.4.

**Theorem 2.7.** Assume that  $h(t)$  is defined by (1.7) and  $\alpha \in (0, 1/e]$ . If for some  $\ell \in \mathbb{N}$

$$\limsup_{t \rightarrow \infty} \int_{h(t)}^t p(s) \exp \left( \int_{\tau(s)}^t p(u) \exp \left( \int_{\tau(u)}^u R_\ell(\xi) d\xi \right) du \right) ds > \frac{1}{D(\alpha)} - 1, \quad (2.23)$$

where  $R_\ell$  is defined by (2.5), then all solutions of (E) are oscillatory.

*Proof.* Assume, for the sake of contradiction, that there exists a nonoscillatory solution  $x$  of (E) and that  $x$  is eventually positive. Then, as in the proof of Theorem 2.4, for sufficiently large  $t$  we have

$$x(\tau(s)) \geq x(t) \exp \left( \int_{\tau(s)}^t p(u) \exp \left( \int_{\tau(u)}^u R_\ell(\xi, \epsilon) d\xi \right) du \right). \quad (2.24)$$

Integrating (E) from  $h(t)$  to  $t$ , we have

$$x(t) - x(h(t)) + \int_{h(t)}^t p(s)x(\tau(s))ds = 0,$$

which, in view of (2.24), gives

$$x(t) - x(h(t)) + \int_{h(t)}^t p(s)x(t) \exp \left( \int_{\tau(s)}^t p(u) \exp \left( \int_{\tau(u)}^u R_\ell(\xi, \epsilon) d\xi \right) du \right) ds \leq 0,$$

or

$$x(t) - x(h(t)) + x(h(t)) \int_{h(t)}^t p(s) \frac{x(t)}{x(h(t))} \exp \left( \int_{\tau(s)}^t p(u) \exp \left( \int_{\tau(u)}^u R_\ell(\xi, \epsilon) d\xi \right) du \right) ds \leq 0.$$

That is, for all sufficiently large  $t$  it holds

$$\int_{h(t)}^t p(s) \exp \left( \int_{\tau(s)}^t p(u) \exp \left( \int_{\tau(u)}^u R_\ell(\xi, \epsilon) d\xi \right) du \right) ds \leq \frac{x(h(t))}{x(t)} - 1$$

and therefore

$$\limsup_{t \rightarrow \infty} \int_{h(t)}^t p(s) \exp \left( \int_{\tau(s)}^t p(u) \exp \left( \int_{\tau(u)}^u R_\ell(\xi, \epsilon) d\xi \right) du \right) ds \leq \limsup_{t \rightarrow \infty} \frac{x(h(t))}{x(t)} - 1. \tag{2.25}$$

By combining Lemmas 2.1 and 2.2, it becomes obvious that inequality (2.2) is fulfilled. So, (2.25) leads to

$$\limsup_{t \rightarrow \infty} \int_{h(t)}^t p(s) \exp \left( \int_{\tau(s)}^t p(u) \exp \left( \int_{\tau(u)}^u R_\ell(\xi, \epsilon) d\xi \right) du \right) ds \leq \frac{1}{D(\alpha)} - 1.$$

Since  $\epsilon$  may be taken arbitrarily small, this inequality contradicts (2.23).

The proof of the theorem is complete. □

**Theorem 2.8.** *Assume that  $h(t)$  is defined by (1.7) and  $\alpha \in (0, 1/e]$ . If for some  $\ell \in \mathbb{N}$*

$$\limsup_{t \rightarrow \infty} \int_{h(t)}^t p(s) \exp \left( \int_{\tau(s)}^{h(s)} p(u) \exp \left( \int_{\tau(u)}^u R_\ell(\xi) d\xi \right) du \right) ds > \frac{1 + \ln \lambda_0}{\lambda_0} - D(\alpha), \tag{2.26}$$

where  $R_\ell$  is defined by (2.5) and  $\lambda_0$  is the smaller root of the transcendental equation  $\lambda = e^{\alpha\lambda}$ , then all solutions of (E) are oscillatory.

*Proof.* Let  $x$  be an eventually positive solution and obtain (2.24) as in Theorem 2.7, i.e.,

$$x(\tau(s)) \geq x(t) \exp \left( \int_{\tau(s)}^t p(u) \exp \left( \int_{\tau(u)}^u R_\ell(\xi, \epsilon) d\xi \right) du \right).$$

Since  $\tau(s) \leq h(s)$ , the above inequality gives

$$x(\tau(s)) \geq x(h(s)) \exp \left( \int_{\tau(s)}^{h(s)} p(u) \exp \left( \int_{\tau(u)}^u R_\ell(\xi, \epsilon) d\xi \right) du \right). \tag{2.27}$$

Observe that (2.3) implies that for each  $\epsilon > 0$  there exists a  $t_\epsilon$  such that

$$\frac{x(h(t))}{x(t)} > \lambda_0 - \epsilon \quad \text{for all } t \geq t_\epsilon \geq t_1. \tag{2.28}$$

Noting that by nondecreasing nature of the function  $\frac{x(h(t))}{x(s)}$  in  $s$ , it holds

$$1 = \frac{x(h(t))}{x(h(t))} \leq \frac{x(h(t))}{x(s)} \leq \frac{x(h(t))}{x(t)}, \quad t_\epsilon \leq h(t) \leq s \leq t,$$

in particular for  $\epsilon \in (0, \lambda_0 - 1)$ , by continuity we see that there exists a  $t^* \in (h(t), t]$  such that

$$1 < \lambda_0 - \epsilon = \frac{x(h(t))}{x(t^*)}. \tag{2.29}$$

Integrating (E) from  $t^*$  to  $t$  we have

$$x(t) - x(t^*) + \int_{t^*}^t p(s)x(\tau(s))ds = 0,$$

so, by using (2.27) along with  $h(s) \leq h(t)$  in combination with the the fact that  $x$  is nonincreasing, we have

$$x(t) - x(t^*) + x(h(t)) \int_{t^*}^t p(s) \exp \left( \int_{\tau(s)}^{h(s)} p(u) \exp \left( \int_{\tau(u)}^u R_\ell(\xi, \epsilon) d\xi \right) du \right) ds \leq 0,$$

or

$$\int_{t^*}^t p(s) \exp \left( \int_{\tau(s)}^{h(s)} p(u) \exp \left( \int_{\tau(u)}^u R_\ell(\xi, \epsilon) d\xi \right) du \right) ds \leq \frac{x(t^*)}{x(h(t))} - \frac{x(t)}{x(h(t))}.$$

In view of (2.29) and Lemma 2.2, for the  $\epsilon$  considered, there exists  $t'_\epsilon \geq t_\epsilon$  such that

$$\int_{t^*}^t p(s) \exp \left( \int_{\tau(s)}^{h(s)} p(u) \exp \left( \int_{\tau(u)}^u R_\ell(\xi, \epsilon) d\xi \right) du \right) ds < \frac{1}{\lambda_0 - \epsilon} - D(\alpha) + \epsilon, \tag{2.30}$$

for  $t \geq t'_\epsilon$ .

Dividing (E) by  $x(t)$  and integrating from  $h(t)$  to  $t^*$  we find

$$\int_{h(t)}^{t^*} p(s) \frac{x(\tau(s))}{x(s)} ds = - \int_{h(t)}^{t^*} \frac{x'(s)}{x(s)} ds$$

and using (2.27), we find

$$\int_{h(t)}^{t^*} p(s) \frac{x(h(s))}{x(s)} \exp \left( \int_{\tau(s)}^{h(s)} p(u) \exp \left( \int_{\tau(u)}^u R_\ell(\xi, \epsilon) d\xi \right) du \right) ds \leq - \int_{h(t)}^{t^*} \frac{x'(s)}{x(s)} ds. \tag{2.31}$$

By (2.3), for  $s \geq h(t) \geq t'_\epsilon$ , we have  $\frac{x(h(s))}{x(s)} > \lambda_0 - \epsilon$ , so from (2.31) we get

$$(\lambda_0 - \epsilon) \int_{h(t)}^{t^*} p(s) \exp \left( \int_{\tau(s)}^{h(s)} p(u) \exp \left( \int_{\tau(u)}^u R_\ell(\xi, \epsilon) d\xi \right) du \right) ds < - \int_{h(t)}^{t^*} \frac{x'(s)}{x(s)} ds.$$



Hence, for all sufficiently large  $t$  we have

$$\int_{h(t)}^{t^*} p(s) \exp \left( \int_{\tau(s)}^{h(s)} p(u) \exp \left( \int_{\tau(u)}^u R_\ell(\xi, \epsilon) d\xi \right) du \right) ds < -\frac{1}{\lambda_0 - \epsilon} \int_{h(t)}^{t^*} \frac{x'(s)}{x(s)} ds = \frac{1}{\lambda_0 - \epsilon} \ln \frac{x(h(t))}{x(t^*)} = \frac{\ln(\lambda_0 - \epsilon)}{\lambda_0 - \epsilon},$$

i.e.,

$$\int_{h(t)}^{t^*} p(s) \exp \left( \int_{\tau(s)}^{h(s)} p(u) \exp \left( \int_{\tau(u)}^u R_\ell(\xi, \epsilon) d\xi \right) du \right) ds < \frac{\ln(\lambda_0 - \epsilon)}{\lambda_0 - \epsilon}. \tag{2.32}$$

Adding (2.30) and (2.32), and then taking the limit as  $t \rightarrow \infty$ , we have

$$\limsup_{t \rightarrow \infty} \int_{h(t)}^t p(s) \exp \left( \int_{\tau(s)}^{h(s)} p(u) \exp \left( \int_{\tau(u)}^u R_\ell(\xi, \epsilon) d\xi \right) du \right) ds \leq \frac{1 + \ln(\lambda_0 - \epsilon)}{\lambda_0 - \epsilon} - D(\alpha) + \epsilon.$$

Since  $\epsilon$  may be taken arbitrarily small, this inequality contradicts (2.26).

The proof of the theorem is complete. □

**Theorem 2.9.** Assume that for some  $\ell \in \mathbb{N}$

$$\liminf_{t \rightarrow \infty} \int_{h(t)}^t p(s) \exp \left( \int_{\tau(s)}^{h(s)} p(u) \exp \left( \int_{\tau(u)}^u R_\ell(\xi) d\xi \right) du \right) ds > \frac{1}{e}, \tag{2.33}$$

where  $R_\ell$  is defined by (2.5). Then all solutions of (E) are oscillatory.

*Proof.* For the sake of contradiction, let  $x$  be a nonincreasing eventually positive solution and  $t_1 > t_0$  be such that  $x(t) > 0$  and  $x(\tau(t)) > 0$  for all  $t \geq t_1$ . We note that we may obtain (2.27) as in previous theorem.

Dividing (E) by  $x(t)$  and integrating from  $h(t)$  to  $t$ , we have

$$\ln \left( \frac{x(h(t))}{x(t)} \right) = \int_{h(t)}^t p(s) \frac{x(\tau(s))}{x(s)} ds \text{ for all } t \geq t_2 \geq t_1,$$

from which in view of  $\tau(s) \leq h(s)$  and by (2.27), we obtain

$$\ln \left( \frac{x(h(t))}{x(t)} \right) \geq \int_{h(t)}^t p(s) \frac{x(h(s))}{x(s)} \exp \left( \int_{\tau(s)}^{h(s)} p(u) \exp \left( \int_{\tau(u)}^u R_\ell(\xi, \epsilon) d\xi \right) du \right) ds.$$

Taking into account that  $x$  is nonincreasing and  $h(s) < s$ , the last inequality leads to

$$\ln \left( \frac{x(h(t))}{x(t)} \right) \geq \int_{h(t)}^t p(s) \exp \left( \int_{\tau(s)}^{h(s)} p(u) \exp \left( \int_{\tau(u)}^u R_\ell(\xi, \epsilon) d\xi \right) du \right) ds. \quad (2.34)$$

From (2.33), it follows that there exists a constant  $c > 0$  such that for a sufficiently large  $t$  holds

$$\int_{h(t)}^t p(s) \exp \left( \int_{\tau(s)}^{h(s)} p(u) \exp \left( \int_{\tau(u)}^u R_\ell(\xi) d\xi \right) du \right) ds \geq c > \frac{1}{e}.$$

Choose  $c'$  such that  $c > c' > 1/e$ . For every  $\epsilon > 0$  such that  $c - \epsilon > c'$  we have

$$\int_{h(t)}^t p(s) \exp \left( \int_{\tau(s)}^{h(s)} p(u) \exp \left( \int_{\tau(u)}^u R_\ell(\xi, \epsilon) d\xi \right) du \right) ds \geq c - \epsilon > c' > \frac{1}{e}. \quad (2.35)$$

Combining inequalities (2.34) and (2.35), we obtain

$$\ln \left( \frac{x(h(t))}{x(t)} \right) \geq c',$$

or

$$\frac{x(h(t))}{x(t)} \geq e^{c'} \geq ec' > 1,$$

which implies

$$x(h(t)) \geq (ec')x(t).$$

Repeating the above procedure, it follows by induction that for any positive integer  $k$ ,

$$\frac{x(h(t))}{x(t)} \geq (ec')^k, \quad \text{for sufficiently large } t.$$

Since  $ec' > 1$ , there is  $k \in \mathbb{N}$  satisfying  $k > \frac{2[\ln 2 - \ln c']}{1 + \ln c'}$  such that for  $t$  sufficiently large

$$\frac{x(h(t))}{x(t)} \geq (ec')^k > \left( \frac{2}{c'} \right)^2. \quad (2.36)$$

Further (cf. [17, 1]), for sufficiently large  $t$ , there exists a  $t_m \in (h(t), t)$  such that

$$\int_{h(t)}^{t_m} p(s) \exp \left( \int_{\tau(s)}^{h(s)} p(u) \exp \left( \int_{\tau(u)}^u R_\ell(\xi, \epsilon) d\xi \right) du \right) ds \geq \frac{c'}{2}, \quad (2.37)$$

$$\int_{t_m}^t p(s) \exp \left( \int_{\tau(s)}^{h(s)} p(u) \exp \left( \int_{\tau(u)}^u R_\ell(\xi, \epsilon) d\xi \right) du \right) ds \geq \frac{c'}{2}.$$

Integrating (E) from  $h(t)$  to  $t_m$ , using (2.27) and the fact that  $x(t) > 0$ , we obtain

$$x(h(t)) > x(h(t_m)) \int_{h(t)}^{t_m} p(s) \exp \left( \int_{\tau(s)}^{h(s)} p(u) \exp \left( \int_{\tau(u)}^u R_\ell(\xi, \epsilon) d\xi \right) du \right) ds,$$

which, in view of the first inequality in (2.37), implies that

$$x(h(t)) > \frac{c'}{2} x(h(t_m)). \tag{2.38}$$

Similarly, integrating (E) from  $t_m$  to  $t$ , using (2.27) and the fact that  $x(t) > 0$ , we have

$$x(t_m) > x(h(t)) \int_{t_m}^t p(s) \exp \left( \int_{\tau(s)}^{h(s)} p(u) \exp \left( \int_{\tau(u)}^u R_\ell(\xi, \epsilon) d\xi \right) du \right) ds,$$

which, in view of the second inequality in (2.37), implies that

$$x(t_m) > \frac{c'}{2} x(h(t)). \tag{2.39}$$

Combining the inequalities (2.38) and (2.39), we obtain

$$x(h(t_m)) < \frac{2}{c'} x(h(t)) < \left( \frac{2}{c'} \right)^2 x(t_m),$$

which contradicts (2.36).

The proof of the theorem is complete. □

## 2.2. ADVANCED DIFFERENTIAL EQUATIONS

Oscillation conditions analogous to those obtained for the delay equation (E) can be derived for the (dual) advanced differential equation (E') by following similar arguments with the ones employed for obtaining Theorems 2.4–2.9. The corresponding theorems are stated below while their proofs are omitted, as they are quite similar to those for Theorems 2.4–2.9.

**Theorem 2.10.** *Assume that  $\rho(t)$  is defined by (1.21) and for some  $\ell \in \mathbb{N}$*

$$\limsup_{t \rightarrow \infty} \int_t^{\rho(t)} q(s) \exp \left( \int_{\rho(t)}^{\sigma(s)} q(u) \exp \left( \int_u^{\sigma(u)} G_\ell(\xi) d\xi \right) du \right) ds > 1, \tag{2.40}$$

where

$$G_\ell(t) = q(t) \left[ 1 + \int_t^{\sigma(t)} q(s) \exp \left( \int_t^{\sigma(s)} q(u) \exp \left( \int_u^{\sigma(u)} G_{\ell-1}(\xi) d\xi \right) du \right) ds \right] \tag{2.41}$$

with

$$G_0(t) = q(t) \left[ 1 + \lambda_0 \int_t^{\sigma(t)} q(s) ds \right],$$

and  $\lambda_0$  is the smaller root of the transcendental equation  $\lambda = e^{\beta\lambda}$ . Then all solutions of  $(E')$  are oscillatory.

**Theorem 2.11.** Assume that  $\rho(t)$  is defined by (1.21) and  $\beta \in (0, 1/e]$ . If for some  $\ell \in \mathbb{N}$

$$\limsup_{t \rightarrow \infty} \int_t^{\rho(t)} q(s) \exp \left( \int_{\rho(t)}^{\sigma(s)} q(u) \exp \left( \int_u^{\sigma(u)} G_\ell(\xi) d\xi \right) du \right) ds > 1 - D(\beta), \quad (2.42)$$

where  $G_\ell$  is defined by (2.41), then all solutions of  $(E')$  are oscillatory.

**Theorem 2.12.** Assume that  $\rho(t)$  is defined by (1.21) and  $\beta \in (0, 1/e]$ . If for some  $\ell \in \mathbb{N}$

$$\limsup_{t \rightarrow \infty} \int_t^{\rho(t)} q(s) \exp \left( \int_t^{\sigma(s)} q(u) \exp \left( \int_u^{\sigma(u)} G_\ell(\xi) d\xi \right) du \right) ds > \frac{1}{D(\beta)} - 1, \quad (2.43)$$

where  $G_\ell$  is defined by (2.41), then all solutions of  $(E')$  are oscillatory.

**Theorem 2.13.** Assume that  $\rho(t)$  is defined by (1.21) and  $\beta \in (0, 1/e]$ . If for some  $\ell \in \mathbb{N}$

$$\limsup_{t \rightarrow \infty} \int_t^{\rho(t)} q(s) \exp \left( \int_{\rho(s)}^{\sigma(s)} q(u) \exp \left( \int_u^{\sigma(u)} G_\ell(\xi) d\xi \right) du \right) ds > \frac{1 + \ln \lambda_0}{\lambda_0} - D(\beta), \quad (2.44)$$

where  $G_\ell$  is defined by (2.41) and  $\lambda_0$  is the smaller root of the transcendental equation  $\lambda = e^{\beta\lambda}$ , then all solutions of  $(E')$  are oscillatory.

**Theorem 2.14.** Assume that  $\rho(t)$  is defined by (1.21). If for some  $\ell \in \mathbb{N}$

$$\liminf_{t \rightarrow \infty} \int_t^{\rho(t)} q(s) \exp \left( \int_{\rho(s)}^{\sigma(s)} q(u) \exp \left( \int_u^{\sigma(u)} G_\ell(\xi) d\xi \right) du \right) ds > \frac{1}{e}, \quad (2.45)$$

where  $G_\ell$  is defined by (2.41), then all solutions of  $(E')$  are oscillatory.

### 2.3. DIFFERENTIAL INEQUALITIES

A slight modification in the proofs of Theorems 2.4–2.14 leads to the following results about differential inequalities.

**Theorem 2.15.** *Assume that all the conditions of Theorem 2.4 [2.10] or 2.5 [2.11] or 2.7 [2.12] or 2.8 [2.13] or 2.9 [2.14] hold. Then*

(i) *the delay [advanced] differential inequality*

$$x'(t) + p(t)x(\tau(t)) \leq 0 \quad [x'(t) - q(t)x(\sigma(t)) \geq 0], \quad t \geq t_0,$$

*has no eventually positive solutions;*

(ii) *the delay [advanced] differential inequality*

$$x'(t) + p(t)x(\tau(t)) \geq 0 \quad [x'(t) - q(t)x(\sigma(t)) \leq 0], \quad t \geq t_0,$$

*has no eventually negative solutions.*

### 3. EXAMPLES AND COMMENTS

The examples below illustrate the significance of our results and indicate high level of improvement in the oscillation criteria. The calculations were made by the use of MATLAB software.

**Example 3.1** (taken and adapted from [5]). Consider the delay differential equation

$$x'(t) + \frac{117}{1000}x(\tau(t)) = 0, \quad t \geq 0, \tag{3.1}$$

with (see Figure 1 (a))

$$\tau(t) = \begin{cases} t - 1, & \text{if } t \in [8k, 8k + 2], \\ -4t + 40k + 9, & \text{if } t \in [8k + 2, 8k + 3], \\ 5t - 32k - 18, & \text{if } t \in [8k + 3, 8k + 4], \\ -4t + 40k + 18, & \text{if } t \in [8k + 4, 8k + 5], \\ 5t - 32k - 27, & \text{if } t \in [8k + 5, 8k + 6], \\ -2t + 24k + 15, & \text{if } t \in [8k + 6, 8k + 7], \\ 6t - 40k - 41, & \text{if } t \in [8k + 7, 8k + 8], \end{cases}$$

where  $k \in \mathbb{N}_0$  and  $\mathbb{N}_0$  is the set of nonnegative integers.

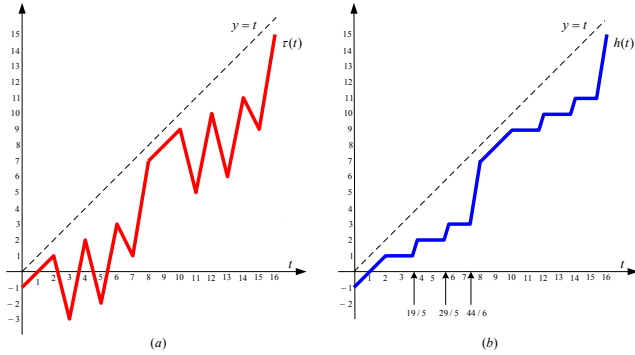
By (1.7), we see (Figure 1 (b)) that

$$h(t) = \begin{cases} t - 1, & \text{if } t \in [8k, 8k + 2], \\ 8k + 1, & \text{if } t \in [8k + 2, 8k + 19/5], \\ 5t - 32k - 18, & \text{if } t \in [8k + 19/5, 8k + 4], \\ 8k + 2, & \text{if } t \in [8k + 4, 8k + 29/5], \\ 5t - 32k - 27, & \text{if } t \in [8k + 29/5, 8k + 6], \\ 8k + 3, & \text{if } t \in [8k + 6, 8k + 44/6], \\ 6t - 40k - 41, & \text{if } t \in [8k + 44/6, 8k + 8]. \end{cases}$$

It is easy to see that

$$\alpha = \liminf_{t \rightarrow \infty} \int_{\tau(t)}^t p(s) ds = \liminf_{k \rightarrow \infty} \int_{8k+1}^{8k+2} \frac{117}{1000} ds = 0.117$$

and therefore, the smaller root of  $e^{0.117\lambda} = \lambda$  is  $\lambda_0 = 1.1431$ .



**Fig. 1.** The graphs of  $\tau(t)$  and  $h(t)$

Observe that the function  $F_\ell : \mathbb{R}_0 \rightarrow \mathbb{R}_+$  defined as

$$F_\ell(t) = \int_{h(t)}^t p(s) \exp \left( \int_{\tau(s)}^{h(t)} p(u) \exp \left( \int_{\tau(u)}^u R_\ell(\xi) d\xi \right) du \right) ds$$

attains its maximum at  $t = 8k + 44/6$ ,  $k \in \mathbb{N}_0$ , for every  $\ell \in \mathbb{N}$ . Specifically

$$F_1(t = 8k + 44/6) = \int_{8k+3}^{8k+44/6} p(s) \exp \left( \int_{\tau(s)}^{8k+3} p(u) \exp \left( \int_{\tau(u)}^u R_1(\xi) d\xi \right) du \right) ds,$$

with

$$R_1(\xi) = p(\xi) \left[ 1 + \int_{\tau(\xi)}^\xi p(v) \exp \left( \int_{\tau(v)}^\xi p(w) \exp \left( \int_{\tau(w)}^w R_0(z) dz \right) dw \right) dv \right]$$

and

$$R_0(z) = p(z) \left[ 1 + \lambda_0 \int_{\tau(z)}^z p(\omega) d\omega \right].$$

By using an algorithm on MATLAB software, we obtain

$$F_1(t = 8k + 44/6) \simeq 1.012$$

and therefore

$$\limsup_{t \rightarrow \infty} F_1(t) \simeq 1.012 > 1.$$

That is, condition (2.4) of Theorem 2.4 is satisfied for  $\ell = 1$ , and therefore all solutions of (3.1) are oscillatory.

Observe, however, that

$$LD = \limsup_{t \rightarrow \infty} \int_{h(t)}^t p(s) ds = \limsup_{k \rightarrow \infty} \int_{8k+3}^{8k+44/6} \frac{117}{1000} ds = 0.507 < 1, \quad \alpha = 0.117 < \frac{1}{e}$$

and

$$0.507 < \frac{1 + \ln \lambda_0}{\lambda_0} - D(\alpha) \simeq 0.984.$$

Noting that the function  $\Phi_j$  defined by

$$\Phi_j(t) = \int_{h(t)}^t p(s) \exp \left( \int_{h(s)}^{h(t)} p(u) \psi_j(u) du \right) ds, \quad (j \geq 2),$$

attains its maximum at  $t = 8k + 44/6$ ,  $k \in \mathbb{N}_0$  for every  $j \geq 2$ . Specifically,

$$\begin{aligned} & \Phi_2(t = 8k + 44/6) \\ &= \int_{8k+3}^{8k+44/6} p(s) \exp \left( \int_{h(s)}^{8k+3} p(u) \psi_2(u) du \right) ds \\ &= \int_{8k+3}^{8k+44/6} \frac{117}{1000} \exp \left( \int_{h(s)}^{8k+3} \frac{117}{1000} \exp \left( \int_{\tau(u)}^u \frac{117}{1000} \cdot 0 dw \right) du \right) ds \\ &= \int_{8k+3}^{8k+44/6} \frac{117}{1000} \exp \left( \int_{h(s)}^{8k+3} \frac{117}{1000} \cdot 1 du \right) ds \\ &= \frac{117}{1000} \cdot \left[ \int_{8k+3}^{8k+19/5} \exp \left( \frac{117}{1000} \int_{8k+1}^{8k+3} du \right) ds + \int_{8k+19/5}^{8k+4} \exp \left( \frac{117}{1000} \int_{5s-32k-18}^{8k+3} du \right) ds \right. \\ & \quad + \int_{8k+4}^{8k+29/5} \exp \left( \frac{117}{1000} \int_{8k+2}^{8k+3} du \right) ds + \int_{8k+29/5}^{8k+6} \exp \left( \frac{117}{1000} \int_{5s-32k-27}^{8k+3} du \right) ds \\ & \quad \left. + \int_{8k+6}^{8k+44/6} \exp \left( \frac{117}{1000} \int_{8k+3}^{8k+3} du \right) ds \right] \simeq 0.5637. \end{aligned}$$

Thus

$$\limsup_{t \rightarrow \infty} \Phi_2(t) \simeq 0.5637 < 1 - D(\alpha) \simeq 0.9922.$$

Also

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \int_{h(t)}^t p(s) \exp \left( \int_{\tau(s)}^{h(t)} p(u) du \right) ds \\ &= \limsup_{k \rightarrow \infty} \int_{8k+3}^{8k+44/6} \frac{117}{1000} \exp \left( \int_{\tau(s)}^{8k+3} \frac{117}{1000} du \right) ds \\ &= \frac{117}{1000} \cdot \limsup_{k \rightarrow \infty} \left[ \int_{8k+3}^{8k+4} \exp \left( \frac{117}{1000} \int_{5s-32k-18}^{8k+3} du \right) ds \right. \\ & \quad + \int_{8k+4}^{8k+5} \exp \left( \frac{117}{1000} \int_{-4s+40k+18}^{8k+3} du \right) ds \\ & \quad + \int_{8k+5}^{8k+6} \exp \left( \frac{117}{1000} \int_{5s-32k-27}^{8k+3} du \right) ds \\ & \quad + \int_{8k+6}^{8k+7} \exp \left( \frac{117}{1000} \int_{-2s+24k+15}^{8k+3} du \right) ds \\ & \quad \left. + \int_{8k+7}^{8k+44/6} \exp \left( \frac{117}{1000} \int_{6s-40k-41}^{8k+3} du \right) ds \right] \simeq 0.6812 < 1, \end{aligned}$$

and

$$0.6812 < 1 - D(\alpha) \simeq 0.9922.$$

Finally, by using algorithms on MATLAB software, we obtain

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \int_{h(t)}^t p(s) \exp \left( \int_{\tau(s)}^{h(t)} p_1(u) du \right) ds \simeq 0.7724 < 1, \\ & 0.7724 < 1 - D(\alpha) \simeq 0.9922, \end{aligned}$$

$$\limsup_{t \rightarrow \infty} \int_{h(t)}^t p(s) \exp \left( \int_{\tau(s)}^{h(t)} p(u) \exp \left( \int_{\tau(u)}^u P_1(\xi) d\xi \right) du \right) ds \simeq 0.9518 < 1,$$



and

$$0.9518 < 1 - D(\alpha) \simeq 0.9922.$$

That is, none of the conditions (1.4)–(1.6), (1.8) (for  $j = 2$ ), (1.9)–(1.10), (1.12)–(1.13) (for  $j = 1$ ) and (1.14)–(1.15) (for  $j = 1$ ) is satisfied.

It is worth noting that the improvement of condition (2.4) to the corresponding condition (1.4) is significant, approximately 99.6%, if we compare the values on the left-side of these conditions. Also, the improvement compared to conditions (1.8), (1.9), (1.12) and (1.14) is very satisfactory, around 79.53%, 48.56%, 31.02% and 6.32%, respectively. In addition, observe that conditions (1.8), (1.12) and (1.14) do not lead to oscillation for first iteration. On the contrary, condition (2.4) is satisfied from the first iteration. This means that our condition is better and much faster than (1.8), (1.12) and (1.14).

**Example 3.2.** Consider the advanced differential equation

$$x'(t) - \frac{1}{8}x(\sigma(t)) = 0, \quad t \geq 0, \tag{3.2}$$

with (see Figure 2 (a))

$$\sigma(t) = \begin{cases} 6t - 35k - 4, & \text{if } t \in [7k + 1, 7k + 2], \\ -2t + 21k + 12, & \text{if } t \in [7k + 2, 7k + 3], \\ 5t - 28k - 9, & \text{if } t \in [7k + 3, 7k + 4], \\ -3t + 28k + 23, & \text{if } t \in [7k + 4, 7k + 5], \\ 7k + 8, & \text{if } t \in [7k + 5, 7k + 6], \\ t + 2, & \text{if } t \in [7k + 6, 7k + 7], \\ 7k + 9, & \text{if } t \in [7k + 7, 7k + 8], \end{cases}$$

where  $k \in \mathbb{N}_0$  and  $\mathbb{N}_0$  is the set of nonnegative integers.

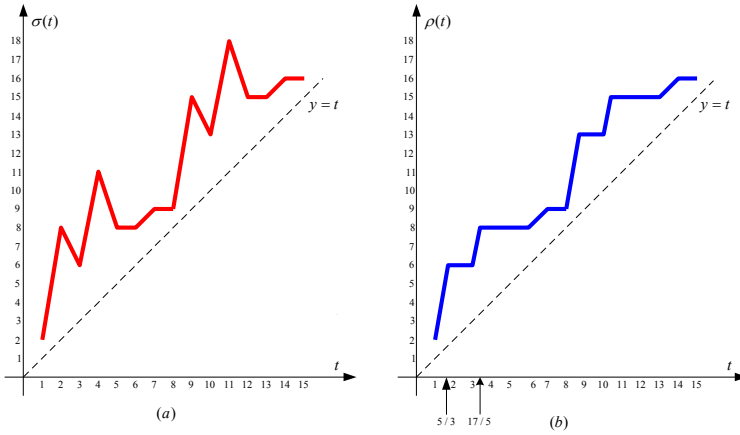
By (1.21), we see (Figure 2 (b)) that

$$\rho(t) = \begin{cases} 6t - 35k - 4, & \text{if } t \in [7k + 1, 7k + 5/3], \\ 7k + 6, & \text{if } t \in [7k + 5/3, 7k + 3], \\ 5t - 28k - 9, & \text{if } t \in [7k + 3, 7k + 17/5], \\ 7k + 8, & \text{if } t \in [7k + 17/5, 7k + 6], \\ t + 2, & \text{if } t \in [7k + 6, 7k + 7], \\ 7k + 9, & \text{if } t \in [7k + 7, 7k + 8]. \end{cases}$$

It is easy to see that

$$\beta = \liminf_{t \rightarrow \infty} \int_{\tau(t)}^t p(s) ds = \liminf_{k \rightarrow \infty} \int_{7k+1}^{7k+2} \frac{1}{8} ds = 0.125$$

and therefore, the smaller solution of  $e^{0.125\lambda} = \lambda$  is  $\lambda_0 = 1.15537$ .



**Fig. 2.** The graphs of  $\sigma(t)$  and  $\rho(t)$

Observe that the function  $F_\ell : \mathbb{R}_0 \rightarrow \mathbb{R}_+$  defined as

$$F_\ell(t) = \int_t^{\rho(t)} q(s) \exp \left( \int_{\rho(t)}^{\sigma(s)} q(u) \exp \left( \int_u^{\sigma(u)} G_\ell(\xi) d\xi \right) du \right) ds$$

attains its maximum at  $t = 7k + 17/5$ ,  $k \in \mathbb{N}_0$ , for every  $\ell \in \mathbb{N}$ . Specifically,

$$F_1(t = 7k + 17/5) = \int_{7k+17/5}^{7k+8} q(s) \exp \left( \int_{7k+8}^{\sigma(s)} q(u) \exp \left( \int_u^{\sigma(u)} G_1(\xi) d\xi \right) du \right) ds,$$

with

$$G_1(\xi) = q(\xi) \left[ 1 + \int_\xi^{\sigma(\xi)} q(v) \exp \left( \int_\xi^{\sigma(v)} q(w) \exp \left( \int_w^{\sigma(w)} G_0(z) dz \right) dw \right) dv \right]$$

and

$$G_0(z) = q(z) \left[ 1 + \lambda_0 \int_z^{\sigma(z)} q(\omega) d\omega \right].$$

By using an algorithm on MATLAB software, we obtain

$$F_1(t = 7k + 17/5) \simeq 1.03233$$

and therefore

$$\limsup_{t \rightarrow \infty} F_1(t) \simeq 1.03233 > 1.$$

That is, condition (2.40) of Theorem 2.10 is satisfied for  $\ell = 1$ , and therefore all solutions of (3.2) are oscillatory.

Observe, however, that

$$LA = \limsup_{t \rightarrow \infty} \int_t^{\rho(t)} q(s) ds = \limsup_{k \rightarrow \infty} \int_{7k+17/5}^{7k+8} \frac{1}{8} ds = 0.575 < 1,$$

$$\beta = 0.125 < \frac{1}{e},$$

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \int_t^{\rho(t)} q(s) \exp \left( \int_{\rho(t)}^{\sigma(s)} q(u) du \right) ds \\ &= \limsup_{k \rightarrow \infty} \int_{7k+17/5}^{7k+8} q(s) \exp \left( \int_{7k+8}^{\sigma(s)} q(u) du \right) ds \\ &= \limsup_{k \rightarrow \infty} \left[ \int_{7k+17/5}^{7k+4} q(s) \exp \left( \int_{7k+8}^{5s-28k-9} q(u) du \right) ds \right. \\ & \quad + \int_{7k+4}^{7k+5} q(s) \exp \left( \int_{7k+8}^{-3s+28k+23} q(u) du \right) ds \\ & \quad + \int_{7k+5}^{7k+6} q(s) \exp \left( \int_{7k+8}^{7k+8} q(u) du \right) ds + \int_{7k+6}^{7k+7} q(s) \exp \left( \int_{7k+8}^{s+2} q(u) du \right) ds \\ & \quad \left. + \int_{7k+7}^{7k+8} q(s) \exp \left( \int_{7k+8}^{7k+9} q(u) du \right) ds \right] \simeq 0.6425 < 1, \end{aligned}$$

$$\begin{aligned} \liminf_{t \rightarrow \infty} \int_t^{\rho(t)} q(s) \exp \left( \int_{\rho(t)}^{\sigma(s)} q(u) du \right) ds &= \liminf_{k \rightarrow \infty} \int_{7k+1}^{7k+2} q(s) \exp \left( \int_{7k+2}^{\sigma(s)} q(u) du \right) ds \\ &= \liminf_{k \rightarrow \infty} \int_{7k+1}^{7k+2} q(s) \exp \left( \int_{7k+2}^{6s-35k-4} q(u) du \right) ds \\ &\simeq 0.186167 < \frac{1}{e}, \end{aligned}$$

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \int_t^{\rho(t)} q(s) \exp \left( \int_{\rho(t)}^{\sigma(s)} q_1(u) du \right) ds \\ &= \limsup_{k \rightarrow \infty} \int_{7k+17/5}^{7k+8} q(s) \exp \left( \int_{7k+8}^{\sigma(s)} q_1(u) du \right) ds \simeq 0.6743, \\ & \qquad \qquad \qquad 0.6743 < 1 - D(\beta) \simeq 0.991, \\ & \limsup_{t \rightarrow \infty} \int_t^{\rho(t)} q(s) \exp \left( \int_{\rho(t)}^{\sigma(s)} q(u) \exp \left( \int_u^{\sigma(u)} Q_1(\xi) d\xi \right) du \right) ds \simeq 0.9211 < 1 \end{aligned}$$

and

$$0.9211 < 1 - D(\beta) \simeq 0.991.$$

That is, none of conditions (1.19)–(1.20), (1.22)–(1.23), (1.24)–(1.25) (for  $j = 1$ ) and (1.26)–(1.27) (for  $j = 1$ ) is satisfied.

It is worth noting that the improvement of condition (2.40) to the corresponding condition (1.19) is significant, approximately 79.54%, if we compare the values on the left-hand side of these conditions. Also, the improvement compared to conditions (1.22), (1.24) and (1.26) is very satisfactory, around 60.67%, 53.1% and 12.08%, respectively. In addition, observe that conditions (1.24) and (1.26) do not lead to oscillation for first iteration. On the contrary, condition (2.40) is satisfied from the first iteration. This means that our condition is better and much faster than (1.24) and (1.26).

**Remark 3.3.** Similarly, one can construct examples to illustrate the other main results.

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