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MINIMAL REALIZATIONS OF GENERALIZED NEVANLINNA FUNCTIONS

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Abstract. Minimal realizations of generalized Nevanlinna functions that carry the information on their generalized poles of nonpositive type in an explicit form are established. These realizations are based on a modification of the basic canonical factorization of generalized Nevanlinna functions whereby the non-minimality problems in realizations that are based directly on the canonical factorization are circumvented.

Keywords: generalized Nevanlinna functions, selfadjoint (multi-valued) operators, (minimal) realizations.

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1. INTRODUCTION

Generalized Nevanlinna functions, introduced and studied by M.G. Kreĭn and H. Langer in a series of papers, see [14–16], as an extension of the class of ordinary Nevanlinna functions, can be realized by means of (multi-valued) operators in Pontryagin spaces. For every generalized Nevanlinna function f there exists a selfadjoint relation A in a Pontryagin space $\{\Pi, [\cdot,\cdot]\}$ with nonempty resolvent set $\rho(A)$ and $\omega \in \Pi$ such that

$$f(z) = \overline{f(z_0)} + (z-\overline{z_0}) \left[\left(I + (z-z_0)(A-z)^{-1} \right) \omega, \omega \right].$$

Such an (operator) realization for f is not unique. Of particular interest are those realization which are minimal; i.e, (operator) realizations for which

$$\Pi = \text{c.l.s.} \left\{ \left(I + (z-z_0)(A-z)^{-1} \right) \omega : z \in \rho(A) \right\}.$$

In that case the analytic behavior of f corresponds to the spectral behavior of A and vice versa; see [8, 14]. This connection shows that explicit minimal (operator) realizations of generalized Nevanlinna functions is an interesting and useful topic. By means of reproducing kernel methods such minimal realizations have been constructed in the literature; the most general case can be found in [1]. However, the nature of those realization is such that they do not provide much explicit information on the generalized poles of the function.

More explicit realizations for generalized Nevanlinna functions have recently been constructed which take into account the following *canonical factorization* of these functions that can be found from [3, 6].

Theorem 1.1. *f is a generalized Nevanlinna function of index κ if and only if there exists an ordinary Nevanlinna function f_0 and a rational function r of degree κ such that $f = rf_0r^{\#}$. Here $r^{\#}(z) = \overline{r(\overline{z})}$.*

Minimal (operator) realizations of generalized Nevanlinna functions based on the above factorization have been constructed by first constructing a minimal realization for the following matrix-valued generalized Nevanlinna function

$$\begin{pmatrix} f_0(z) & 0 & 0 \\ 0 & 0 & r^{\#}(z) \\ 0 & r(z) & 0 \end{pmatrix},$$

where f_0 is an ordinary Nevanlinna function and r is any rational function. By properly restricting such realizations, a realization of the generalized Nevanlinna function $rf_0r^{\#}$ is obtained, see [2, p. 3800] or [7, Theorem 3.2]. However, the realization so obtained need not be minimal; the non-minimality of those realizations is caused by the fact that f_0 might have a generalized zero (pole) at $\beta \in \mathbb{R} \cup \{\infty\}$, whilst r has a pole (zero) at β; see [2, Theorem 4.1] or [7, Corollary 4.3]; cf. Theorem 4.2 below. In the latter paper, this possible non-minimal part is taken care of by going to quotient spaces, while in the former paper the minimality is characterized. Neither procedure explicitly presents the structure of the nonpositive eigenspaces.

The problem with constructing minimal realization from the product factorization of generalized Nevanlinna is illustrated by the following example.

Example 1.2. The function $f(z) = \frac{1}{z}$ belongs to the class of generalized Nevanlinna functions of index 1. The canonical factorization of this function is given by

$$\frac{1}{z} = r(z)f_0(z)r^{\#}(z) = \frac{1}{z}\,z\,\frac{1}{z}.$$

A factorization model for the function f uses a minimal realization of the rational function $r(z)r^{\#}(z) = \frac{1}{z^2}$ in a 2-dimensional Pontryagin space, say $\mathfrak{H}_r$, and a minimal realization of the function $f_0(z) = z$ in a 1-dimensional Hilbert space, say $\mathfrak{H}_0$. The space that then appears as the corresponding realization space for f is the orthogonal sum of these spaces: $\mathfrak{H} = \mathfrak{H}_r \oplus \mathfrak{H}_0$. It is clear that such a realization for f is not minimal: a minimal realization for f can be established in a 1-dimensional Pontryagin space with negative index 1 (i.e. in an anti-Hilbert space).

In this paper minimal realizations of generalized Nevanlinna functions are established in such a manner that the realization takes into account the canonical factorization described in Theorem 1.1 while the minimality of the realization is not violated. These minimal realizations contain full information on generalized poles of nonpositive type of the functions in an explicit form. In particular, the negative index of the functions can be determined from the constructed minimal realizations. After a couple of basic notions and some central properties of generalized Nevanlinna functions are presented in Section 2, we introduce a modification of the canonical factorization which can be used to construct minimal realizations. The main results for the construction of such minimal models can be found in the last two Sections. In Section 3 so-called minimal representations of generalized Nevanlinna functions are introduced, see Theorem 3.5, and in Section 4, corresponding minimal realizations are established; see especially Theorems 4.2 and 4.3.

2. PRELIMINARIES

2.1. ORDINARY AND GENERALIZED NEVANLINNA FUNCTIONS

A symmetric function f, $\overline{f(z)} = f(\overline{z})$, meromorphic on $\mathbb{C}\backslash\mathbb{R}$ is *a generalized Nevanlinna function* if the kernel $N_f(z,w) := (f(z) - \overline{f(w)})(z - \overline{w})^{-1}$ has finitely many negative squares. This means that for arbitrary $z_1, \ldots, z_n$ contained in the intersection of $\mathbb{C}^+$ with the domain of holomorphy of f, the Hermitian matrix $(N_f(z_i, z_j))_{i,j=1,\ldots,n}$ has finitely many negative eigenvalues. The maximum number of negative eigenvalues of all such Hermitian matrices is called the index of f. If f is a generalized Nevanlinna function with index κ, one writes $f \in \mathfrak{N}_\kappa$.

The class of generalized Nevanlinna functions with index zero, $\mathfrak{N}_0$, coincides precisely with the class of (ordinary) Nevanlinna functions $\mathfrak{N}$. Recall that a symmetric complex function f holomorphic on $\mathbb{C} \setminus \mathbb{R}$ is an *ordinary Nevanlinna function* if $\operatorname{Im}(f(z)) \cdot \operatorname{Im} z > 0$ for all $z \in \mathbb{C} \setminus \mathbb{R}$. Ordinary Nevanlinna functions are characterized by their integral representation: $f \in \mathfrak{N}$ if and only if there exists $a \in \mathbb{R}$, $b > 0$ and a nonnegative measure $\mathrm{d}\sigma$, called the *spectral measure* of f, such that

$$f(z) = a + bz + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) \mathrm{d}\sigma(t), \quad \int_{\mathbb{R}} \frac{\mathrm{d}\sigma(t)}{1+t^2} < \infty; \tag{2.1}$$

here b is understood to be the spectral mass at ∞: $b = \mathrm{d}\sigma(\{\infty\})$. Recall that the notation $z \widehat{\to} z_0$ denotes the non-tangential limit from the upper half-plane to a point of the extended real line if $z_0 \in \mathbb{R} \cup \{\infty\}$ and otherwise it denotes an ordinary limit. Using this notation the following basic facts about ordinary Nevanlinna functions can be derived; for the last statement in Lemma 2.1 below see e.g. [13, Theorem on p. 109].

Lemma 2.1. *Let $f \in \mathfrak{N}$ have the representation in* (2.1)*. Then* [2)]

$$\lim_{z \widehat{\to} x \in \mathbb{R}\cup\{\infty\}} (x-z) f(z) = \mathrm{d}\sigma(\{x\}) \in [0, \infty);$$

$$\lim_{z \widehat{\to} x \in \mathbb{R}\cup\{\infty\}} \frac{f(z)}{x-z} \in (-\infty, 0) \quad \textit{or} \quad \lim_{z \widehat{\to} x \in \mathbb{R}} \left| \frac{f(z)}{x-z} \right| = \infty.$$

[2)] Here $(x-z)f(z)$ and $\frac{f(z)}{x-z}$ should be understood to be $f(z)/z$ and $zf(z)$, respectively, if $x = \infty$.

Moreover, for all $x \in \mathbb{R}$ such that $\sigma'(x)$ exists one has that

$$\lim_{z \widehat{\to} x \in \mathbb{R}} \operatorname{Im} f(z) = \pi\sigma'(x) \geq 0.$$

2.2. RELATIONS IN PONTRYAGIN SPACES

A linear space Π together with a sesqui-linear form $[\cdot,\cdot]$ defined on it, is a *Pontryagin space* if there exists an orthogonal decomposition $\Pi^+ + \Pi^-$ of Π such that $\{\Pi^+, [\cdot,\cdot]\}$ and $\{\Pi^-, -[\cdot,\cdot]\}$ are Hilbert spaces, at least one of which is finite-dimensional; here orthogonal means that $[f^+, f^-] = 0$ for all $f^+ \in \Pi^+$ and $f^- \in \Pi^-$. For our purposes it suffices to consider only Pontryagin spaces for which Π^- is finite-dimensional, its dimension (which is independent of the orthogonal decomposition $\Pi^+ + \Pi^-$) is *the negative index* of Π. Recall that the dimension of all negative, nonpositive and neutral subspaces of $\{\Pi, [\cdot,\cdot]\}$ is less than or equal to this negative index.

A mapping H in $\{\Pi, [\cdot,\cdot]\}$ is a *multi-valued operator* (or *relation*) if H is defined on a subset (called $\operatorname{dom} H$) of Π, maps each element $x \in \operatorname{dom} H$ to a subset $Hx := H(x)$ of Π and is linear; i.e., linear relations on Π can be identified with subspaces of $\Pi \times \Pi$. A subspace $\mathfrak{L}$ ($\mathfrak{L} \subset \operatorname{dom} H \oplus \operatorname{mul} H$) of Π is said to be *invariant* under H if

$$H(\mathfrak{L} \cap \operatorname{dom} H) \subseteq \mathfrak{L} + \operatorname{mul} H,$$

where $\operatorname{mul} H = H(0)$ is *the multi-valued part of* H. This is equivalent for $\mathfrak{L}$ to be invariant under the resolvents $(H - zI)^{-1}$, $z \in \rho(H)$. For any relation H in $\{\Pi, [\cdot,\cdot]\}$, its adjoint, denoted as $H^{[*]}$, is defined via its graph as

$$\operatorname{gr} H^{[*]} = \{\{f, f'\} \in \Pi \times \Pi : [f, g'] = [f', g] \text{ for all } \{g, g'\} \in \operatorname{gr} H\}.$$

In particular, if H is a densely defined operator, then $H^{[*]}$ is the operator such that $[f, Hg] = [H^{[*]}f, g]$, for all $f \in \operatorname{dom} H^{[*]}$ and $g \in \operatorname{dom} H$. A relation A in $\{\Pi, [\cdot,\cdot]\}$ is *selfadjoint* if $A = A^{[*]}$ and an operator V from (a Pontryagin space) $\{\Pi_1, [\cdot,\cdot]_1\}$ to (a Pontryagin space) $\{\Pi_2, [\cdot,\cdot]_2\}$ is *isometric* if $[f, g]_1 = [Vf, Vg]_2$ for all $f, g \in \operatorname{dom} V$. An isometric operator U from $\{\Pi_1, [\cdot,\cdot]_1\}$ to $\{\Pi_2, [\cdot,\cdot]_2\}$ is a *standard unitary operator* if $\operatorname{dom} U = \Pi_1$ and $\operatorname{ran} U = \Pi_2$.

2.3. MINIMAL REALIZATIONS OF GENERALIZED NEVANLINNA FUNCTIONS

If A is a selfadjoint relation in (a Pontryagin space) $\{\Pi, [\cdot,\cdot]\}$ with nonempty resolvent set $\rho(A)$ and $\omega \in \Pi$ is such that $f \in \mathfrak{N}_\kappa$ can be written as

$$f(z) = \overline{f(z_0)} + (z - \overline{z_0}) \left[\left(I + (z - z_0)(A - z)^{-1} \right) \omega, \omega \right], \tag{2.2}$$

for some $z_0 \in \rho(A)$, then the pair $\{A, \omega\}$ realizes f (and $f + c$ for every $c \in \mathbb{R}$). In particular, in the expression that $\{A, \omega\}$ realizes f, the *realizing space* $\{\Pi, [\cdot,\cdot]\}$ and the selection of the arbitrary but fixed point $z_0 \in \rho(A)$ are suppressed. To a realization $\{A, \omega\}$ we associated a *γ-field*, γ_z, via

$$\gamma_z = \left(I + (z - z_0)(A - z)^{-1} \right) \omega, \qquad z \in \rho(A). \tag{2.3}$$

By means of the γ-field and the resolvent identity, (2.2) can also be written as

$$\frac{f(z) - \overline{f(w)}}{z - \overline{w}} = \gamma_w^{[*]}\gamma_z = [\gamma_z, \gamma_w], \qquad z, w \in \rho(A).$$

A pair $\{A, \omega\}$ realizes f *minimally* if

$$\Pi = \text{c.l.s.}\ \{\gamma_z : z \in \rho(A)\} = \text{c.l.s.}\ \left\{\left(I + (z - z_0)(A - z)^{-1}\right)\omega : z \in \rho(A)\right\}.$$

If the realization is minimal, then $\rho(A)$ coincides with the domain of holomorphy of f, see [8, Theorem 1.1]. Moreover, if $\rho(A) \neq \emptyset$, then $\rho(A)$ contains all of $\mathbb{C} \setminus \mathbb{R}$ except at most finitely many points, see e.g. [9, Proposition 4.4]. The preceding implies that if f is not identically equal to zero, then z_0 in (2.2) can be taken such that $f(z_0)$ is invertible.

All minimal realizations of generalized Nevanlinna functions are essentially equal: if $f \in \mathfrak{N}_\kappa$ is minimally realized by $\{A_i, \omega_i\}$, where $\rho(A_i) \neq \{0\}$, for $i = 1, 2$. Then there exists a standard unitary operator from $\{\Pi_1, [\cdot,\cdot]_1\}$ to $\{\Pi_2, [\cdot,\cdot]_2\}$ such that $A_2 = UA_1U^{-1}$ and $\omega_2 = U\omega_1$, see e.g. [11, Theorem 3.2]. The existence of a pair $\{A, \omega\}$ minimally realizing an arbitrary $f \in \mathfrak{N}_\kappa$ has essentially been established in [14]; cf. [8, Section 2]. Since those minimal realizations are in a Pontryagin space whose negative index is κ, the preceding facts imply that all minimal realizations of $f \in \mathfrak{N}_\kappa$ are in Pontryagin spaces with negative index κ. Non-minimal realizations can always be reduced to minimal ones by going to a quotient space invariant under the realizing operator, see [19, Proposition 2.2]. The following criterium for the minimality of a realization illustrates the preceding.

Corollary 2.2 ([19, Corollary 2.3])**.** *Let $\{A, \omega\}$ realize $f \in \mathfrak{N}_\kappa$. Then $\{A, \omega\}$ realizes f minimally if and only if there exists no non-trivial A-invariant subspace $\mathfrak{L}$ such that $[h, \omega] = 0$ for all $h \in \mathfrak{L}$.*

Theorem 1.1 shows that if $f \in \mathfrak{N}_\kappa \setminus \{0\}$, then also $-f^{-1} \in \mathfrak{N}_\kappa$. There is an explicit connection between (minimal) realizations of those two generalized Nevanlinna functions if f is not constant: let A be a selfadjoint relation in $\{\Pi, [\cdot,\cdot]\}$ and $\omega \in \Pi$ be such that $\{A, \omega\}$ realizes $f \in \mathfrak{N}_\kappa$. Note here that since f is not constant $\rho(A) \neq \emptyset$ and, in particular, there exists $z_0 \in \rho(A)$ such that $f(z_0) \neq 0$. Using any such z_0 define $\widehat{\omega} \in \Pi$ and the selfadjoint relation $\widehat{A}$ in $\{\Pi, [\cdot,\cdot]\}$ via

$$\widehat{\omega} = -(f(z_0))^{-1}\omega \quad \text{and} \quad (\widehat{A} - z)^{-1} = (A - z)^{-1} - \gamma_z (f(z))^{-1}[\cdot, \gamma_{\overline{z}}], \tag{2.4}$$

see (2.3). Then $\{A, \omega\}$ realizes f (minimally) if and only if $\{\widehat{A}, \widehat{\omega}\}$ realizes $-f^{-1}$ (minimally), see e.g. [18].

2.4. GENERALIZED POLES

Recall, see [17], that if f is a generalized Nevanlinna function, then $\beta \in \mathbb{R}$ or $\beta = \infty$ is a *generalized pole of positive type* of f if

$$\lim_{z \widehat{\to} \beta} (\beta - z) f(z) \in (0, \infty) \quad \text{or} \quad \lim_{z \widehat{\to} \beta = \infty} \frac{f(z)}{z} \in (0, \infty), \tag{2.5}$$

cf. Lemma 2.1. Furthermore, $\beta \in \mathbb{C} \cup \{\infty\}$ is a *generalized pole of nonpositive type* (*GPNT*) of f if i) $\beta \in \mathbb{C} \setminus \mathbb{R}$ is a singularity of f in which case the (positive) order of the pole β is its multiplicity, or if ii) $\beta \in \mathbb{R}$ or $\beta = \infty$ and the smallest nonnegative number π such that

$$\lim_{z \widehat{\to} \beta} (\beta - z)^{2\pi+1} f(z) \in [0, \infty) \quad \text{or} \quad \lim_{z \widehat{\to} \infty} f(z) z^{-2\pi-1} \in [0, \infty) \tag{2.6}$$

is positive; in this case $\pi\ (= \pi_\beta(f))$ is the multiplicity of β. In particular, $\beta \in \mathbb{R} \cup \{\infty\}$ is a *generalized pole* of f if it is either a generalized pole of positive or nonpositive type. Similarly, $\alpha \in \mathbb{C} \cup \{\infty\}$ is a *generalized zero* (*of positive type, or of nonpositive type* (*GZNT*)) of f if and only if α is a generalized pole (of positive or nonpositive type) of $-f^{-1} \in \mathfrak{N}_\kappa$, respectively. The multiplicity of the GZNT α is defined by $\kappa_\alpha(f) = \pi_\alpha(-f^{-1})$. Using the introduced terminology, the following well-known, see [17], characterization of the index of a generalized Nevanlinna function is obtained from Theorem 1.1 and Lemma 2.1.

Corollary 2.3. *Let $f \in \mathfrak{N}_\kappa$. Then f has precisely κ GPNTs and κ GZNTs in $\mathbb{C} \cup \{\infty\}$ when taking into account their multiplicities.*

The following basic result slightly extending [4, Lemmas 2.2 & 2.3] shows that the generalized poles of generalized Nevanlinna functions correspond in their operator realization to the spectral points of the realization.

Lemma 2.4. *Let $\{A, \omega\}$, $\rho(A) \neq \emptyset$, realize $f \in \mathfrak{N}_\kappa$ minimally. Then $\gamma \in \mathbb{C} \cup \{\infty\}$ is a generalized pole of f if and only if $\gamma \in \sigma_p(A)$.*

For $\gamma = \infty$ the equivalence should be interpreted as follows: ∞ is a generalized pole of f if and only if A has a non-trivial multi-valued part.

Proof. The cases $\gamma = \infty$ and $\gamma \in \mathbb{R}$ are contained in [4, Lemmas 2.2 & 2.3]. Hence, only the case $\gamma \in \mathbb{C} \setminus \mathbb{R}$ needs to be considered. Since $\rho(A)$ coincides with the domain of holomorphy of f, see e.g. [8, Theorem 8.8], and $\rho(A)$ contains all of $\mathbb{C} \setminus \mathbb{R}$ except finitely many points, f has an isolated pole at γ if $\gamma \in \sigma_p(A)$ and is holomorphic at γ if $\gamma \notin \sigma_p(A)$. This proves the statement in this case by the definition of a non-real generalized pole. □

Finally, for a generalized Nevanlinna function f define $\mathcal{P}_f$ to be the set of all $\beta \in \mathbb{R} \cup \{\infty\}$ for which there exists a positive integer π such that

$$\lim_{z \widehat{\to} \beta} (\beta - z)^{2\pi-1} f(z) \in (-\infty, 0) \quad \text{or} \quad \lim_{z \widehat{\to} \infty} f(z) z^{-2\pi+1} \in (-\infty, 0). \tag{2.7}$$

In particular, $\mathcal{P}_f$ is a subset of the set of all GPNTs of f, see (2.6). Furthermore, define $\mathcal{Z}_f$ to be $\mathcal{P}_{-f^{-1}}$. These sets are more easily understood when expressed by means of the product factorization of generalized Nevanlinna functions in Theorem 1.1; for an analogous characterization, see [5, Section 3.2 & Proposition 6.6].

Proposition 2.5. *Let $rf_0r^{\#}$, where $f_0 \in \mathfrak{N}$, be a factorization of $f \in \mathfrak{N}_\kappa$ as in Theorem 1.1. Then*

$$\begin{aligned}\mathcal{P}_f &= \{\beta \in \mathbb{R} \cup \{\infty\} : \beta \text{ is a pole of } r \text{ and a generalized zero of } f_0\},\\ \mathcal{Z}_f &= \{\alpha \in \mathbb{R} \cup \{\infty\} : \alpha \text{ is a zero of } r \text{ and a generalized pole of } f_0\}.\end{aligned} \tag{2.8}$$

Proof. It suffices to prove the expression for $\mathcal{P}_f$. If $\beta \in \mathbb{R}$, rewrite r as $(z-\beta)^{-n}s(z)$ where $n \in \mathbb{N}$ and s is a rational function not having β as a pole or zero, then

$$\begin{aligned}\lim_{z \widehat{\to} \beta} (\beta - z)^{2\pi-1} f(z) &= \lim_{z \widehat{\to} \beta} (\beta - z)^{2\pi-1} r(z) f_0(z) r^{\#}(z)\\ &= s(\beta)s^{\#}(\beta) \lim_{z \widehat{\to} \beta} \frac{f_0(z)}{(\beta - z)^{2(n-\pi)+1}}.\end{aligned} \tag{2.9}$$

Recall that $\lim_{z \widehat{\to} \beta} (\beta - z) f_0(z) \in [0,\infty)$ for all β and that $\lim_{z \widehat{\to} \beta} \frac{f_0(z)}{\beta - z}$ is negative if β is a generalized zero of positive type of f_0 and that $\lim_{z \widehat{\to} \beta} \left|\frac{f_0(z)}{\beta - z}\right| = \infty$ otherwise, see Lemma 2.1 and (2.5). With this in mind, (2.9) shows that if $\beta \in \mathcal{P}_f$, then $n = \pi > 0$ (β is a pole of r) and β is a generalized zero of positive type of f_0. Conversely, if β is a pole of r and a generalized zero of positive type of f_0, then (2.9) shows that (2.7) holds with $\pi = n > 0$. Since a similar argument holds if $\beta = \infty$, the expression for $\mathcal{P}_f$ has been proven. □

3. A MINIMAL REPRESENTATION OF GENERALIZED NEVANLINNA FUNCTIONS

Theorem 4.2 below shows that a minimal realization for f is readily obtained when $\mathcal{P}_f$ and $\mathcal{Z}_f$, see Proposition 2.5, are empty. When they are nonempty, a minimal realization for f cannot directly be constructed from the product representation of f as in Theorem 1.1. Instead in those cases that representation needs to be rewritten into what we propose to call *a minimal representation*, see Theorem 3.5 below. This minimal representation of f is obtained from the product representation of $f \in \mathfrak{N}_\kappa$ in Theorem 1.1 by making explicit how that representation can be modified if $\mathcal{P}_f \neq \emptyset$. This is done by means of three statements starting with the following basic lemma. In this first lemma the equivalence of (i) and (ii) can be considered as folklore.

Lemma 3.1. *Let $f \in \mathfrak{N}$ have spectral measure $\mathrm{d}\sigma$ and let $f_\infty(z) := f(z) - bz$, where b is the spectral mass at infinity; see (2.1). Then equivalent are*

(i) *there exists $c \in \mathbb{R}$ such that $\lim_{z \widehat{\to} \infty} z\,(f_\infty(z) - c)$ exists;*
(ii) *there exists $c \in \mathbb{R}$ such that $\lim_{z \widehat{\to} \infty} f_\infty(z) = c$ and $\int_{\mathbb{R}} \mathrm{d}\sigma(t) < \infty$;*
(iii) *for every $\alpha \in \mathbb{R}$ there exists $h \in \mathfrak{N}$ for which ∞ is not a generalized pole (of positive type), and $m_1, m_2, m_3 \in \mathbb{R}$, such that*

$$(z-\alpha)^2 f(z) = m_3 z^3 + m_2 z^2 + m_1 z + h(z). \tag{3.1}$$

Moreover, if any of the above three equivalent conditions holds, then, with $c := \lim_{z \widehat{\to} \infty} f_\infty(z)$*, the constant* m_1*,* m_2 *and* m_3 *in* (3.1) *are*

$$m_3 = b, \qquad m_2 = c - 2\alpha b,$$
$$m_1 = \alpha^2 b - 2\alpha c + \lim_{z \widehat{\to} \infty} z\,(f_\infty(z) - c) = \alpha^2 b - 2\alpha c - \int_{\mathbb{R}} \sigma(t),$$

and the spectral measure of h *is* $(t-\alpha)^2 \mathrm{d}\sigma(t)$.

Proof. We make use of the integral representation of f in (2.1):

$$f(z) = a + bz + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) \mathrm{d}\sigma(t), \qquad \int_{\mathbb{R}} \frac{\mathrm{d}\sigma(t)}{1+t^2} < \infty. \tag{3.2}$$

(i)⇒(ii). If (i) holds, then certainly there exists c as in (ii). Let $f_1(z) := (f(z) - bz - c) \in \mathfrak{N}$, then (i) implies that $\lim_{h\to\infty} ihf_1(ih)$ exists. Therefore, by [12, Lemma S.1.1.1] the spectral measure $\mathrm{d}\sigma_1$ of f_1 satisfies $\int_{\mathbb{R}} \mathrm{d}\sigma_1(t) < \infty$ and, hence, (ii) holds, because $\mathrm{d}\sigma_1(t) \equiv \mathrm{d}\sigma(t)$ for all $t \in \mathbb{R}$, cf. Lemma 2.1.

(ii)⇒(iii). If (ii) holds, then a in (3.2) is equal to $c + \int_{\mathbb{R}} \frac{t\mathrm{d}\sigma(t)}{1+t^2}$. Hence,

$$(z-\alpha)^2 f(z) = (z-\alpha)^2 \left(bz + c + \int_{\mathbb{R}} \frac{\mathrm{d}\sigma(t)}{t-z} \right).$$

The integral condition satisfied by $\mathrm{d}\sigma$ shows that

$$\begin{aligned}(z-\alpha)^2 \int_{\mathbb{R}} \frac{\mathrm{d}\sigma(t)}{t-z} &= (z-\alpha) \left(-\int_{\mathbb{R}} \mathrm{d}\sigma(t) + \int_{\mathbb{R}} \frac{(t-\alpha)\mathrm{d}\sigma(t)}{t-z} \right) \\ &= -(z-\alpha) \int_{\mathbb{R}} \mathrm{d}\sigma(t) + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) (t-\alpha)^2 d\sigma(t) \\ &\quad + \int_{\mathbb{R}} \left(\frac{t(t-\alpha)}{1+t^2} - 1 \right) (t-\alpha)\mathrm{d}\sigma(t).\end{aligned}$$

Note that the integrant of the last expression is equal to

$$-\frac{(1+\alpha t)(t-\alpha)}{1+t^2}\,\mathrm{d}\sigma(t)$$

and, therefore, integrable by the assumption on $\mathrm{d}\sigma$ in (ii). Combining the preceding calculations shows that (iii) holds, where the spectral measure of h is $(t-\alpha)^2\mathrm{d}\sigma(t)$.

(iii) ⇒ (i). If (iii) holds, then clearly $m_3 = b$; see (3.2) and Lemma 2.1. Therefore

$$(z-\alpha)^2 (f(z) - bz) = (m_2 + 2\alpha b) z^2 + (m_1 - \alpha^2 b) z + h(z).$$

This shows that $c := \lim_{z \hat{\to} \infty}(f(z) - bz) = (m_2 + 2\alpha b)$, because $\lim_{z \hat{\to} \infty} \frac{h(z)}{(z-\alpha)^2} = 0$ by Lemma 2.1. Consequently,

$$(z - \alpha)^2 (f(z) - bz - c) = (m_1 - \alpha^2 b + 2\alpha c)z - \alpha^2 c + h(z).$$

From this it is easily seen that (i) holds. □

Transforming Lemma 3.1 to any point of the real line by means of the transformation $\tau(z) = (\beta - z)^{-1}$, $\beta \in \mathbb{R}$, yields the following result.

Lemma 3.2. *Let $f \in \mathfrak{N}$ have spectral measure* $\mathrm{d}\sigma$ *and for $\beta \in \mathbb{R}$ let $f_\beta(z) := f(z) - \frac{\sigma(\{\beta\})}{\beta - z}$. Then equivalent are*

(i) *there exists $c \in \mathbb{R}$ such that $\lim_{z \hat{\to} \beta} \frac{f_\beta(z) - c}{\beta - z}$ exists;*
(ii) *there exists $c \in \mathbb{R}$ such that $\lim_{z \hat{\to} \beta} f_\beta(z) = c$ and $\int_{0<|t-\beta|<1} \frac{\mathrm{d}\sigma(t)}{(t-\beta)^2} < \infty$;*
(iii) *for every $\alpha \in (\mathbb{C} \cup \{\infty\}) \setminus \{\beta\}$ there exists $h \in \mathfrak{N}$ for which β is not a generalized pole (of positive type), and $m_1, m_2, m_3 \in \mathbb{R}$ such that*

$$\frac{(z-\alpha)(z-\overline{\alpha})}{(z-\beta)^2} f(z) = \frac{m_3}{(\beta - z)^3} + \frac{m_2}{(\beta - z)^2} + \frac{m_1}{\beta - z} + h(z). \tag{3.3}$$

Moreover, if any of the above three equivalent conditions holds, then, with $c := \lim_{z \hat{\to} \beta} f_\beta(z)$, the constant m_1, m_2 and m_3 in (3.3) *are*

$$m_3 = |\beta - \alpha|^2 \sigma(\{\beta\}), \qquad m_2 = |\beta - \alpha|^2 c + (\alpha + \overline{\alpha} - 2\beta)\sigma(\{\beta\}),$$
$$m_1 = |\beta - \alpha|^2 \lim_{z \hat{\to} \beta} \frac{f_\beta(z) - c}{\beta - z} + \sigma(\{\beta\}) + (\alpha + \overline{\alpha} - 2\beta)c,$$

and the spectral measure of h is $\frac{(t-\alpha)^2}{(t-\beta)^2}\,\mathrm{d}\sigma(t)$.

If $\alpha = \infty$ in Lemma 3.2, then $(z - \alpha)(z - \overline{\alpha})$ in (3.3) should be interpreted to be one. Moreover, in that case $m_3 = \sigma(\{\beta\})$, $m_2 = c$ and $m_1 = \lim_{z \hat{\to} \beta} \frac{f_\beta(z) - c}{\beta - z}$.

By means of the preceding lemmas, a certain class of generalized Nevanlinna functions can be rewritten into a form being more amendable to a minimal realization.

Proposition 3.3. *Let $f \in \mathfrak{N}$ and let r be a rational function of degree n which has n distinct poles $\beta_1, \ldots, \beta_n \in \mathbb{R} \cup \{\infty\}$. If these β_i are generalized zeros (of positive type) of f, i.e. if $d_i := \lim_{z \hat{\to} \beta_i} \frac{f(z)}{\beta_i - z} \in (-\infty, 0)$, then*

$$r(z) f(z) r^{\#}(z) = \sum_{i=1}^{n} \frac{c_i}{\beta_i - z} + h(z),$$

where $c_i = d_i \lim_{z \to \beta_i}(z - \beta_i)^2 r(z) r^{\#}(z) \in (-\infty, 0)$ and β_i is not a generalized pole of $h \in \mathfrak{N}$, for $i = 1, \ldots, n$.

If one of the constants β_i, say β_n is ∞, then d_n, $\frac{c_n}{\beta_n - z}$ and c_n should be interpreted to be $\lim_{z \widehat{\to} \infty} zf(z)$, $c_n z$ and $d_n \lim_{z\to\infty} r^{\#}(z) r(z) z^{-2}$, respectively. Observe that

$$r_0(z) = -\sum_{i=1}^{n} \frac{c_i}{\beta_i - z}$$

is a rational Nevanlinna function whose degree is equal to the degree of r.

Proof. First assume that all β_i are finite. According to Theorem 1.1 the function $f_n(z) := r(z)f(z)r^{\#}(z)$ is a generalized Nevanlinna function with index $\kappa = n$ and it can be written as

$$f_n(z) = \frac{\prod_{i=1}^{n}(z-\alpha_i)(z-\overline{\alpha_i})}{\prod_{i=1}^{n}(z-\beta_i)^2} f(z),$$

where the $\alpha_i \in (\mathbb{C} \cup \{\infty\}) \setminus \{\beta_1, \dots, \beta_n\}$. Here $(z - \alpha_i)$ should be interpreted as one if $\alpha_i = \infty$. The assumption $d_i = \lim_{z \widehat{\to} \beta_j} \frac{f(z)}{\beta_j - z} \in (-\infty, 0)$ implies that $\sigma(\{\beta_j\}) = 0$ and that $\lim_{z \widehat{\to} \beta_i} f(z) = 0$; see Lemma 2.1. Applying Lemma 3.2 n times yields that there exists $f_j \in \mathfrak{N}$ for which β_j is not a generalized pole such that

$$f_n(z) = p_j(z) \left(\frac{d_j |\beta_j - \alpha_j|^2}{\beta_j - z} + f_j(z) \right), \tag{3.4}$$

where

$$p_j(z) = \prod_{i=1,\dots,j-1,j+1,\dots,n} \frac{(z-\alpha_i)(z-\overline{\alpha_i})}{(z-\beta_i)^2}, \quad j = 1, \dots, n;$$

see (3.3) with $m_3 = m_2 = 0$ and $m_1 = |\beta_j - \alpha_j|^2 d_i < 0$. Next define

$$f_l(z) := -\sum_{j=1}^{n} \frac{d_j |\beta_j - \alpha_j|^2 p_j(\beta_j)}{\beta_j - z}. \tag{3.5}$$

Then $f_l \in \mathfrak{N}$, because $-d_j$ and $p_j(\beta_j)$ are nonnegative. Therefore $h := f_n + f_l$ being the sum of a generalized Nevanlinna function and an ordinary Nevanlinna function is a generalized Nevanlinna functions whose GPNTs are a subset of the GPNTs of f_n; cf. (2.6). For every β_j we have that

$$\lim_{z \widehat{\to} \beta_j} (\beta_j - z) h(z) = \lim_{z \widehat{\to} \beta_j} (\beta_j - z) f_n(z) + \lim_{z \widehat{\to} \beta_j} (\beta_j - z) f_l(z) = 0,$$

see (3.4) and (3.5). This shows that no β_j is a generalized pole of h. Since the β_j are the only GPNTs of f_n, we conclude that h has no GPNTs (see (2.6)). Therefore, $h \in \mathfrak{N}$ by Corollary 2.3 and the statement holds in this case.

The case that one β_i is infinite can be treated analogously by making again use of Lemma 3.1; the details will be omitted here. □

Remark 3.4. If f has the integral representation in (2.1), then Lemma 3.1 and Lemma 3.2 (or Lemma 2.1) yield that h in Proposition 3.3 has the integral representation

$$h(z) = \operatorname{Re} h(i) + b_r z + \int_{\Delta_r} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) r^{\#}(t) r(t) d\sigma(t),$$

where $\Delta_r = \mathbb{R} \setminus \{\beta_1, \ldots, \beta_n\}$ and $b_r = \lim_{z \widehat{\to} \infty} b\, r^{\#}(z) r(z)$ if ∞ is not a pole of r and zero otherwise.

Making use of Proposition 3.3, a minimal representation of generalized Nevanlinna functions is established.

Theorem 3.5. *A function f is a generalized Nevanlinna function with index κ satisfying $\#\mathcal{Z}_f \leq \#\mathcal{P}_f$ (see (2.8)) if and only if*

$$f(z) = r_1(z) \left(h(z) - r_0(z) \right) r_1^{\#}(z), \tag{3.6}$$

where

(i) $r_0 \in \mathfrak{N}$ *is a rational function automatically of degree $\#\mathcal{P}_f$ with poles at $\mathcal{P}_f$;*
(ii) $h \in \mathfrak{N}$ *and no pole of r_0 is a generalized pole of h;*
(iii) r_1 *is a rational function of degree $\kappa - \#\mathcal{P}_f$ and if $\beta \in \mathbb{C} \cup \{\infty\}$ is a zero (pole) of r_1, then β is not a generalized pole (zero) of $h - r_0$.*

Proof. ($\Rightarrow$) By Theorem 1.1 there exists a rational function r of degree κ and $f_0 \in \mathfrak{N}$ such that $f = r f_0 r^{\#}$. Let $r_1 r_2$ be any decomposition of r such that r_2 is a rational function of degree $\#\mathcal{P}_f$ whose set of poles is $\mathcal{P}_f$ and whose set of zeros contains $\mathcal{Z}_f$. Then by Proposition 3.3 $r_2 f_0 r_2^{\#}$ can be written as $h - r_0$, where h and r_0 have the properties (i)–(iii) listed in the statement.

($\Leftarrow$) Assume that f has the representation (3.6) and that (i)–(iii) hold. Clearly, $h - r_0$ is a generalized Nevanlinna function and hence by Theorem 1.1 it admits a (unique) factorization of the form

$$h - r_0 = r_2 f_0 r_2^{\#}. \tag{3.7}$$

It follows from (i) and (ii) that the index of the generalized Nevanlinna function $h - r_0$ is equal the degree of r_0, see Corollary 2.3. In particular, $\deg r_2 = \deg r_0$, see Theorem 1.1. Moreover, (iii) implies that r_1 and r_2 are relatively prime, i.e., the poles (zeros) of r_1 cannot be zeros (poles) of r_2. This means that f can be written as

$$f = r_1 r_2 f_0 r_2^{\#} r_1^{\#} = r f_0 r^{\#}. \tag{3.8}$$

Here $r = r_1 r_2$ is a rational function of degree κ. Thus $f \in \mathfrak{N}_\kappa$ by Theorem 1.1.

Next it is shown that the set of poles of r_0 coincides with the set $\mathcal{P}_f$ and, in particular, that $\#\mathcal{P}_f = \deg r_0$. Let $\beta \in \mathbb{R}$ be a pole of r_0, then there exists $n \in \mathbb{N}_0$ such that r_1 can be written as $(z-\beta)^{-n} s_1(z)$, where s_1 does not have a pole or zero at β. Now (3.6) shows in light of conditions (i) and (ii) that

$$\lim_{z \widehat{\to} \beta} (\beta - z)^{2(n+1)-1} f(z) = \lim_{z \widehat{\to} \beta} (\beta - z) s_1(z) \left(h(z) - r_0(z) \right) s_1^{\#}(z) \in (-\infty, 0).$$

Therefore $\beta \in \mathcal{P}_f$, see (2.7). Conversely, if $\beta \in \mathcal{P}_f$, then β is a generalized zero of f_0 as in (3.7) and (3.8) by Proposition 2.5 while by (iii) it cannot be a generalized zero of $h - r_0 = r_2 f_0 r_2^{\#}$. Hence, β must be a GPNT of $r_2 f_0 r_2^{\#}$ and, therefore, also a pole of r_0. Since a similar argument holds for $\beta = \infty$, we have shown that the set of poles of r_0 coincides with the set $\mathcal{P}_f$. In particular, one has $\#\mathcal{P}_f = \deg r_0 (= \deg r_2)$ and $\kappa = \deg r_1 + \#\mathcal{P}_f$.

Finally, if $\alpha \in \mathcal{Z}_f$, then, in view of (3.8), Proposition 2.5 implies that $r(\alpha) = 0$. If $r_2(\alpha) \neq 0$ then $r_1(\alpha) = 0$ and, moreover, α as a generalized pole of f_0 is also a generalized pole of $r_2 f_0 r_2^{\#}$; a contraction to the assumption (iii). Therefore, $r_2(\alpha) = 0$ must hold and consequently $\#\mathcal{Z}_f \leq \deg r_2 = \#\mathcal{P}_f$. This completes the proof. □

4. A MINIMAL REALIZATION

Theorem 3.5 shows that any generalized Nevanlinna function can be represented as

$$f(z) = r_1(z)\,(h(z) - r_0(z))\,r_1^{\#}(z),$$

where

(i) $h, r_0 \in \mathfrak{N}$, where r_0 is a rational function, and no generalized pole (of positive type) of h is a generalized pole (of positive type) of r_0;
(ii) no pole of the rational function r_1 is a generalized zero of $h - r_0$ and no zero of r_1 is a generalized pole of $h - r_0$.

According to the above decomposition an explicit minimal realization for f can be constructed in the following manner:

(Step 1) a minimal realization for h is constructed;
(Step 2) given a minimal realization for h, a minimal realization for $h - r_0$ is constructed;
(Step 3) given a minimal realization for any $g \in \mathfrak{N}_\kappa$ and any rational function r of degree one, whose pole is not a generalized zero of g and whose zero is not a generalized pole of g of nonpositive type, a minimal realization for $rgr^{\#}$ is explicitly constructed;
(Step 4) repeating the procedure in Step 3.

4.1. A MINIMAL REALIZATION FOR ORDINARY NEVANLINNA FUNCTIONS

There are several (unitarily equivalent) minimal realizations of ordinary Nevanlinna functions stated in the literature. Here we present the $\mathrm{L}^2(d\sigma)$-realization from [10]. Recall that if $h \in \mathfrak{N}$, then it has the integral representation (2.1):

$$h(z) = a + bz + \int_{\mathbb{R}} \left(\frac{1}{t - z} - \frac{t}{1 + t^2} \right) \mathrm{d}\sigma(t), \quad \int_{\mathbb{R}} \frac{\mathrm{d}\sigma(t)}{1 + t^2} < \infty,$$

where $a \in \mathbb{R}$, $b \geq 0$ and $d\sigma$ is a nonnegative measure. Using these objects define $\{\mathfrak{H}, (\cdot,\cdot)\}$ as $\mathfrak{H} = L^2(\mathrm{d}\sigma) \times \mathbb{C}$ with the inner product being given by

$$(\{f(t), f_\infty\}, \{g(t), g_\infty\}) := \int_{\mathbb{R}} \overline{g(t)} f(t) \mathrm{d}\sigma(t) + \overline{g_\infty} b f_\infty, \quad \{f(t), f_\infty\}, \{g(t), g_\infty\} \in \mathfrak{H}.$$

Here $\{\mathfrak{H}, (\cdot,\cdot)\}$ should be interpreted to be $L^2(\mathrm{d}\sigma)$ if $b = 0$. In this space define the operator A_h via

$$\mathrm{gr}(A_h) = \{\{\{f(t), 0\}, \{tf(t), f_\infty\}\} : f(t), tf(t) \in L^2(\mathrm{d}\sigma), \;\; f_\infty \in \mathbb{C}\},$$

and let $\omega_h(t) := \{(t - z_0)^{-1}, 1\}$ (again these objects should be interpreted properly if $b = 0$). Then $\{A_h, \omega_h\}$ realizes h minimally, see [10, Theorem 2.5].

4.2. A MINIMAL REALIZATION FOR THE DIFFERENCE OF ORDINARY NEVANLINNA FUNCTIONS

In this second step we construct a realization for $h - r_0$ where h is an ordinary Nevanlinna function and r_0 is a rational ordinary Nevanlinna function from the explicit realization of h presented in the preceding section.

Proposition 4.1. *Let $\{A_h, \omega_h\}$ realize $h \in \mathfrak{N}$ minimally* (*see Section 4.1*) *and assume that the distinct points $b_1, \ldots, b_n \in \mathbb{R} \cup \{\infty\}$ are not generalized poles* (*of positive type*) *of h: $\lim_{z \hat{\to} b_i} (b_i - z)h(z) = 0$, $1 \leq i \leq n$. Then*

$$h(z) - \sum_{i=1}^{n} \frac{c_i}{b_i - z}, \quad c_i > 0.$$

is minimally realized by $\{A, \omega\}$ defined as

$$A := A_h \oplus \mathrm{diag}\,(b_1, \ldots, b_n) \quad \textit{and} \quad \omega := \omega_h \oplus ((b_1 - z_0)^{-1}, \ldots, (b_n - z_0)^{-1}).$$

The corresponding realizing space is the Pontryagin space $\{\Pi, [\cdot,\cdot]\}$ with negative index n defined as $\Pi = \mathfrak{H} \oplus \mathbb{C}^n$ and

$$[g, h] := [g_0 \oplus \{g_1, \ldots, g_n\}, h_0 \oplus \{h_1, \ldots, h_n\}] = (g_0, h_0) - \sum_{i=1}^{n} c_i g_i \overline{h_i}, \quad g, h \in \Pi.$$

If any of the b_i, say b_n, is ∞, then $\lim_{z \hat{\to} b_n} (z - b_n)h(z) = 0$ should be interpreted to be $\lim_{z \hat{\to} \infty} h(z)/z = 0$ and A should be interpreted as

$$\mathrm{gr}(A) = \{\{f_0 \oplus \{f_1, \ldots, f_{n-1}, 0\}, A_h f_0 \oplus \{b_1 f_1, \ldots, b_{n-1} f_{n-1}, f_n\}\} \in \Pi \times \Pi : \\ f_0 \in \mathrm{dom}\, A_h, \;\; f_1, \ldots, f_{n-1}, f_n \in \mathbb{C}\}.$$

Since $\lim_{z \hat{\to} b_n} h(z)/z = 0$, A_h itself is in this case an operator; see Lemma 2.4.

Proof. Only the minimality of the realization will be shown to hold; all other assertions are easily established. If $\{A,\omega\}$ is not minimal, then by Corollary 2.2 there exists an eigenvector $x=\{x_{\mathfrak{H}},x_1,\ldots,x_n\}$, $x_{\mathfrak{H}}\in\mathfrak{H}$ and $x_i\in\mathbb{C}$, of A such that $[x,\omega]=0$. Since $b_i\notin\sigma_p(A_h)$ by the assumptions, see Lemma 2.4, for x to be an eigenvalue of A either i) all the x_i should be zero or ii) $x_{\mathfrak{H}}$ and all but one of the x_i should be zero. Taking into account $[x,\omega]=0$ and the assumption of minimality on $\{A_h,\omega_h\}$ both cases are easily seen to be impossible. This contradiction shows that $\{A,\omega\}$ realizes $h(z)-\sum_{i=1}^n c_i(b_i-z)^{-1}$ minimally. □

4.3. MULTIPLYING WITH A RATIONAL TERM OF DEGREE ONE

Let $r(z)=\frac{z-\alpha}{z-\beta}$ where $\alpha,\beta\in\mathbb{C}\cup\{\infty\}$ and $\alpha\neq\beta\neq\overline{\alpha}$. Here it is shown how a minimal realization for $rgr^{\#}$ can be obtained given a minimal realization for $g\in\mathfrak{N}_\kappa$ in two cases:

(a) the pole and zero of r are not a generalized zero and pole of g, respectively;
(b) the pole of r is not a generalized zero of g, but the zero of r is a generalized pole of positive type of g.

Using this technique inductively one obtains an explicit representation for any generalized Nevanlinna function.

Case (a): The following two theorems show how a minimal realization of $rgr^{\#}$ can be explicitly constructed from any minimal realization of $g\in\mathfrak{N}_\kappa$ under the assumptions in (a); these two theorems are extensions of [19, Theorem 3.2].

Theorem 4.2. *Let $f\in\mathfrak{N}_\kappa\setminus\{0\}$ be minimally realized by $\{A,\omega\}$. For $g=\{g_l,g_c,g_r\}$, $h=\{h_l,h_c,h_r\}\in\Pi_r:=\mathbb{C}\times\Pi\times\mathbb{C}$, define $[g,h]_r:=[g_c,h_c]+g_r\overline{h_l}+g_l\overline{h_r}$, and for $\alpha,\beta,\mu\in\mathbb{C}$, $\alpha\neq\beta\neq\overline{\alpha}$, and any $z_0\in\rho(A)\setminus\{\beta,\overline{\beta},\alpha,\overline{\alpha}\}$, define A_r and ω_r as*

$$A_r=\begin{pmatrix}\beta & [\cdot,\omega] & -e\\ 0 & A & \omega\\ 0 & 0 & \overline{\beta}\end{pmatrix},\quad \omega_r=\begin{pmatrix} d-\frac{\mu(\overline{\alpha}-\beta)^{-1}}{(z_0-\beta)(\overline{z_0}-\beta)}\\ \omega\\ \overline{\beta}-\alpha\end{pmatrix},$$

where

$$d=\tfrac{(2z_0-\overline{\beta}-\alpha)[\omega,\omega]}{2(z_0-\overline{\beta})(\overline{z_0}-\beta)}-\tfrac{r(z_0)f(z_0)}{(z_0-\beta)(\overline{z_0}-\beta)},$$
$$e=\tfrac{f(z_0)+\overline{f(z_0)}+(\beta+\overline{\beta}-z_0-\overline{z_0})[\omega,\omega]}{2(z_0-\overline{\beta})(\overline{z_0}-\beta)}.$$

Then $\{\Pi_r,[\cdot,\cdot]_r\}$ is a Pontryagin space with negative index $\kappa+1$, A_r is a selfadjoint relation in this space and $\{A_r,\omega_r\}$ realizes

$$f_r(z):=a+\frac{\mu}{\beta-z}+\frac{\overline{\mu}}{\overline{\beta}-z}+r(z)f(z)r^{\#}(z),\qquad r(z)=\frac{z-\alpha}{z-\beta},$$

for every $a\in\mathbb{R}$. Moreover, $\{A_r,\omega_r\}$ realizes f_r minimally if and only if α is not a generalized pole of f and β is not a generalized zero of $f(z)+\frac{\mu(\overline{\beta}-\beta)}{(\beta-\alpha)(\beta-\overline{\alpha})}$.

Proof. The proof consists out of two steps. In the first step all statements except the characterization of the minimality of $\{A_r, \omega_r\}$ are proven.

Step 1: By construction, $\{\Pi_r, [\cdot,\cdot]_r\}$ is a Pontryagin space with negative index $\kappa+1$. The selfadjointness of A_r can be easily checked after noting that $e \in \mathbb{R}$. Clearly,

$$(A_r - z)^{-1} = \begin{pmatrix} -\frac{1}{z-\beta} & \frac{[(A-z)^{-1}\cdot,\omega]}{z-\beta} & \frac{e+[(A-z)^{-1}\omega,\omega]}{(z-\beta)(z-\overline{\beta})} \\ 0 & (A-z)^{-1} & \frac{(A-z)^{-1}\omega}{z-\overline{\beta}} \\ 0 & 0 & -\frac{1}{z-\overline{\beta}} \end{pmatrix}. \tag{4.1}$$

In order to simplify the calculations the realization $\{A_r, \omega_r\}$ is transformed by the standard unitary mapping U (in the Pontryagin space $\{\Pi_r, [\cdot,\cdot]_r\}$) defined as

$$U = \begin{pmatrix} 1 & -[\cdot,\omega] & -[\omega,\omega]/2 \\ 0 & 1 & \omega \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \overline{z_0} - \beta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (z_0 - \overline{\beta})^{-1} \end{pmatrix}.$$

Recall that $\{A_s, \omega_s\} := \{UA_rU^{-1}, U\omega_r\}$ realizes the same generalized Nevanlinna function as $\{A_r, \omega_r\}$, see Section 2.3. Let $\gamma_z := (I + (z - z_0)(A-z)^{-1})\omega$ be the γ-field of f, then one obtains

$$(A_s - z)^{-1} = \begin{pmatrix} -\frac{1}{z-\beta} & -\frac{[\cdot,\gamma_{\overline{z}}]}{z-\beta} & \frac{f(z)}{(z-\beta)(z-\overline{\beta})} \\ 0 & (A-z)^{-1} & \frac{-\gamma_z}{z-\overline{\beta}} \\ 0 & 0 & -\frac{1}{z-\overline{\beta}} \end{pmatrix}, \ \omega_s = \begin{pmatrix} -\frac{(\overline{\alpha}-\beta)^{-1}\mu + r(z_0)f(z_0)}{z_0-\beta} \\ r(z_0)\omega \\ -\frac{\alpha-\overline{\beta}}{z_0-\overline{\beta}} \end{pmatrix}.$$

In the calculation to establish the above identity one has to make use of the identity $f(z_0) - \overline{f(z_0)} = (z_0 - \overline{z_0})[\omega,\omega]$, see (2.2). Now a further calculation shows that

$$\gamma_z^s := (I + (z - z_0)(A_s - z)^{-1})\omega_s = \begin{pmatrix} -\frac{(\overline{\alpha}-\beta)^{-1}\mu + r(z)f(z)}{z-\beta} & r(z)\gamma_z & -\frac{\alpha-\overline{\beta}}{z-\overline{\beta}} \end{pmatrix}^T.$$

Recall that $N_f(z,w)$, the kernel of f, is defined to be $\frac{f(z)-\overline{f(w)}}{z-\overline{w}}$ and that for this kernel the following identity holds

$$N_{rfr^\#}(z,w) = r(z)N_f(z,w)\overline{r(w)} + f(z)r(z)\frac{\overline{r(\overline{z})} - \overline{r(w)}}{z - \overline{w}} + \overline{f(w)r(w)}\frac{r(z) - r(\overline{w})}{z - \overline{w}},$$

see e.g. [3, (3.14)]. Using this identity, one gets

$$\begin{aligned}[][\gamma_z^s, \omega_s]_r &= N_{rfr^\#}(z, z_0) + \left(\frac{\mu}{(z-\beta)(\overline{z_0}-\beta)} + \frac{\overline{\mu}}{(z-\overline{\beta})(\overline{z_0}-\overline{\beta})}\right) \\ &= N_{rfr^\#}(z,z_0) + N_{f_l}(z,z_0) = N_{f_r}(z,z_0);\end{aligned}$$

here $f_l(z) := \mu(\beta - z)^{-1} + \overline{\mu}(\overline{\beta} - z)^{-1}$. By definition, $\gamma_z^s = U\gamma_z^r$ and $\omega_s = U\gamma_s$, and, hence,

$$[\gamma_z^s, \omega_s]_r = [U\gamma_z^r, U\omega_r]_r = [\gamma_z^r, \omega_r]_r,$$

because by construction U is a unitary operator in the Pontryagin space $\{\Pi_r, [\cdot,\cdot]_r\}$. Combining the above two results shows that $\{A_s, \omega_s\}$ realizes f_r, see (2.2).

Step 2: Corollary 2.2 implies that the realization $\{A_s, \omega_s\}$ (and, hence, the realization $\{A_r, \omega_r\}$) is non-minimal if and only if there exists an A_s-invariant subspace $\mathfrak{L}$ orthogonal to ω_s. Let $\{0, x_c, 0\} \in \Pi_r$ be such that

$$0 = [\gamma_z^s, \{0, x_c, 0\}]_r = [r(z)\gamma_z, x_c], \quad z \in \rho(A_s) \subseteq \rho(A) \setminus \{\alpha, \overline{\alpha}, \beta, \overline{\beta}\}.$$

Then the assumed minimality of $\{A, \omega\}$ yields that $x_c = 0$. This shows that $\mathfrak{L} \subseteq (\text{c.l.s.}\{\gamma_z^s : z \in \rho(A_s)\})^{[\perp]}$ is at most two-dimensional. Therefore, the realization $\{A_s, \omega_s\}$ is non-minimal if and only if there exists no eigenvector $x = \{x_l, x_c, x_r\} \in \Pi_r$ of A_s satisfying $[x, \omega_s] = 0$, and $x_l \neq 0$ and/or $x_r \neq 0$. If there does exist such an eigenvector, then there exists a $\delta \in \mathbb{C} \cup \{\infty\}$ such that x satisfies for all $z \in \rho(A_s) \setminus \{\delta\}$

$$\begin{gathered} \frac{x_l}{\beta - z} + \frac{[x_c, \gamma_{\overline{z}}]}{\beta - z} + \frac{f(z)x_r}{(\overline{\beta} - z)(\beta - z)} = \frac{x_l}{\delta - z}, \quad (A - z)^{-1}x_c + \frac{\gamma_z x_r}{\overline{\beta} - z} = \frac{x_c}{\delta - z}; \\ \frac{x_r}{\overline{\beta} - z} = \frac{x_r}{\delta - z}, \quad \frac{(\alpha - \overline{\beta})^{-1}\overline{\mu}}{\overline{\beta} - \overline{z_0}} x_r + \overline{r(z_0)} \left[[x_c, \omega] + \frac{f(\overline{z_0})x_r}{\overline{\beta} - \overline{z_0}} \right] = \frac{\beta - \overline{\alpha}}{\beta - \overline{z_0}} x_l. \end{gathered} \tag{4.2}$$

The third equality shows that there are two cases to consider: $\delta = \overline{\beta}$ or $x_r = 0$. In the latter case the remaining equalities reduce to

$$[x_c, \gamma_{\overline{z}}] = \frac{\beta - \delta}{\delta - z} x_l, \quad (A - z)^{-1}x_c = \frac{x_c}{\delta - z}, \quad (\overline{\alpha} - \overline{z_0})[x_c, \omega] = (\beta - \overline{\alpha})x_l. \tag{4.3}$$

The second equality shows that the first one is satisfied if and only if $(\delta - \overline{z_0})[x_c, \omega] = (\beta - \delta)x_l$. Thus the equalities in (4.3) are satisfied if and only if $\delta = \overline{\alpha}$ and $\{x_c, \overline{\alpha}x_c\} \in \text{gr}(A)$; here it was used that $[x_c, \omega] \neq 0$, because of the assumed minimality of $\{A, \omega\}$, see Corollary 2.2. Recall that $\{x_c, \overline{\alpha}x_c\} \in \text{gr}(A)$ if and only if $\overline{\alpha}$ (and, hence, also α) is a generalized pole of f, see Lemma 2.4.

On the other hand, if $\delta = \overline{\beta}$, then the equalities in (4.2) reduce to

$$\begin{gathered} [x_c, \gamma_{\overline{z}}] + \frac{f(z)x_r}{\overline{\beta} - z} = \frac{\beta - \overline{\beta}}{\overline{\beta} - z} x_l, \quad (A - z)^{-1}x_c + \frac{\gamma_z x_r}{\overline{\beta} - z} = \frac{x_c}{\overline{\beta} - z}; \\ \frac{(\alpha - \overline{\beta})^{-1}\overline{\mu}}{\overline{\beta} - \overline{z_0}} x_r + \frac{\overline{\alpha} - \overline{z_0}}{\beta - \overline{z_0}} \left[[x_c, \omega] + \frac{f(\overline{z_0})x_r}{\overline{\beta} - \overline{z_0}} \right] = \frac{\beta - \overline{\alpha}}{\beta - \overline{z_0}} x_l. \end{gathered} \tag{4.4}$$

Taking z to be $\overline{z_0}$ in the first equality and comparing with the third equation allows one to simplify the above set of equalities to

$$\begin{gathered} [x_c, \gamma_{\overline{z}}] + \frac{f(z)x_r}{\overline{\beta} - z} = \frac{\beta - \overline{\beta}}{\overline{\beta} - z} x_l, \quad (A - z)^{-1}x_c + \frac{\gamma_z x_r}{\overline{\beta} - z} = \frac{x_c}{\overline{\beta} - z}; \\ \overline{\mu} x_r = (\overline{\beta} - \overline{\alpha})(\alpha - \overline{\beta})x_l. \end{gathered} \tag{4.5}$$

If $\beta \in \mathbb{R}$, then the first identify in (4.5) can be rewritten as $\frac{x_r}{\beta - z} = -(f(z))^{-1}[x_c, \gamma_{\overline{z}}]$. Here one should note that f is invertible on $\mathbb{C} \setminus \mathbb{R}$ for all but finitely many points, cf. Theorem 1.1. Combining the preceding identity with the second one in (4.5) yields that $(\overline{\beta} - z)^{-1} x_c = (A - z)^{-1} x_c + \gamma_{z_0}[x_c, \widehat{\gamma_{\overline{z}}}]$. Thus, we can solve (4.4) for $\beta \in \mathbb{R}$ if and only if $\overline{\beta} \in \sigma_p(\widehat{A})$. Here $\widehat{A}$ is as in (2.4): it is a selfadjoint operator such that $\{\widehat{A}, -(f(z_0))^{-1}\omega\}$ realizes $-(f(z))^{-1}$ minimally. Hence, Lemma 2.4 shows that the minimality statement holds if $\beta \in \mathbb{R}$.

If $\beta \in \mathbb{C} \setminus \mathbb{R}$ and $\overline{\beta} \in \sigma_p(A)$, then also $\beta \in \sigma_p(A)$. Let $y_\beta \neq 0$ be an eigenvector of A corresponding to β. Rewrite the second equality in (4.5) in the form $\gamma_z x_r = (I + (z - \overline{\beta})(A - z)^{-1}) x_c$. This implies that for all $z \in \rho(A)$

$$[\gamma_z x_r, y_\beta] = [x_c, (I + (\overline{z} - \beta)(A - \overline{z})^{-1}) y_\beta] = [x_c, y_\beta + (\overline{z} - \beta)(\beta - \overline{z})^{-1}) y_\beta] = 0$$

and thus the assumed minimality of the realization $\{A, \omega\}$ of f implies that $x_r = 0$. Since $\alpha \neq \beta \neq \overline{\alpha}$, the third equality in (4.5) gives $x_l = 0$. Thus the first equation in (4.5) cannot be solved in this case. Consequently, the minimality statement holds in this case.

Finally, if $\beta \in \mathbb{C} \setminus \mathbb{R}$ and, additionally, $\overline{\beta} \in \rho(A)$, then the second condition in (4.5) can be rewritten to be $x_c = \gamma_{\overline{\beta}} x_r$. Plugging that identity in the first equality in (4.5) and using that $(\overline{\beta} - z)[\gamma_{\overline{\beta}}, \gamma_{\overline{z}}] = f(\overline{\beta}) - f(z)$, cf. (2.2), yields that the equalities in (4.5) are in this case equivalent to

$$f(\overline{\beta}) x_r = (\beta - \overline{\beta}) x_l, \quad x_c = \gamma_{\overline{\beta}} x_r, \quad \overline{\mu} x_r = (\overline{\beta} - \overline{\alpha})(\alpha - \overline{\beta}) x_l.$$

Clearly, this system of equations has a solution if and only if $\overline{f(\beta)} = f(\overline{\beta}) = \frac{\overline{\mu}(\beta - \overline{\beta})}{(\overline{\beta} - \overline{\alpha})(\alpha - \overline{\beta})}$; note that here $(\overline{\beta} - \overline{\alpha})(\alpha - \overline{\beta}) \neq 0$ by assumption. Thus the minimality statement also holds in this case. □

The above statement also holds if α or β is ∞. The former case is presented below.

Theorem 4.3. *Let $f \in \mathfrak{N}_\kappa \setminus \{0\}$ be minimally realized by $\{A, \omega\}$. For $g = \{g_l, g_c, g_r\}$, $h = \{h_l, h_c, h_r\} \in \Pi_r := \mathbb{C} \times \Pi \times \mathbb{C}$, define $[g, h]_r := [g_c, h_c] + g_r \overline{h_l} + g_l \overline{h_r}$, and for $\beta, \mu \in \mathbb{C}$, $\overline{\beta} \neq \infty$, and any $z_0 \in \rho(A) \setminus \{\beta, \overline{\beta}, \infty\}$, define A_r and ω_r as*

$$A_r = \begin{pmatrix} \beta & [\cdot, \omega] & -e \\ 0 & A & \omega \\ 0 & 0 & \overline{\beta} \end{pmatrix}, \quad \omega_r = \begin{pmatrix} d + \frac{\mu}{(z_0 - \beta)(\overline{z_0} - \beta)} \\ 0 \\ 1 \end{pmatrix},$$

where

$$d = -\frac{r(z_0) f(z_0)}{(z_0 - \beta)(\overline{z_0} - \beta)} + \frac{[\omega, \omega]}{2(\overline{z_0} - \beta)(z_0 - \overline{\beta})},$$
$$e = \frac{f(z_0) + \overline{f(z_0)} + (\beta + \overline{\beta} - z_0 - \overline{z_0})[\omega, \omega]}{2(z_0 - \overline{\beta})(\overline{z_0} - \beta)}.$$

Then $\{\Pi_r, [\cdot, \cdot]_r\}$ is a Pontryagin space with negative index $\kappa + 1$, A_r is a selfadjoint relation in this space and $\{A_r, \omega_r\}$ realizes

$$f_r(z) := a + \frac{\mu}{\beta - z} + \frac{\overline{\mu}}{\overline{\beta} - z} + r(z) f(z) r^{\#}(z), \qquad r(z) = \frac{1}{z - \overline{\beta}},$$

for every $a \in \mathbb{R}$. Moreover, $\{A_r, \omega_r\}$ realizes f_r minimally if and only if α is not a generalized pole of f and β is not a generalized zero of $f + \mu(\overline{\beta} - \beta)$.

Case (b): Let $g \in \mathfrak{N}_\kappa$ and let $r(z) = \frac{z-\alpha}{z-\beta}$ where $\alpha, \beta \in \mathbb{C} \cup \{\infty\}$ and $\alpha \neq \beta \neq \overline{\alpha}$. Moreover, assume that the zero (α) of r is a generalized pole of positive type of g and the pole (β) is not a generalized zero of g. Under these assumptions we will here show how to construct a minimal realization for $rgr^{\#}$ given a minimal realization for g. In order to do this one must start by rewriting the product $rgr^{\#}$ such that Theorem 4.2 or Theorem 4.3 is applicable; the following basic lemmas show how that can be done.

Lemma 4.4. *Let $\alpha \in \mathbb{R}$ be a generalized pole of positive type of $g \in \mathfrak{N}_\kappa$: $\lim_{z \widehat{\to} \alpha}(\alpha - z)g(z) = m > 0$. Then for every $\beta \in (\mathbb{C} \cup \{\infty\}) \setminus \{\alpha, \overline{\alpha}\}$*

$$\frac{(\alpha - z)^2}{(\beta - z)(\overline{\beta} - z)} g(z) = \begin{cases} \frac{\alpha-\beta}{\overline{\beta}-\beta}\frac{m}{\beta-z} + \frac{\alpha-\overline{\beta}}{\beta-\overline{\beta}}\frac{m}{\overline{\beta}-z} + \frac{(\alpha-z)^2}{(\beta-z)(\overline{\beta}-z)} h(z), & \beta \in \mathbb{C} \setminus \mathbb{R}; \\ \frac{m}{\beta-\alpha} - \frac{m}{\beta-z} + \frac{(\alpha-z)^2}{(\beta-z)^2}\left(h(z) - \frac{m}{\beta-\alpha}\right), & \beta \in \mathbb{R}; \\ m\alpha - mz + (\alpha - z)^2 h(z), & \beta = \infty, \end{cases}$$

where $h(z) := g(z) - \frac{m}{\alpha-z} \in \mathfrak{N}_\kappa$ does not have α as a generalized pole.

Here $(z - \beta)$ should be interpreted to be one if $\beta = \infty$.

Proof. Clearly, $g_\alpha(z) := -\frac{m}{\alpha-z} \in \mathfrak{N}_1(z)$. Hence, $h := g + g_\alpha$ is a generalized Nevanlinna function whose index is κ by Corollary 2.3. If $\beta \in \mathbb{C} \setminus \mathbb{R}$, then

$$\frac{(\alpha - z)^2}{(\beta - z)(\overline{\beta} - z)}(h(z) - g_\alpha(z)) = \frac{(\alpha - z)^2}{(\beta - z)(\overline{\beta} - z)} h(z) + m\frac{\alpha - z}{(\beta - z)(\overline{\beta} - z)},$$

which shows that the statement holds in this case. If $\beta \in \mathbb{R}$, then

$$\frac{m}{\alpha - z} = \frac{m}{\beta - \alpha}\frac{\beta - \alpha}{\alpha - z} = \frac{m}{\beta - \alpha}\left(\frac{\beta - z}{\alpha - z} - 1\right).$$

Therefore in this case

$$\frac{(\alpha - z)^2}{(\beta - z)^2}(h(z) - g_\alpha(z)) = \frac{(\alpha - z)^2}{(\beta - z)^2}\left(h(z) - \frac{m}{\beta - \alpha}\right) + \frac{m}{\beta - \alpha}\frac{\alpha - z}{\beta - z}.$$

This shows that the statement also holds in this case. The final case $(\beta = \infty)$ can be obtained by similar arguments. □

Remark 4.5. Let g and h be as in Lemma 4.4. Then α is not a generalized pole of h, if α is not a generalized pole of g. Moreover, if β is not a generalized zero of g, then β is not a generalized zero of $h + \frac{m}{\alpha-\beta}$, if $\beta \in \mathbb{C}$.

For $\alpha = \infty$ the preceding statement takes the following form.

Lemma 4.6. *Let ∞ be a generalized pole of positive type of $g \in \mathfrak{N}_\kappa$: $\lim_{z \widehat{\to} \infty} \frac{g(z)}{z} = m > 0$. Then for every $\beta \in \mathbb{C}$*

$$\frac{g(z)}{(\beta - z)(\overline{\beta} - z)} = \begin{cases} \frac{\beta}{\overline{\beta}-\beta}\frac{m}{\beta-z} + \frac{\overline{\beta}}{\beta-\overline{\beta}}\frac{m}{\overline{\beta}-z} + \frac{1}{(\beta-z)(\overline{\beta}-z)} h(z), & \beta \in \mathbb{C} \setminus \mathbb{R}; \\ -\frac{m}{\beta-z} + \frac{1}{(\beta-z)^2}\left(h(z) + m\beta\right), & \beta \in \mathbb{R}; \end{cases}$$

where $h(z) := g(z) - mz \in \mathfrak{N}_\kappa$ *does not have* ∞ *as a generalized pole.*

Remark 4.7. Let g and h be as in Lemma 4.4. Then ∞ is not a generalized pole of h, if ∞ is not a generalized pole of g. Moreover, if β is not a generalized zero of g, then β is not a generalized zero of $h + m\beta$, if $\beta \in \mathbb{C}$.

Evidently, if $\{A_g, \Gamma_g\}$ realizes g minimally, then h as in Lemma 4.4 or 4.6 (and, hence, $h + c$ for any real c) is minimally realized by $\{PA_gP, \Gamma_gP\}$ where P is the orthogonal projection (in the realizing space $\{\Pi_g, [\cdot,\cdot]_g\}$) onto the orthogonal complement of $\ker(A - \alpha)$; hereby the new realizing space becomes $\{P\Pi_g, [\cdot,\cdot]_g\}$. The afore-mentioned projection is well-defined because if $x \in \ker(A - \alpha) \setminus \{0\}$, then $[x, x]_g > 0$ by definition of a generalized pole of positive type. From the minimal realization of h so obtained, a minimal realization of $rgr^{\#}$ can be obtained by means of Theorem 4.2 or Theorem 4.3, see Remark 4.5 and Remark 4.7.

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