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On the state estimation for nonlinear continuous-time fuzzy systems

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A large class of nonlinear systems can be represented or well approximated by Takagi-Sugeno (TS) fuzzy models, which in theory can approximate a general nonlinear system to an arbitrary degree of accuracy. The TS fuzzy model consists of a fuzzy rule base. The rule antecedents partition a given subspace of the model variables into fuzzy regions, while the consequent of each rule is usually a linear or affine model, valid locally in the corresponding region. In this paper, the observer design problem for a T-S fuzzy system subject to Lypschitz perturbation is investigated. First, an observer of Kalman type is designed to estimate the unknown system states. Then, the class of one-sided Lipschitz for a TS fuzzy system subject to a sufficient condition on the bound is studied. The challenges are discussed and some analysis oriented tools are provided. An example is given to show the applicability of the main result.

Key words: fuzzy systems, Lipschitz analysis, observer design

1. Introduction

A state observer is a dynamical system which provides the estimation of the internal states of the model. In most practical cases, the physical state of the system cannot be determined by direct observation. The problem of state observation for nonlinear systems is of main importance in automatic control. Nonlinear state observer design has been an area of important research for the last decades and despite important progress. In recent years many contributions have been presented in literature that solve the observer design problem for classes of nonlinear systems (see [4, 5, 9–11, 13]). Lipschitz analysis is extensively used in control theory where the Lipschitz condition is one of the central concepts for the

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construction of nonlinear observer for control systems. Unlike the linear case, the conception of observer is still a difficult task for nonlinear systems. The observer synthesis is first suggested by Thau [19] for Lipschitz systems, he obtained a sufficient condition to ensure the asymptotic stability of the observer. Thau's condition is a very useful analysis tool but does not address the fundamental design problem. Encouraged by Thau's result, several authors studied observer design for Lipschitz systems (see [9, 11] and references therein). However, the real physical systems are often nonlinear. The design of state observers for non-linear systems using Takagi-Sugeno models has been actively considered during the last decades. TS models are currently being used for a large class of physical and industrial processes, such as electrical machines and robot manipulators. As it is delicate to synthesize an observer for an unspecified nonlinear system, it is preferable to represent this system with the TS fuzzy model ([3, 6–8, 12, 14, 15, 23]). With a fuzzy observer, the estimated states error system is described as two parts: unknown premise variable caused terms and observer error terms (see [18]). Then based on the Lyapunov function method (see [1]), a series of linear matrix inequality conditions are proposed to asymptotically stabilize the system, the observer gain matrices are used to overcome the uncertainties. The approach used by [2] for the analysis and design of observers for Takagi-Sugeno fuzzy systems is based on extending sliding mode observer schemes to the case of interpolated multiple local affine linear models. In [15], a new approach to build an interval observer for nonlinear uncertain systems was presented for nonlinear systems modeled in the Takagi-Sugeno form. In [20], the authors studied the problem of fault detection and diagnosis of Takagi-Sugeno fuzzy expert model based soft fault diagnosis for two tank interacting system. In [21], an observer and controller errors augmented system is obtained based on the Lyapunov function method. This work deals with an extension to this problem by introducing a more general family of nonlinear functions, namely one-sided Lipschitz functions. The corresponding class of systems is a superset of its well-known Lipschitz counterpart and possesses inherent advantages with respect to conservativeness. In this paper, an approach combining a Lyapunov function and Lipschitz analysis concept is proposed in order to overcome the observer design problem using the Kalman like configuration related to the Lipschitz constant and one sided-Lipschitz fuzzy systems. We give some sufficient conditions to ensure that the error fuzzy equation is globally exponentially stable provided that the Lipschitz constant is small.

2. Preliminarily results

In control theory, a state observer is a system that provides an estimate of the internal state of a given real system, from measurements of the input and output of the real system. It is typically computer-implemented, and provides the basis of many practical applications. Knowing the system state is necessary to solve many control theory problems; for example, stabilizing a system using state feedback. In most practical cases, the physical state of the system cannot be determined by direct observation. Instead, indirect effects of the internal state are observed by way of the system outputs.

A class of nonlinear systems that has seen much attention in the literature is the class of Lipschitz systems:

$$\dot{x}(t) = Ax(t) + Bu + f(x,u), \qquad x(0) = x_0, \qquad y(t) = Cx(t), \tag{1}$$

where the function $f(x,u)$ satisfies a uniform Lipschitz condition globally in x, i.e,

$$\|f(x,u) - f(\hat{x},u)\| \leqslant k\|x - \hat{x}\|$$

for all u and for all x and $\hat{x}$, $k > 0$ is referred to as the Lipschitz constant which is independent of x, u.

Given a nonlinear system (1), one can estimate the states by using an observer, whose structure is as follows:

$$\dot{x}(t) = G(x(t), u(t), y(t)),$$

where $x(t)$ is the state of the observer. It is needed that the estimation error, $e(t) = \hat{x}(t) - x(t)$ has to converge as fast as possible to zero. Most current methods lead to the design of an exponential observer, exponential stability is the most wanted. With the model given in (1), the problem is to design a continuous observer with input $y(t)$ such that the estimates denoted by $\hat{x}(t)$ converge to $x(t)$ exponentially fast. We shall assume that the pair (A, C) is observable. Suppose the observability matrix for the time invariant associated linear system. Then, there exists a gain matrix $L(n \times p)$ such that the matrix $(A - LC)$ is Horwitz. In this condition, one can design an exponential observer for system (1) as:

$$\dot{x}(t) = A\hat{x}(t) + Bu + f(\hat{x},u) - L(C\hat{x}(t) - y(t)). \tag{2}$$

In presence of the function f, the observer (2) can be designed provided the Lipschitz constant k is small enough. In such away, the system (2) is an exponential observer for system (1), where the matrix L is chosen such that $(A-LC)^T P + P(A-LC) = -Q$, with P and Q are $(n \times n)$ positive definite symmetric matrices and

$$e^T Qe > 2e^T P(\Delta f),$$

$$\Delta f = f(\hat{x},u) - f(x,u).$$

Here, we just consider the error equation:

$$\dot{e}(t) = (A - LC)e(t) + f(\hat{x},u) - f(x,u),$$

where $e(t) = \hat{x}(t) - x(t)$ and use the condition imposed on Lipschitz constant k.

This observer design incorporates only the bound of the nonlinearities (uncertainties), and does not require exact knowledge concerning the structure of the plant nonlinearities $f(x,u)$. A simple condition imposed on the Lipschitz constant k is that,

$$k < \frac{1}{2}\frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}.$$

Therefore, using a Kalman like observer, we can also design a state observer for (1) as follows (see [9] and [10]):

$$\dot{x}(t) = A\hat{x}(t) + Bu + f(\hat{x},u) - S_\theta^{-1}C^T\left(C\hat{x}(t) - y(t)\right),$$

where S_θ satisfies the following stationary equation:

$$0 = -\theta S_\theta - A^T S_\theta - S_\theta A + C^T C, \quad \theta > 0,$$

$$k < \frac{\theta}{2}\frac{\lambda_{\min}(S_\theta)}{\lambda_{\max}(S_\theta)} \qquad \text{and} \qquad S_\theta = \lim_{t\to+\infty} S_t$$

with $S_t \in \mathcal{S}^+$ the cone of symmetric positive definite matrices on $\mathbb{R}^n$ which satisfies

$$\dot{S}_t = -\theta S_t - A^T S_t - S_t A + C^T C.$$

In the next section, we will consider the case of fuzzy control systems. It is well known that, Takagi-Sugeno fuzzy models are nonlinear systems described by a set of if-then rules which gives local linear approximations of an underlying system. Such models can approximate or describe a wide class of nonlinear systems.

3. T.S fuzzy dynamic model

The mathematical model of a system can be in different forms, such as algebraic equations, differential equations, finite state machines, etc. In the modeling framework considered on rule based fuzzy models, the relationships between variables are described by means of if- then rules, such as:

If input is high then output will increase fast.

These rules establish logical relations between the system's variables by relating qualitative values of one variable to qualitative values of another variable.

The structure of the fuzzy system is composed of a set of if-then rules, where qualitative knowledge can be expressed in the form of rules IF "*condition*" THEN "*action*". The condition part (premise) contains facts in the form of symptoms as inputs and the conclusion part includes events as a logical cause of the facts. In the T-S model, the inference is reduced to a simple algebraic expression, similar to the fuzzy-mean defuzzification formula (Takagi and Sugeno [16, 17]). The algorithm for the development of T-S fuzzy model has the following steps:

(i) The optimal number of fuzzy rules is determined.

(ii) The relevant input variables as antecedents are selected.

(iii) The membership function parameters are estimated.

(iv) The consequent structure is selected.

(v) The consequent parameters are estimated.

Consider the following T.S fuzzy dynamic model:

$$\dot{x} = \sum_{i=1}^{r} \mu_i(z)\big(A_i x + B_i u\big), \tag{3}$$

$$y = \sum_{i=1}^{r} \mu_i(z) C_i x, \tag{4}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, and $y \in \mathbb{R}^q$ is the output. The matrices A_i, B_i and C_i are of appropriate dimension, $r \geqslant 2$ is the number of rules, z is the premise vector which may include unmeasurable variables. It is assumed that $\mu_i(z) \geqslant 0$, for all $i = 1, \ldots, r$ and $\sum_{i=1}^{r} \mu_i(z) = 1$, for all $t \geqslant 0$.

In many practical control problems, the physical state variables of systems are partially or fully unavailable for measurement, since the state variables are not accessible by sensing devices and transducers are not available or very expensive. In such cases, observer based control schemes should be designed to estimate the state for (4), (7). Taking $\hat{y}$ defined by

$$\hat{y} = \sum_{i=1}^{r} \mu_i(z) C_i \hat{x}.$$

In this case, an observer can be designed which has the form:

$$\dot{\hat{x}} = \sum_{i=1}^{r} \mu_i(z)\big(A_i \hat{x} + B_i u\big) - \sum_{i=1}^{r} \mu_i(z) L_i(\hat{y} - y). \tag{5}$$

Takagi-Sugeno model has proved its effectiveness in the study of nonlinear systems. Indeed, it gives a simpler formulation from the mathematical point of view to represent the behavior of nonlinear systems. Thanks to the convex sum property of the weighing functions, it is possible to generalize some tools developed in the linear domain to the nonlinear systems. This representation is very interesting in the sense that it simplifies the problem of the observer design.

3.1. Case of Lipschitiz systems

The topic on control and state estimation of nonlinear systems satisfying a Lipschitz condition has been studied for almost four decades, resulting in abundant amount of literature. Especially for the observer synthesis problem on Lipschitz nonlinear system, it is often accomplished by using pseudo-linear techniques which is based on the Lipschitz continuity assumption providing a norm-based form of a nonlinear inequality substituted into the observer error dynamics and the observer error dynamics turning out in a numerically tractable format that is determined by a linear term. The proposed design method is dependent on the solution of a Riccati equation.

In this part, we consider the following fuzzy model:

$$\dot{x}(t) = \sum_{i=1}^{r} \mu_i(z)\big(A_i x + B_i u + f_i(x,u)\big) \tag{6}$$

with the output y defined as in (4).

We assume that:

$(\mathcal{H}_1)$

$$\|f_i(\hat{x},u) - f_i(x,u)\| \leqslant k_i \|\hat{x} - x\|, \qquad k_i > 0, \quad i = 1,2,\ldots,r, \tag{7}$$

for all $(\hat{x}, x) \in \mathbb{R}^n \times \mathbb{R}^n$ and $u \in \mathbb{R}^m$.

Denotes,

$$\sum_{i=1}^{r} k_i = k > 0.$$

Let consider for (6) an observer of the form:

$$\dot{\hat{x}} = \sum_{i=1}^{r} \mu_i(z)\big(A_i \hat{x} + B_i u + f_i(\hat{x},u)\big) - \sum_{i=1}^{r} \mu_i(z) L_i(\hat{y} - y), \tag{8}$$

where $\hat{y}$ is given by:

$$\hat{y} = \sum_{i=1}^{r} \mu_i(z) C_i \hat{x}.$$

Taking into account (6) and (8), the system error is given by:

$$\dot{e} = \sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i(z)\mu_j(z)\big(A_i - L_j C_i\big)e + \sum_{i=1}^{r} \mu_i\big(f_i(\hat{x},u) - f_i(\hat{x} - e,u)\big). \tag{9}$$

Thus,

$$\dot{e} = \sum_{i=1}^{r} \mu_i^2 \Upsilon_{ij} e + 2 \sum_{i<j} \mu_i \mu_j(z) \Upsilon_{ij} e + \sum_{i=1}^{r} \mu_i(z) \big(f_i(\hat{x}, u) - f_i(\hat{x} - e, u) \big),$$

where

$$\Upsilon_{ii} = A_i - L_i C_i,$$

and

$$\Upsilon_{ij} = \frac{1}{2} \big(A_i - L_j C_i + A_j - L_j C_i \big).$$

Now, we can state the following theorem.

Theorem 1 *Suppose that* $(\mathcal{H}_1)$ *holds and there exist positive symmetric definite matrices* $\widetilde{P}$, $\widetilde{Q}$ *and some matrices* L_i, $i = 1, \ldots, r$, *such that the following inequalities hold,*

$$\Upsilon_{ii}^T \widetilde{P} + \widetilde{P} \Upsilon_{ii} < -\widetilde{Q}, \quad i = 1, \ldots, r, \tag{10}$$

and

$$\Upsilon_{ij}^T \widetilde{P} + \widetilde{P} \Upsilon_{ij} < -\widetilde{Q}, \quad 1 \leqslant i < j \leqslant r, \tag{11}$$

then the system error (9) *is guaranteed to be globally uniformly exponentially stable provided that* $k < \dfrac{1}{2} \dfrac{\widetilde{\lambda}_0}{\|\widetilde{P}\|}$, *where* $\widetilde{\lambda}_0 = \lambda_{\min}(\widetilde{Q})$, $\lambda_{\min}$ *denoting the smallest eigenvalue of the matrix.*

Remark 1 *(10) and (11) can be written as LMIs by a simple congruence as in (Tanaka et al. [16]), with the terms* $\widetilde{X} = \widetilde{P}^{-1}$, $L_j = \widetilde{P} N_j$ *and* $\widetilde{H} = \widetilde{X} \widetilde{Q} \widetilde{X}$.

Proof. Consider the Lyapunov function candidate $V(e) = e^T \widetilde{P} e$. It's derivative with respect to time is given by:

$$\dot{V}(e) = \sum_{i=1}^{r} \mu_i^2 e^T \left(\Upsilon_{ii}^T \widetilde{P} + \widetilde{P} \Upsilon_{ii} \right) e + 2 \sum_{i<j} \mu_i \mu_j e^T \left(\Upsilon_{ij}^T \widetilde{P} + \widetilde{P} \Upsilon_{ij} \right) e$$
$$+ 2 e^T \widetilde{P} \sum_{i=1}^{r} \mu_i \big(f_i(\hat{x}, u) - f_i(\hat{x} - e, u) \big).$$

On the one hand, we have

$$e^T \left(\Upsilon_{ii}^T \widetilde{P} + \widetilde{P} \Upsilon_{ii} \right) e \leqslant -\widetilde{\lambda}_0 \|e\|^2, \quad i = 1, \ldots, r,$$

and

$$e^T\left(\Upsilon_{ij}^T\widetilde{P}+\widetilde{P}\Upsilon_{ij}\right)e \leqslant -\widetilde{\lambda}_0\|e\|^2, \quad 1<i<j<r.$$

Then, one gets

$$\dot{V}(e) \leqslant -\widetilde{\lambda}_0\|e\|^2\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j + 2e^T\widetilde{P}\sum_{i=1}^{r}\mu_i\big(f_i(\hat{x},u)-f_i(\hat{x}-e,u)\big).$$

It follows that

$$\dot{V}(e) \leqslant -\widetilde{\lambda}_0\|e\|^2 + 2e^T\widetilde{P}\sum_{i=1}^{r}\mu_i\big(f_i(\hat{x},u)-f_i(\hat{x}-e,u)\big).$$

On the other hand, we have

$$\left\|\sum_{i=1}^{r}\mu_i\big(f_i(\hat{x},u)-f_i(\hat{x}-e,u)\big)\right\| \leqslant \sum_{i=1}^{r}k_i\|e\|.$$

Taking into account the above expressions, it follows that

$$\dot{V}(e) \leqslant -\widetilde{\lambda}_0\|e\|^2 + 2\|\widetilde{P}\|k\|e\|^2.$$

Since

$$V(e) = e^T\widetilde{P}e \leqslant \lambda_{\max}(\widetilde{P})\|e\|^2,$$

one gets

$$\dot{V}(e) \leqslant -\frac{(\widetilde{\lambda}_0-2k\|\widetilde{P}\|)}{\lambda_{\max}(\widetilde{P})}V(e).$$

Thus, we obtain the following estimation:

$$\|e(t)\| \leqslant \left(\frac{\lambda_{\max}(\widetilde{P})}{\lambda_{\min}(\widetilde{P})}\right)^{1/2}\|e(0)\|e^{-\frac{1}{2}\frac{(\widetilde{\lambda}_0-2k\|\widetilde{P}\|)}{\lambda_{\max}(\widetilde{P})}t}.$$

Using the fact that the Lipschitz constant k satisfies,

$$k < \frac{1}{2}\frac{\widetilde{\lambda}_0}{\|\widetilde{P}\|},$$

then, we deduce that (8) is an exponential observer for (6). □

Proposition 1 *Suppose that* $(\mathcal{H}_1)$ *is satisfied, there exist positive symmetric definite matrices* $\widetilde{P}$, $\widetilde{Q}$ *and some matrices* L_i, $i = 1, \dots, r$, *such that the inequalities* (10) *and* (11) *hold and there exists a nonnegative constant* $\alpha > 0$, *such that*

$$\left(-\widetilde{\lambda}_0 + 2\alpha k^2\right) I + \frac{1}{\alpha}\widetilde{P}^T\widetilde{P} < 0,$$

then the system error (9) *is guaranteed to be globally uniformly exponentially stable.*

Proof. Consider the Lyapunov function candidate $V(x) = e^T\widetilde{P}e$. It's derivative with respect to time along the trajectories of (9) is given by,

$$\dot{V}(e) = \sum_{i=1}^{r}\mu_i^2 e^T\left(\Upsilon_{ii}^T\widetilde{P} + \widetilde{P}\Upsilon_{ii}\right)e + 2\sum_{i<j}\mu_i\mu_j e^T\left(\Upsilon_{ij}^T\widetilde{P} + \widetilde{P}\Upsilon_{ij}\right)e$$
$$+ 2e^T\widetilde{P}\sum_{i=1}^{r}\mu_i\big(f_i(\hat{x},u) - f_i(\hat{x}-e,u)\big).$$

Therefore, by the young inequality, we have for all $\alpha > 0$,

$$2e^T\widetilde{P}\sum_{i=1}^{r}\mu_i\big(f_i(\hat{x},u)-f_i(\hat{x}-e,u)\big)\leqslant 2\sum_{i=1}^{r}k_i\alpha\|f_i(\hat{x},u)-f_i(\hat{x}-e,u)\|^2+\frac{1}{\alpha}\|\widetilde{P}e\|^2.$$

Thus,

$$2e^T\widetilde{P}\sum_{i=1}^{r}\mu_i\big(f_i(\hat{x},u) - f_i(\hat{x}-e,u)\big)\leqslant 2\alpha k^2\|e\|^2 + \frac{1}{\alpha}\|\widetilde{P}e\|^2.$$

Taking into account the above expression with the fact that

$$\alpha k^2\|e\|^2 + \frac{1}{\alpha}\|\widetilde{P}e\|^2 = e^T\left(\alpha k^2 I + \frac{1}{\alpha}\widetilde{P}^T\widetilde{P}\right)e,$$

it follows that

$$\dot{V}(e) \leqslant e^T\left(-\widetilde{\lambda}_0 I + 2\alpha k^2 I + \frac{1}{\alpha}\widetilde{P}^T\widetilde{P}\right)e.$$

Hence, because of

$$\left(-\widetilde{\lambda}_0 + 2\alpha k^2\right) I + \frac{1}{\alpha}\widetilde{P}^T\widetilde{P} < 0,$$

for a certain $\alpha > 0$, then the estimation error of equation (9) converges exponentially to the origin. □

Note that, to design a Kalman type observer, one can take the gain matrices as $L_i = S_\theta^{-1}C_i^T$, $i = 1, \dots, r$, where S_θ satisfies the following stationary equations:

$$0 = -\theta S_\theta - A_i^T S_\theta - S_\theta A_i + C_i^T C_i, \qquad \theta > 0.$$

3.2. One sided Lipschitz systems

Control and state estimation of nonlinear systems satisfying a Lipschitz condition have been important topics in nonlinear system theory for over three decades, resulting in a substantial amount of literature. Inspired by the above section, we extend this concept to the nonlinear observer design problem and consider stabilization of the observer error dynamics based on the one-sided Lipschitz condition. The advantages gained through this approach is that the broad family on nonlinear systems includes the well-known Lipschitz systems as a special case which can guarantee stability only for small values of Lipschitz constants which directly translates into small stability regions.

Let consider the following input-output nonlinear system.

$$\dot{x}(t) = Ax(t) + Bu + f(x(t), u(t)), \qquad y(t) = Cx(t), \tag{12}$$

where $x(0) = x_0$, and the function $f(x, u)$ satisfies a uniform one-sided Lipschitz condition globally in x, i.e,

$$\langle f(\hat{x}, u) - f(x, u), \hat{x} - x\rangle \leqslant \widetilde{k}\|\hat{x} - x\|^2$$

for all u and for all $\hat{x}$ and x, $\widetilde{k} > 0$ is referred to as the one-sided Lipschitz constant which is independent of x, u (see [22]).

Note that, wile the Lipschitz constant must be strictly positive, the one-sided Lipschitz constant can be positive, zero or even negative. Moreover, remark that any Lipschitz function is also one-sided Lipschitz, however the converse is not true.

Similarly to the Lipschitz property, the one-sided Lipschitz condition might be local or global and one has,

$$|\langle f(\hat{x}, u) - f(x, u), \hat{x} - x\rangle| \leqslant \|f(\hat{x}, u) - f(x, u)\|\|\hat{x} - x\|,$$

it follows that, if $f(.,.)$ is Lipschitz, then

$$|\langle f(\hat{x}, u) - f(x, u), \hat{x} - x\rangle| \leqslant k\|\hat{x} - x\|.$$

Therefore, any Lipschitz function is also one-sided Lipschitz. The converse is not true, it suffices to take for example the function:

$$f(x) = -\mathrm{sgn}\,(x)\sqrt{|x|}$$

where sgn(.) denotes the sign function.

Next, we consider the following fuzzy model:

$$\dot{x}(t) = \sum_{i=1}^{r} \mu_i(z)\big(A_i x + B_i u + f_i(x, u)\big) \tag{13}$$

with the output y defined as in (4).

Let consider for (13) an observer of the form:

$$\dot{\hat{x}} = \sum_{i=1}^{r} \mu_i(z)\big(A_i\hat{x} + B_i u + f_i(\hat{x}, u)\big) - \sum_{i=1}^{r} \mu_i(z) L_i(\hat{y} - y). \tag{14}$$

with

$$f_i(x, u) = \widetilde{P}^{-1}\widetilde{f}_i(x, u), \qquad i = 1, \ldots, r,$$

where $\widetilde{P}$ is $(n \times n)$ is the unique symmetric positive definite solution of the Lyapunov equations (10) and (11) for a given symmetric positive definite matrix $\widetilde{Q}$, for all L_i, $i = 1, \ldots, r$, and $\widetilde{f}_i$ satisfy the following one-sided Lipschitz condition:
$(\mathcal{H}_2)$

$$\left\langle \widetilde{f}_i(\hat{x}, u) - \widetilde{f}_i(x, u), \hat{x} - x \right\rangle \leqslant \widetilde{k}\|\hat{x} - x\|^2, \qquad \widetilde{k}_i > 0, \quad i = 1, \ldots, r, \tag{15}$$

for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ and $u \in \mathbb{R}^m$.

Let

$$\sum_{i=1}^{r} \widetilde{k}_i = \widetilde{k} > 0.$$

Theorem 2 *Under the assumption* $(\mathcal{H}_2)$ *with* $\widetilde{k} < \dfrac{1}{2}\widetilde{\lambda}_0$*, the system (14) is an exponential fuzzy observer for the fuzzy system (13).*

Proof. Consider the Lyapunov function candidate $V(e) = e^T\widetilde{P}e$. It's derivative with respect to time is given by:

$$\dot{V}(e) = \sum_{i=1}^{r} \mu_i^2 e^T \left(\Upsilon_{ii}^T\widetilde{P} + \widetilde{P}\Upsilon_{ii}\right) e + 2\sum_{i<j} \mu_i\mu_j e^T \left(\Upsilon_{ij}^T\widetilde{P} + \widetilde{P}\Upsilon_{ij}\right) e$$
$$+ 2e^T\widetilde{P}\sum_{i=1}^{r} \mu_i\big(f_i(\hat{x}, u) - f_i(\hat{x} - e, u)\big).$$

On the one hand, we have

$$e^T \left(\Upsilon_{ii}^T\widetilde{P} + \widetilde{P}\Upsilon_{ii}\right) e \leqslant -\widetilde{\lambda}_0\|e\|^2, \qquad i = 1, \ldots, r,$$

and

$$e^T \left(\Upsilon_{ij}^T\widetilde{P} + \widetilde{P}\Upsilon_{ij}\right) e \leqslant -\widetilde{\lambda}_0\|e\|^2, \qquad 1 < i < j < r.$$

Then, one gets

$$\dot{V}(e) \leqslant -\widetilde{\lambda}_0 \|e\|^2 \sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i \mu_j + 2e^T \widetilde{P} \sum_{i=1}^{r} \mu_i \big(f_i(\hat{x}, u) - f_i(\hat{x} - e, u) \big).$$

Thus,

$$\dot{V}(e) \leqslant -\widetilde{\lambda}_0 \|e\|^2 \sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i \mu_j + 2e^T \sum_{i=1}^{r} \mu_i \widetilde{P} \left(\widetilde{P}^{-1} \widetilde{f}_i(\hat{x}, u) - \widetilde{P}^{-1} \widetilde{f}_i(\hat{x} - e, u) \right).$$

It follows that

$$\dot{V}(e) \leqslant -\widetilde{\lambda}_0 \|e\|^2 + 2 \sum_{i=1}^{r} \mu_i e^T \left(\widetilde{f}_i(\hat{x}, u) - \widetilde{f}_i(\hat{x} - e, u) \right).$$

On the other hand, from $(\mathcal{H}_2)$, we have

$$\left\| \sum_{i=1}^{r} \mu_i e^T \left(\widetilde{f}_i(\hat{x}, u) - \widetilde{f}_i(\hat{x} - e, u) \right) \right\| \leqslant \sum_{i=1}^{r} \widetilde{k}_i \|e\|^2.$$

Taking into account the above expressions, it follows that

$$\dot{V}(e) \leqslant \left(2\widetilde{k} - \widetilde{\lambda}_0 \right) \|e\|^2.$$

The last expression in conjunction with the fact that the Lyapunov function is quadratic, yields

$$\dot{V}(e) \leqslant -\frac{(\widetilde{\lambda}_0 - 2\widetilde{k})}{\lambda_{\max}(\widetilde{P})} V(e).$$

Thus, we obtain the following estimation:

$$\|e(t)\| \leqslant \left(\frac{\lambda_{\max}(\widetilde{P})}{\lambda_{\min}(\widetilde{P})} \right)^{1/2} \|e(0)\| e^{-\frac{1}{2} \frac{(tilde\lambda_0 - 2\widetilde{k})}{\lambda_{\max}(\widetilde{P})} t}.$$

Using the fact that the one-sided Lipschitz constant $\widetilde{k}$ satisfies, $\widetilde{k} < \frac{1}{2}\widetilde{\lambda}_0$, then the system (14) is an exponential fuzzy observer for the fuzzy system (13). □

Remark 2 *For many problems, one-sided Lipschitz constant can be found which are significantly smaller than the classical Lipschitz constant. This makes the one-sided Lipschitz constant much more appropriate for estimating the influence of non-linear terms.*

4. Example

Consider the following nonlinear fuzzy planar system,

$$\begin{cases} \dot{x} = \sum_{i=1}^{2} \mu_i(z)\big(A_i x + B_i u + f_i(x,u)\big) \\ y = \sum_{i=1}^{2} \mu_i(z) C_i x, \end{cases}$$

where $x(t) = \begin{bmatrix} x_1(t) & x_2(t) \end{bmatrix}^T$, is the state vector, $u(t)$ is the input vector, $y(t)$ is the output vector, $t \geqslant 0$.

$$z = \sin(x_1), \qquad A_1 = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix},$$

$$B_1 = B_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad C_1 = C_2 = \begin{bmatrix} -1 & 1 \end{bmatrix},$$

$$f_1(x,u) = \begin{bmatrix} -\epsilon \sin u \cos x_2 \\ -\epsilon(x_1^2 + x_2^2)^{1/2} \end{bmatrix}, \qquad f_2(x,u) = \begin{bmatrix} -\epsilon \sin u \cos x_2 \\ \epsilon(x_1^2 + x_2^2)^{1/2} \end{bmatrix},$$

We define the membership functions as:

$$\mu_1(t) = \frac{1 - \sin(x_1(t))}{2} \qquad \text{and} \qquad \mu_2(t) = 1 - \mu_1(t).$$

A classical T-S fuzzy model would require 8 rules for 3 nonlinearities, and it would depend on unmeasured variables. The nonlinearities are ϵ-Lipschitizian and for the conception of the observer we can take $\epsilon < 0.01$. Using an LMI optimization algorithm, we obtain the following gains matrices:

$$L_1 = \begin{bmatrix} 422.7991 & 483.2676 \end{bmatrix}^T \qquad \text{and} \qquad L_2 = \begin{bmatrix} 85.9898 & 101.2911 \end{bmatrix}^T.$$

Consider the following fuzzy observer:

$$\begin{cases} \dot{\hat{x}} = \sum_{i=1}^{2} \mu_i(\hat{z}) \left(A_i \hat{x} + B_i u + f_i(t, \hat{x}, u) - L_i(\hat{y} - y)\right), \\ \hat{y} = \sum_{i=1}^{2} \mu_i(\hat{z}) C_i \hat{x}. \end{cases}$$

Therefore, the T-S fuzzy error equation:

$$\dot{e} = \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j\left(A_i - L_jC_i\right)e + \sum_{i=1}^{r}\mu_i\left(f_i\left(\hat{x}, u\left(\hat{x}\right)\right) - f_i\left(\hat{x} - e, u\left(\hat{x}\right)\right)\right)$$

is globally exponentially stable.

5. Conclusion

In this paper a new way to simplify the design of observers for T-S fuzzy models is presented. It concerns the cases of non linearity that either meets a Lipschitz condition or simply one-sided Lipschitz. Some results are obtained, the observer can therefore be designed under some sufficient conditions. Moreover, an illustrative example is given.

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