Opuscula Math. 35, no. 5 (2015), 739–773
http://dx.doi.org/10.7494/OpMath.2015.35.5.739

ANALYTIC CONTINUATION OF SOLUTIONS OF SOME NONLINEAR CONVOLUTION PARTIAL DIFFERENTIAL EQUATIONS

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Communicated by P.A. Cojuhari

Abstract. The paper considers a problem of analytic continuation of solutions of some nonlinear convolution partial differential equations which naturally appear in the summability theory of formal solutions of nonlinear partial differential equations. Under a suitable assumption it is proved that any local holomorphic solution has an analytic extension to a certain sector and its extension has exponential growth when the variable goes to infinity in the sector.

Keywords: convolution equations, partial differential equations, analytic continuation, summability, sector.

Mathematics Subject Classification: 45K05, 45G10, 35A20.

1. INTRODUCTION

The multisummability of formal solutions of general ordinary differential equations was first proved by Braaksma [3]; different proofs were given by many authors (see Balser [1, 2], Ramis-Sibuya [10] and their references). In the proof of Braaksma [3], the key point of the proof is that he proved an analytic continuation property of a solution of the convolution equation which is obtained by Borel transformation of the ordinary differential equation.

In the case of partial differential equations, the way of proof by Braaksma was followed by Ouchi [8, 9], Tahara-Yamazawa [11] and Luo-Chen-Zhang [6] in treating various types of partial differential equations. But still there are many types of partial differential equations which have formal solutions but the summability has not been proved yet.

In this situation, it will be worthy to study the analytic continuation problem itself for convolution partial differential equations, apart from the application to the summability theory.

Thus, in this paper we consider the following problem:

Problem 1.1. Find such a class of convolution partial differential equations that a local holomorphic solution has an analytic extension to a suitable sector and its extension has exponential growth in the sector when variable goes to infinity.

As is mentioned above, the arguments in [6, 8, 9, 11] have given some answers to this problem. In this paper, we will introduce a new class of nonlinear convolution partial differential equations which has a nice application: the typical feature of this class is that the structure is very close to Maillet type theorems developed in Gérard-Tahara [4] and so we can apply a similar argument. In the case of linear equations, this class is the same as the one introduced in [11]. The application will be given in a forthcoming paper.

Throughout this paper, we let t be the variable in $\mathbb{C}_t$ (or in $\mathcal{R}(\mathbb{C}_t \setminus \{0\})$ the universal covering space of $\mathbb{C}_t \setminus \{0\}$), and let $x = (x_1, \ldots, x_K)$ be the variable in $\mathbb{C}_x^K$. We denote by $\mathcal{O}_R$ the set of all holomorphic functions in x in a neighborhood of $D_R = \{x \in \mathbb{C}^K \,;\, |x_i| \leq R \ (i = 1, \ldots, K)\}$, and by $\mathcal{O}_R[[t]]$ the ring of formal power series in t with coefficients in $\mathcal{O}_R$. We often denote by $\mathbb{C}\{t\}$ the ring of convergent power series in t with complex coefficients. We set $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{N}^* = \{1, 2, \ldots\}$.

2. MAIN THEOREM

Let $k > 0$, $I = (\theta_1, \theta_2)$ be an open interval of $\mathbb{R}$, and we write $S_I = \{t \in \mathcal{R}(\mathbb{C}_t \setminus \{0\});\ \theta_1 < \arg t < \theta_2\}$, and $S_I(r) = \{t \in S_I \,;\, 0 < |t| < r\}$ for $0 < r \leq \infty$. For holomorphic functions $f(t, x)$ and $g(t, x)$ on $S_I(r) \times D_R$, we define the k-convolution $(f *_k g)(t, x)$ with respect to t by

$$(f *_k g)(t, x) = \int_0^t f(\tau, x) g((t^k - \tau^k)^{1/k}, x) d\tau^k, \quad (t, x) \in S_I(r) \times D_R.$$

For basic properties of k-convolution, see Balser [1, 2], Ouchi [8, 9] and Tahara-Yamazawa [11]. For simplicity, we use the notations:

$$u^{*_k 2} = u *_k u, \quad u^{*_k 3} = u *_k u *_k u \quad \text{and so on},$$

$$\prod_{i=1,2}^{*_k} u_i = u_1 *_k u_2, \qquad \prod_{i=1,2,3}^{*_k} u_i = u_1 *_k u_2 *_k u_3 \quad \text{and so on.}$$

For $(i, \alpha) \in \mathbb{N} \times \mathbb{N}^K$, we write

$$\mathscr{M}_{i,\alpha}[w] = \begin{cases} \frac{t^{k|\alpha|-k}}{\Gamma(|\alpha|)} *_k \left[(kt^k)^i w \right], & \text{if } |\alpha| > 0, \\ (kt^k)^i w, & \text{if } |\alpha| = 0, \end{cases}$$

where $\alpha = (\alpha_1, \ldots, \alpha_K) \in \mathbb{N}^K$ and $|\alpha| = \alpha_1 + \ldots + \alpha_K$. As is often used in [11], $\mathscr{M}_{i,\alpha}[w]$ is nothing but the k-Borel transform of

$$t^{k|\alpha|} \Big(t^{k+1} \frac{\partial}{\partial t} \Big)^i W \quad \text{under } w = \mathcal{B}_k[W].$$

One answer to Problem 1.1 is to consider the convolution partial differential equation

$$P(kt^k,x)u = f(t,x) + \sum_{i+|\alpha|\le m} a_{i,\alpha}(t,x) *_k \big(\mathscr{M}_{i,\alpha}[\partial_x^\alpha u]\big) + \sum_{|\nu|\ge 2} b_\nu(t,x) *_k \prod_{i+|\alpha|\le m}^{*_k} \big(\mathscr{M}_{i,\alpha}[\partial_x^\alpha u]\big)^{*_k \nu_{i,\alpha}} \tag{2.1}$$

(where $\nu=\{\nu_{j,\alpha}\}_{j+|\alpha|\le m}\in\mathbb{N}^N$ with $N=\#\{(j,\alpha)\in\mathbb{N}\times\mathbb{N}^K\,;\,j+|\alpha|\le m\}$, and $|\nu|=\sum_{j+|\alpha|\le m}\nu_{j,\alpha}$) under the following assumptions:

(A_1) $k\ge 1$ is an integer, and $0<|I|<2\pi/k$;
(A_2) l and m are integers with $0\le l\le m$;
(A_3) $P(\lambda,x)=\lambda^l+c_1(x)\lambda^{l-1}+\ldots+c_{l-1}(x)\lambda+c_l(x)$, and the coefficients $c_i(x)$ ($i=1,\ldots,l$) are holomorphic functions in a neighborhood of D_{R_0} for some $R_0>0$;
(A_4) $f(t,x)$, $a_{i,\alpha}(t,x)$ ($i+|\alpha|\le m$) and $b_\nu(t,x)$ ($|\nu|\ge 2$) are all holomorphic functions on $S_I\times D_{R_0}$;
(A_5) there are integers $\mu\ge 1$, $p_{i,\alpha}\ge 1$ ($i+|\alpha|\le m$) and $q_\nu\ge 1$ ($|\nu|\ge 2$) such that the estimates

$$|f(t,x)|\le \frac{F}{\Gamma(\mu/k)}|t|^{\mu-k}\exp(c|t|^k)\ \text{ on } S_I\times D_{R_0},$$
$$|a_{i,\alpha}(t,x)|\le \frac{A_{i,\alpha}}{\Gamma(p_{i,\alpha}/k)}|t|^{p_{i,\alpha}-k}\exp(c|t|^k)\ \text{ on } S_I\times D_{R_0}\ \ (i+|\alpha|\le m),$$
$$|b_\nu(t,x)|\le \frac{B_\nu}{\Gamma(q_\nu/k)}|t|^{q_\nu-k}\exp(c|t|^k)\ \text{ on } S_I\times D_{R_0}\ \ (|\nu|\ge 2)$$

hold for some $c>0$, $F\ge 0$, $A_{i,\alpha}\ge 0$ ($i+|\alpha|\le m$) and $B_\nu\ge 0$ ($|\nu|\ge 2$);
(A_6) moreover, the sum

$$\sum_{|\nu|\ge 2} B_\nu t^{q_\nu} X^{|\nu|}$$

is convergent in a neighborhood of $(t,X)=(0,0)\in\mathbb{C}^2$.

If $b_\nu(t,x)\equiv 0$ holds for all $|\nu|\ge 2$, (2.1) is a linear equation and it is just the same as the one treated in [11].

To show that (2.1) is an answer to Problem 1.1 we must show that (2.1) satisfies the analytic continuation property posed in Problem 1.1. To do so, let us define two indices s_a and s_b. For $x\in\mathbb{R}$ we write $[x]_+=\max\{x,0\}$. For $\nu=\{\nu_{i,\alpha}\}_{i+|\alpha|\le m}\in\mathbb{N}^N$ we set $m_\nu=\max\{i+|\alpha|\,;\,\nu_{i,\alpha}>0\}$ and

$$\langle\nu\rangle_l=\sum_{i+|\alpha|\le m}[i+|\alpha|-l]_+\nu_{i,\alpha}=\sum_{l+1\le i+|\alpha|\le m}(i+|\alpha|-l)\nu_{i,\alpha}.$$

Under the assumptions (A_1)–(A_6), we set

$$\Delta_a=\{(i,\alpha)\in\mathbb{N}\times\mathbb{N}^K\,;\,l+1\le i+|\alpha|\le m, a_{i,\alpha}(t,x)\not\equiv 0\},$$

$$\Delta_b = \{\nu \in \mathbb{N}^N \,;\, |\nu| \geq 2, m_\nu \geq l+1, b_\nu(t,x) \not\equiv 0\},$$
$$s_a = 1 + \max\left[0, \max_{(i,\alpha)\in\Delta_a} \left(\frac{i+|\alpha|-l}{p_{i,\alpha}+k(i+|\alpha|-l)}\right)\right],$$
$$s_b = 1 + \max\left[0, \max_{\nu\in\Delta_b} \left(\frac{m_\nu - l}{q_\nu + k\langle\nu\rangle_l + \mu(|\nu|-1)}\right)\right].$$

If $l = m$ holds, we have $\Delta_a = \emptyset$ and $\Delta_b = \emptyset$. This means that $s_a = 1$ and $s_b = 1$. Now, we define $\kappa > 0$ by

$$1/\kappa = 1/k - (s_0 - 1) \quad \text{with} \quad s_0 = \max\{s_a, s_b\}. \tag{2.2}$$

Lemma 2.1. *If $\Delta = \Delta_a \cup \Delta_b = \emptyset$, we have $s_0 = 1$, and so we have $\kappa = k$. If $\Delta \neq \emptyset$, we have $0 < s_0 - 1 < 1/k$, and so we have $\kappa > k$.*

Proof. The first half is clear. Let us show the latter half. If $\Delta_a \neq \emptyset$, we have

$$s_a - 1 = (i+|\alpha|-l)/(p_{i,\alpha} + k(i+|\alpha|-l))$$

for some $(i,\alpha) \in \Delta_a$, and so

$$0 < s_a - 1 = \frac{i+|\alpha|-l}{p_{i,\alpha}+k(i+|\alpha|-l)} < \frac{i+|\alpha|-l}{k(i+|\alpha|-l)} = 1/k.$$

If $\Delta_b \neq \emptyset$, we have

$$s_b - 1 = (m_\nu - l)/(q_\nu + \mu(|\nu|-1) + k\langle\nu\rangle_l)$$

for some $\nu \in \Delta_b$. Since $m_\nu = i + |\alpha|$ holds for some (i,α) with $\nu_{i,\alpha} > 0$, we have

$$0 < s_b - 1 = \frac{i+|\alpha|-l}{q_\nu + \mu(|\nu|-1) + k\langle\nu\rangle_l} < \frac{i+|\alpha|-l}{k(\ldots + (i+|\alpha|-l)\nu_{i,\alpha} + \ldots)} < \frac{1}{k}.$$

Thus, we have seen that if $\Delta \neq \emptyset$, we have $0 < s_0 - 1 < 1/k$, and so $\kappa > k$. □

The following result is the main theorem of this paper.

Theorem 2.2. *Suppose the conditions (A_1)–(A_6). Let $\lambda_1(x), \ldots, \lambda_l(x)$ be the roots of $P(\lambda, x) = 0$, and assume that*

$$\lambda_i(0) = 0 \quad \text{or} \quad \lambda_i(0) \in \mathbb{C} \setminus \overline{\pi(S_{kI})} \quad \text{for } i = 1, 2, \ldots, l \tag{2.3}$$

(where π is the projection $\pi : \mathcal{R}(\mathbb{C}\setminus\{0\}) \longrightarrow \mathbb{C}$). Let $\kappa > 0$ be as in (2.2). If $u(t,x)$ is a holomorphic solution of equation (2.1) on $S_I(\delta) \times D_{R_0}$ for some $\delta > 0$, and if it satisfies $|u(t,x)| \leq M_0|t|^{\mu-k}$ on $S_I(\delta) \times D_{R_0}$ for some $M_0 > 0$, then $u(t,x)$ has an analytic continuation $u^*(t,x)$ on $S_I \times D_R$ for some $0 < R < R_0$ such that

$$|u^*(t,x)| \leq \frac{M}{(|t|^k+1)^l}\,|t|^{\mu-k}\exp(b|t|^\kappa) \quad \text{on } S_I \times D_R \tag{2.4}$$

holds for some $M > 0$ and $b > 0$.

We note that $0 < |I| < 2\pi/k$ implies $\mathbb{C} \setminus \overline{\pi(S_{kI})} \neq \emptyset$, and so the condition (2.3) makes sense. The rest part of this paper is organized as follows. The proof of Theorem 2.2 will be given in Sections 3 and 4. In the next Section 3 we will prove Theorem 2.2 in the case

$$\lambda_1(0), \ldots, \lambda_m(0) \in \mathbb{C} \setminus \overline{\pi(S_{kI})}, \tag{2.5}$$

and in Section 4 we will show Theorem 2.2 in the general case (2.3). In Section 5, we will give a generalization to the case where the constants $k > 0$, $\mu > 0$, $p_{j,\alpha} > 0$ and $q_\nu > 0$ in the assumptions (A_1) and (A_5) are not necessarily integers.

3. PROOF OF THEOREM 2.2 UNDER (2.5)

In this section, we will prove Theorem 2.2 under the condition:

$$\lambda_1(0), \ldots, \lambda_l(0) \in \mathbb{C} \setminus \overline{\pi(S_{kI})}. \tag{3.1}$$

The meaning of this condition lies in the following lemma:

Lemma 3.1. *If* (3.1) *is satisfied, we have the estimate*

$$|P(kt^k, x)| \geq \sigma(|t|^k + 1)^l \quad \text{on } \overline{S_I} \times D_{R_1}$$

for some $\sigma > 0$ *and* $R_1 > 0$ *sufficiently small.*

The plan of the proof of Theorem 2.2 is as follows. In Subsection 3.1 we construct a formal solution of equation (2.1), in Subsections 3.2 and 3.3 we give some estimates of this formal solution: in this proof we can see that the structure of (2.1) is very similar to that of Maillet type theorem developed in Gérard-Tahara [4]. By using this formal solution, in Subsection 3.4 we show the existence of a holomorphic solution $u^*(t, x)$ of (2.1) on $S_I \times D_R$ for some $R > 0$. In Subsection 3.5, we will show the uniqueness of the local solution of (2.1), and complete the proof of Theorem 2.2.

3.1. CONSTRUCTION OF A FORMAL SOLUTION

Let us look for a formal solution of the form

$$u(t, x) = \sum_{n \geq \mu} u_n(t, x). \tag{3.2}$$

We substitute this formal series into equation (2.1) and then we collect the terms of the same weight in the both sides of the equation: the weight is defined by the following (we denote by $w(f)$ the weight of f): $w(P(kt^k, x)) = 0$, $w(u_n) = n$ $(n \geq \mu)$, $w(f) = \mu$, $w(a_{i,\alpha}) = p_{i,\alpha}$ $(i + |\alpha| \leq m)$, $w(\mathscr{M}_{i,\alpha}) = k[i + |\alpha| - l]_+$, $w(\partial_x^\alpha) = 0$ and $w(b_\nu) = q_\nu$ $(|\nu| \geq 2)$. Then, we can decompose our equation (2.1) into the following recurrent formulas:

$$P(kt^k, x)u_\mu = f(t, x), \tag{3.3}$$

and for $n \geq \mu + 1$

$$P(kt^k, x)u_n = \sum_{i+|\alpha| \leq m} a_{i,\alpha}(t,x) *_k \left(\mathscr{M}_{i,\alpha}[\partial_x^\alpha u_{n-p_{i,\alpha}-k[i+|\alpha|-l]_+}]\right)$$
$$+ \sum_{2 \leq |\nu| \leq n-q_\nu} \sum_{\substack{q_\nu + |n(\nu)| \\ + k\langle \nu \rangle_l = n}} b_\nu(t,x) *_k \prod_{i+|\alpha|\leq m}^{*_k} \prod_{j=1}^{\nu_{i,\alpha}}{}^{*_k} \left(\mathscr{M}_{i,\alpha}[\partial_x^\alpha u_{n_{i,\alpha}(j)}]\right), \tag{3.4}$$

where

$$n(\nu) = (n_{i,\alpha}(j)\,;\, i + |\alpha| \leq m, 1 \leq j \leq \nu_{i,\alpha}), \quad n_{i,\alpha}(j) \in \mathbb{N}^*,$$

and

$$|n(\nu)| = \sum_{i+|\alpha|\leq m} (n_{i,\alpha}(1) + \ldots + n_{i,\alpha}(\nu_{i,\alpha})).$$

In the formula (3.4) we used the convension: $u_p(t,x) = 0$ if $p < \mu$. We denote by $\mathcal{O}(W)$ the set of all holomorphic functions on W. By Lemma 3.1, we can see that the following result holds.

Proposition 3.2. *Let $R_1 > 0$ be sufficiently small. We have a unique solution $u_n(t,x) \in \mathcal{O}(S_I \times D_{R_1})$ $(n \geq \mu)$ which solves the system* (3.3) *and* (3.4) $(n \geq \mu + 1)$.

Moreover, we have another result.

Proposition 3.3. *The above $u_n(t,x)$ $(n \geq \mu)$ satisfy the following estimates: there are $C > 0$, $h > 0$ and $\rho > 0$ such that*

$$|u_n(t,x)| \leq \frac{Ch^n}{(|t|^k+1)^l} \frac{n!^{s-1}}{\Gamma(n/k)} |t|^{n-k} \exp(c|t|^k) \ \text{ on } S_I \times D_\rho$$

holds for any $n \geq \mu$ and $s \geq \max\{s_a, s_b\}$.

Before we give the proof of this proposition, in Subsection 3.2 we present some lemmas which are needed in the proof of Proposition 3.3, and then in Subsection 3.3 we give a proof of Proposition 3.3.

3.2. SOME LEMMAS

We write $D_R^\circ = \{x \in \mathbb{C}^K\,;\, |x_i| < R \ (i = 1, \ldots, K)\}$, for the interior of D_R. For a holomorphic function $\varphi(x)$ on D_R°, we set

$$\|\varphi\|_\rho = \max_{|x|\leq\rho} |\varphi(x)|, \quad 0 < \rho < R.$$

For $a > 0$ and $c \geq 0$, we set

$$\phi_a(t;c) = \frac{|t|^{a-k}}{\Gamma(a/k)} \exp(c|t|^k).$$

Then, the estimates in (A_5) are expressed as $|f(t,x)| \leq F\phi_\mu(t;c)$, $|a_{i,\alpha}(t,x)| \leq A_{i,\alpha}\phi_{p_{i,\alpha}}(t;c)$ and $|b_\nu(t,x)| \leq B_\nu\phi_{q_\nu}(t;c)$ on $S_I \times D_{R_0}$. By [8, Lemma 1.4] and [11, Lemma 7.2] (with $\sigma = 1$ and $\xi_0 = 0$), we have the following lemmas.

Lemma 3.4. *Let $f(t,x) \in \mathcal{O}(S_I \times D_R^\circ)$ and $g(t,x) \in \mathcal{O}(S_I \times D_R^\circ)$. Then we have $(f *_k g)(t,x) \in \mathcal{O}(S_I \times D_R^\circ)$. If they satisfy the estimates $\|f(t)\|_\rho \le A\phi_a(t;c)$ and $\|g(t)\|_\rho \le B\phi_b(t;c)$ on S_I for some $0 < \rho < R$, $A > 0$, $a > 0$, $B > 0$ and $b > 0$, we have the estimate $\|(f *_k g)(t)\|_\rho \le AB\phi_{a+b}(t;c)$ on S_I.*

Lemma 3.5. *Suppose that $c \ge l$ holds. Then for any $\mu > 0$ there is a constant $\beta > 0$ which satisfies the following condition: if $w(t,x) \in \mathcal{O}(S_I \times D_\rho)$ for some $\rho > 0$ and if*

$$\|w(t)\|_\rho \le \frac{A}{(|t|^k+1)^l}\phi_N(t;c) \quad \textit{on } S_I$$

for some $A > 0$ and $N \ge \mu$, we have

$$\|\mathscr{M}_{i,\alpha}[w](t)\|_\rho \le \frac{\beta N^{[i+|\alpha|-l]_+}}{N^{|\alpha|}} A\phi_{N+k[i+|\alpha|-l]_+}(t;c) \quad \textit{on } S_I$$

for any $i + |\alpha| \le m$.

The following lemma is very useful (for the proof, see [7] or Lemma 5.1.3 in [5]).

Lemma 3.6. *If a holomorphic function $\varphi(x)$ on D_R° satisfies*

$$\|\varphi\|_\rho \le \frac{A}{(R-\rho)^a} \quad \textit{for any } 0 < \rho < R$$

for some $A > 0$ and $a \ge 0$, we have the estimates

$$\left\|\partial_{x_i}\varphi\right\|_\rho \le \frac{(a+1)eA}{(R-\rho)^{a+1}} \quad \textit{for any } 0 < \rho < R \textit{ and } i = 1, \dots, K.$$

3.3. PROOF OF PROPOSITION 3.3

Take any $s \ge \max\{s_a, s_b\}$ and any R with $0 < R < \min\{1, R_1\}$. Since u_μ is a solution of (3.3), by (A_5) and Lemma 3.1 we have

$$|u_\mu(t,x)| = \left|\frac{f(t,x)}{P(kt^k,x)}\right| \le \frac{F}{\sigma(|t|^k+1)^l}\phi_\mu(t;c) \quad \text{on } S_I \times D_{R_1}$$

and by Lemma 3.6 we have

$$\|\partial_x^\alpha u_\mu(t)\|_R \le \frac{F}{\sigma(|t|^k+1)^l}\phi_\mu(t;c) \times \frac{|\alpha|!\, e^{|\alpha|}}{(R_1-R)^{|\alpha|}} \quad \text{on } S_I.$$

Thus, by taking $A > 0$ sufficiently large we have

$$\|\partial_x^\alpha u_\mu(t)\|_R \le \frac{A}{\mu^{m-|\alpha|}}\frac{1}{(|t|^k+1)^l}\phi_\mu(t;c) \ \text{ on } S_I \ \text{ for any } |\alpha| \le m. \tag{3.5}$$

Now, let us consider the following functional equation with respect to Y:

$$
\begin{aligned}
Y = &\frac{A}{(R-\rho)^{m(\mu-1)}} t^{\mu} \\
&+ \frac{1}{\sigma(R-\rho)^m} \Bigg[\sum_{i+|\alpha|\le m} \frac{\beta A_{i,\alpha}}{(R-\rho)^{m(p_{i,\alpha}+k[i+|\alpha|-l]_+-1)}} \\
&\qquad\qquad \times \frac{(\mu+p_{i,\alpha}+k[i+|\alpha|-l]_+)^m}{\mu^{m-[i+|\alpha|-l]_+}} t^{p_{i,\alpha}+k[i+|\alpha|-l]_+} \eta Y \\
&+ \sum_{|\nu|\ge 2} \frac{B_\nu (q_\nu + k\langle\nu\rangle_l + \mu|\nu|)^m}{(R-\rho)^{m(q_\nu+k\langle\nu\rangle_l+|\nu|-2)}} \, t^{q_\nu} \prod_{i+|\alpha|\le m} \left[\beta\, t^{k[i+|\alpha|-l]_+} \eta Y\right]^{\nu_{i,\alpha}} \Bigg],
\end{aligned} \tag{3.6}
$$

where ρ is a parameter with $0 < \rho < R$, σ is the one in Lemma 3.1, and $\eta = (me)^m$. Since this equation (3.6) is an analytic functional equation, by the implicit function theorem we see that (3.6) has a unique holomorphic solution $Y = Y(t)$ with $Y(t) = O(t^\mu)$ (as $t \longrightarrow 0$). If we expand it into Taylor series $Y = \sum_{n\ge\mu} Y_n t^n$, we see that the coefficients Y_n $(n \ge \mu)$ are determined by the following recurrent formulas:

$$
Y_\mu = \frac{A}{(R-\rho)^{m(\mu-1)}}, \tag{3.7}
$$

and for $n \ge \mu + 1$

$$
\begin{aligned}
Y_n = &\frac{1}{\sigma(R-\rho)^m} \Bigg[\sum_{i+|\alpha|\le m} \frac{\beta A_{i,\alpha}}{(R-\rho)^{m(p_{i,\alpha}+k[i+|\alpha|-l]_+-1)}} \\
&\qquad\qquad \times \frac{(\mu+p_{i,\alpha}+k[i+|\alpha|-l]_+)^m}{\mu^{m-[i+|\alpha|-l]_+}} \eta Y_{n-p_{i,\alpha}-k[i+|\alpha|-l]_+} \\
&+ \sum_{2\le|\nu|\le n-q_\nu} \sum_{\substack{q_\nu+|n(\nu)| \\ +k\langle\nu\rangle_l = n}} \frac{B_\nu (q_\nu + k\langle\nu\rangle_l + \mu|\nu|)^m}{(R-\rho)^{m(q_\nu+k\langle\nu\rangle_l+|\nu|-2)}} \prod_{i+|\alpha|\le m} \prod_{j=1}^{\nu_{i,\alpha}} \left[\beta\eta Y_{n_{i,\alpha}(j)}\right] \Bigg],
\end{aligned} \tag{3.8}
$$

where we used the convention: $Y_p = 0$ if $p < \mu$. Moreover, by induction on n we can see that Y_n has the form

$$
Y_n = \frac{C_n}{(R-\rho)^{m(n-1)}}, \quad n \ge \mu, \tag{3.9}
$$

where $C_\mu = A$ and $C_n > 0$ $(n \ge \mu + 1)$ are constants which are independent of the parameter ρ. Since Y_n depends on the parameter ρ, we sometimes write $Y_n = Y_n(\rho)$ (if we hope to emphasize that it depends on ρ).

The following lemma guarantees that $Y(t)$ is a majorant series of our formal solution $u(t,x)$ in (3.2).

Lemma 3.7. *For any $n \ge \mu$ we have*

$$
\|\partial_x^\alpha u_n(t)\|_\rho \le \frac{(n-\mu)!^{s-1}}{n^{m-|\alpha|}} \frac{\eta}{(|t|^k+1)^l} Y_n \phi_n(t;c) \quad \text{on } S_I \tag{3.10$_n$}
$$

for any $0 < \rho < R$ and $|\alpha| \le m$.

Proof of Lemma 3.7. By the definition of A in (3.5), and the conditions (3.7), $0 < R < 1$ and $\eta > 1$ we have

$$\begin{aligned}\|\partial_x^\alpha u_\mu(t)\|_\rho &\le \|\partial_x^\alpha u_\mu(t)\|_R \\ &\le \frac{A}{\mu^{m-|\alpha|}} \frac{1}{(|t|^k+1)^l} \phi_\mu(t;c) \le \frac{1}{\mu^{m-|\alpha|}} \frac{\eta}{(|t|^k+1)^l} Y_\mu \phi_\mu(t;c) \quad \text{on } S_I\end{aligned}$$

for any $0 < \rho < R$ and $|\alpha| \le m$. This proves $(3.10)_\mu$. Let us show the general case by induction on n.

Let $n \ge \mu+1$, and suppose that $(3.10)_N$ is already proved for all N with $\mu \le N \le n-1$. By $(3.10)_N$ and Lemma 3.5, we have

$$\|\mathscr{M}_{i,\alpha}[\partial_x^\alpha u_N](t)\|_\rho \le \frac{\beta(N-\mu)!^{s-1}}{N^{m-[i+|\alpha|-l]_+}} \eta Y_N \phi_{N+k[i+|\alpha|-l]_+}(t;c) \quad \text{on } S_I$$
$$\text{for any } 0<\rho<R \text{ and } i+|\alpha| \le m$$

for any $\mu \le N \le n-1$. We note that by the assumption (A_5) we have

$$\begin{aligned}\|f(t)\|_R &\le F\phi_\mu(t;c) \quad \text{on } S_I, \\ \|a_{i,\alpha}(t)\|_R &\le A_{i,\alpha}\phi_{p_{i,\alpha}}(t;c) \quad \text{on } S_I \quad (i+|\alpha|\le m), \\ \|b_\nu(t)\|_R &\le B_\nu \phi_{q_\nu}(t;c) \quad \text{on } S_I \quad (|\nu|\ge 2).\end{aligned}$$

Therefore, by applying these estimates to (3.4), by using Lemma 3.4 and by setting

$$p^*_{i,\alpha} = p_{i,\alpha} + k[i+|\alpha|-l]_+, \quad q^*_\nu = q_\nu + k\langle\nu\rangle_l, \tag{3.11}$$

we have

$$\begin{aligned}&\|P(kt^k)u_n(t)\|_\rho \\ &\le \phi_n(t;c)\Bigg[\sum_{i+|\alpha|\le m} A_{i,\alpha}\frac{\beta(n-p^*_{i,\alpha}-\mu)!^{s-1}}{(n-p^*_{i,\alpha})^{m-[i+|\alpha|-l]_+}}\eta Y_{n-p_{i,\alpha}-k[i+|\alpha|-l]_+} \\ &\qquad + \sum_{2\le|\nu|\le n-q_\nu}\ \sum_{q^*_\nu+|n(\nu)|=n} B_\nu \prod_{i+|\alpha|\le m}\prod_{j=1}^{\nu_{i,\alpha}}\Big[\frac{\beta(n_{i,\alpha}(j)-\mu)!^{s-1}}{n_{i,\alpha}(j)^{m-[i+|\alpha|-l]_+}}\eta Y_{n_{i,\alpha}(j)}\Big]\Bigg] \\ &= \phi_n(t;c)[I_1+I_2].\end{aligned} \tag{3.12}$$

We note that $Y_{n-p_{i,\alpha}-k[i+|\alpha|-l]_+} \ne 0$ implies $n - p_{i,\alpha} - k[i+|\alpha|-l]_+ \ge \mu$ and so in I_1 we may suppose that $n - p^*_{i,\alpha} \ge \mu$ holds. We also note that if I_2 we have $n = q^*_\nu + |n(\nu)| \ge q^*_\nu + \mu|\nu|$.

Lemma 3.8. *Under the above situation we have*

$$\frac{n^m}{(n-\mu)!^{s-1}}\frac{(n-p^*_{i,\alpha}-\mu)!^{s-1}}{(n-p^*_{i,\alpha})^{m-[i+|\alpha|-l]_+}} \le \frac{(\mu+p^*_{i,\alpha})^m}{\mu^{m-[i+|\alpha|-l]_+}} \qquad \text{in } I_1, \tag{3.13}$$

$$\frac{n^m}{(n-\mu)!^{s-1}}\prod_{i+|\alpha|\le m}\prod_{j=1}^{\nu_{i,\alpha}}\Big(\frac{(n_{i,\alpha}(j)-\mu)!^{s-1}}{n_{i,\alpha}(j)^{m-[i+|\alpha|-l]_+}}\Big) \le (q^*_\nu+\mu|\nu|)^m \qquad \text{in } I_2. \tag{3.14}$$

Proof of Lemma 3.8. The proof of (3.13) is as follows. If $0 \le i+|\alpha| \le l$, we have $[i+|\alpha|-l]_+ = 0$ and so by using the condition $n - p^*_{i,\alpha} \ge \mu$ we have

$$\begin{aligned}&\frac{n^m}{(n-\mu)!^{s-1}} \frac{(n-p^*_{i,\alpha}-\mu)!^{s-1}}{(n-p^*_{i,\alpha})^{m-[i+|\alpha|-l]_+}} = \frac{n^m}{(n-\mu)!^{s-1}} \frac{(n-p^*_{i,\alpha}-\mu)!^{s-1}}{(n-p^*_{i,\alpha})^{m}} \\ &\le \frac{n^m}{(n-p^*_{i,\alpha})^m} = \Big(1+\frac{p^*_{i,\alpha}}{n-p^*_{i,\alpha}}\Big)^m \le \Big(1+\frac{p^*_{i,\alpha}}{\mu}\Big)^m = \Big(\frac{\mu+p^*_{i,\alpha}}{\mu}\Big)^m.\end{aligned}$$

If $l+1 \le i+|\alpha| \le m$ holds, by the condition $s \ge s_a$ we have $p^*_{i,\alpha}(s-1) \ge [i+|\alpha|-l]_+$, and so we have

$$\begin{aligned}&\frac{n^m}{(n-\mu)!^{s-1}} \frac{(n-p^*_{i,\alpha}-\mu)!^{s-1}}{(n-p^*_{i,\alpha})^{m-[i+|\alpha|-l]_+}} \\ &\quad\le \frac{n^m}{(n-p^*_{i,\alpha})^{m-[i+|\alpha|-l]_+}} \times \frac{1}{(n-\mu-p^*_{i,\alpha}+1)^{p^*_{i,\alpha}(s-1)}} \\ &\quad= \Big(\frac{n}{n-p^*_{i,\alpha}}\Big)^m \Big(\frac{n-p^*_{i,\alpha}}{n-\mu-p^*_{i,\alpha}+1}\Big)^{[i+|\alpha|-l]_+} \frac{(n-\mu-p^*_{i,\alpha}+1)^{[i+|\alpha|-l]_+}}{(n-\mu-p^*_{i,\alpha}+1)^{p^*_{i,\alpha}(s-1)}} \\ &\quad\le \Big(\frac{n}{n-p^*_{i,\alpha}}\Big)^m \Big(\frac{n-p^*_{i,\alpha}}{n-\mu-p^*_{i,\alpha}+1}\Big)^{[i+|\alpha|-l]_+} \\ &\quad= \Big(1+\frac{p^*_{i,\alpha}}{n-p^*_{i,\alpha}}\Big)^m \Big(1+\frac{\mu-1}{n-\mu-p^*_{i,\alpha}+1}\Big)^{[i+|\alpha|-l]_+} \\ &\quad\le \Big(1+\frac{p^*_{i,\alpha}}{\mu}\Big)^m \Big(1+\frac{\mu-1}{1}\Big)^{[i+|\alpha|-l]_+} = \frac{(\mu+p^*_{i,\alpha})^m}{\mu^{m-[i+|\alpha|-l]_+}}.\end{aligned}$$

This proves (3.13).

Let us show (3.14). We note: if $n_i \ge 1$ $(i=1,\dots,|\nu|)$ and $n_1+\dots+n_{|\nu|} = n-q^*_\nu$ hold, we have $n_i \le (n_1 \dots n_{|\nu|})$ for $i=1,\dots,|\nu|$ and so $n-q^*_\nu = n_1+\dots+n_{|\nu|} \le |\nu|(n_1\dots n_{|\nu|})$ which yields $n \le (q^*_\nu+|\nu|)(n_1\dots n_{|\nu|})$, that is,

$$\frac{1}{n_1\dots n_{|\nu|}} \le \frac{q^*_\nu+|\nu|}{n}.$$

Therefore, by the same argument we have

$$\prod_{i+|\alpha|\le m} \prod_{j=1}^{\nu_{i,\alpha}} \frac{1}{n_{i,\alpha}(j)} \le \frac{(q^*_\nu+|\nu|)}{n} \quad \text{in the case } I_2.$$

Since $s \ge s_b$ holds, we have $(q^*_\nu+\mu(|\nu|-1))(s-1) \ge [m_\nu-l]_+$, and so

$$\frac{n^m}{(n-\mu)!^{s-1}} \prod_{i+|\alpha|\le m} \prod_{j=1}^{\nu_{i,\alpha}} \Big(\frac{(n_{i,\alpha}(j)-\mu)!^{s-1}}{n_{i,\alpha}(j)^{m-[i+|\alpha|-l]_+}}\Big)$$

$$
\begin{aligned}
&\le \frac{n^m}{(n-\mu)!^{s-1}} \times (|n(\nu)|-\mu|\nu|)!^{s-1} \prod_{i+|\alpha|\le m} \prod_{j=1}^{\nu_{i,\alpha}} \Bigl(\frac{1}{n_{i,\alpha}(j)^{m-[m_\nu-l]_+}}\Bigr) \\
&\le \frac{n^m}{(n-\mu)!^{s-1}} \times (|n(\nu)|-\mu|\nu|)!^{s-1} \times \Bigl(\frac{(q_\nu^*+|\nu|)}{n}\Bigr)^{m-[m_\nu-l]_+} \\
&= \frac{n^{[m_\nu-l]_+}}{(n-\mu)!^{s-1}} \times (n-q_\nu^*-\mu|\nu|)!^{s-1} \times (q_\nu^*+|\nu|)^{m-[m_\nu-l]_+} \\
&\le \frac{n^{[m_\nu-l]_+}}{(n-q_\nu^*-\mu|\nu|+1)^{(q_\nu^*+\mu(|\nu|-1))(s-1)}} \times (q_\nu^*+|\nu|)^{m-[m_\nu-l]_+} \\
&\le \Bigl(\frac{n}{n-q_\nu^*-\mu|\nu|+1}\Bigr)^{[m_\nu-l]_+} \times (q_\nu^*+|\nu|)^{m-[m_\nu-l]_+} \\
&= \Bigl(1+\frac{q_\nu^*+\mu|\nu|-1}{n-q_\nu^*-\mu|\nu|+1}\Bigr)^{[m_\nu-l]_+} \times (q_\nu^*+|\nu|)^{m-[m_\nu-l]_+} \\
&\le \Bigl(1+\frac{q_\nu^*+\mu|\nu|-1}{1}\Bigr)^{[m_\nu-l]_+} \times (q_\nu^*+|\nu|)^{m-[m_\nu-l]_+} \\
&= (q_\nu^*+\mu|\nu|)^{[m_\nu-l]_+} \times (q_\nu^*+|\nu|)^{m-[m_\nu-l]_+} \le (q_\nu^*+\mu|\nu|)^m.
\end{aligned}
$$

This proves (3.14). □

Hence, by applying Lemma 3.8 to (3.12) we have

$$
\begin{aligned}
&\|P(kt^k)u_n(t)\|_\rho \\
&\le \frac{(n-\mu)!^{s-1}}{n^m} \phi_n(t;c) \Biggl[\sum_{i+|\alpha|\le m} A_{i,\alpha}\beta \frac{(\mu+p_{i,\alpha}^*)^m}{\mu^{m-[i+|\alpha|-l]_+}} \, \eta Y_{n-p_{i,\alpha}-k[i+|\alpha|-l]_+} \\
&\quad + \sum_{2\le|\nu|\le n-q_\nu} \sum_{q_\nu^*+|n(\nu)|=n} B_\nu (q_\nu^*+\mu|\nu|)^m \prod_{i+|\alpha|\le m} \prod_{j=1}^{\nu_{i,\alpha}} \Bigl[\beta\, \eta Y_{n_{i,\alpha}(j)}\Bigr] \Biggr]. \qquad (3.15)
\end{aligned}
$$

By comparing (3.8) and (3.15) under the equalities (3.11), and then by using the conditions $1/(R-\rho) > 1$ and (3.9) we have

$$
\begin{aligned}
\|P(kt^k)u_n(t)\|_\rho &\le \frac{(n-\mu)!^{s-1}}{n^m} \phi_n(t;c) \times \sigma (R-\rho)^m Y_n \\
&= \frac{(n-\mu)!^{s-1}}{n^m} \phi_n(t;c) \times \sigma \frac{C_n}{(R-\rho)^{m(n-2)}},
\end{aligned}
$$

and so by Lemma 3.1 we have

$$
\|u_n(t)\|_\rho \le \frac{(n-\mu)!^{s-1}}{n^m} \phi_n(t;c) \times \frac{1}{(|t|^k+1)^l} \frac{C_n}{(R-\rho)^{m(n-2)}} \quad \text{on } S_I
$$

for any $0 < \rho < R$. Hence, by Lemma 3.6, we have

$$
\|\partial_x^\alpha u_n(t)\|_\rho \le \frac{(n-\mu)!^{s-1}}{n^m} \phi_n(t;c) \frac{1}{(|t|^k+1)^l} \frac{(mn)^{|\alpha|} e^{|\alpha|} C_n}{(R-\rho)^{m(n-2)+|\alpha|}}
$$

$$\leq \frac{(n-\mu)!^{s-1}}{n^{m-|\alpha|}} \phi_n(t;c) \frac{1}{(|t|^k+1)^l} \frac{(me)^m C_n}{(R-\rho)^{m(n-2)+m}}$$
$$= \frac{(n-\mu)!^{s-1}}{n^{m-|\alpha|}} \phi_n(t;c) \frac{1}{(|t|^k+1)^l}\, \eta Y_n \quad \text{on } S_I$$

for any $0 < \rho < R$ and $|\alpha| \leq m$. This proves $(3.10)_n$. □

Completion of the proof of Proposition 3.3. Take any $0 < \rho < R$ and fix it. Since $Y = \sum_{n \geq \mu} Y_n t^n$ is a holomorphic function in a neighborhood of $t = 0$, we have the estimates $Y_n \leq Ch^n$ $(n \geq \mu)$ for some $C > 0$ and $h > 0$. Therefore, applying this to $(3.10)_n$ we have the estimate

$$\|u_n(t)\|_\rho \leq Ch^n \frac{(n-\mu)!^{s-1}\eta}{n^m(|t|^k+1)^l} \phi_n(t;c)$$

for any $n \geq \mu$. This proves Proposition 3.3. □

3.4. EXISTENCE OF A SOLUTION ON $S_I \times D_R$

Let us show the existence of a holomorphic solution $u(t,x)$ of (2.1) on $S_I \times D_R$ with some exponential growth: we have

Theorem 3.9. *Suppose the conditions* (A_1)–(A_6) *and* (3.1). *Let* $\kappa > 0$ *be the one in* (2.2). *Then, equation* (2.1) *has a holomorphic solution* $u(t,x)$ *on* $S_I \times D_\rho$ *for some* $\rho > 0$ *which satisfies the estimate*

$$|u(t,x)| \leq \frac{M}{(|t|^k+1)^l} |t|^{\mu-k} \exp(b|t|^\kappa) \quad \text{on } S_I \times D_\rho \tag{3.16}$$

for some $M > 0$ *and* $b > 0$.

As is seen in the proof given below, this result is valid also for (s,κ) satisfying $\max\{s_a, s_b\} \leq s < 1 + 1/k$ and $1/\kappa = 1/k - (s-1)$.

To prove Theorem 3.9, we will need the following lemma.

Lemma 3.10. *Let* $\alpha > 0$ *and* $k > 0$. *For any* $d > 1$ *there is a* $C > 0$ *such that*

$$\sum_{n \geq 0} \frac{t^n}{\Gamma((\alpha+n)/k)} \leq C \exp(dt^k) \quad \text{for } t > 0. \tag{3.17}$$

Precisely, for any $d > 1$ *we can take* C *as*

$$C = 1 + \frac{B(\alpha/k, 1/k)}{\sqrt{2\pi}} \sum_{n \geq 1} \sqrt{\frac{n}{k}} \Big(\frac{1}{d}\Big)^{n/k},$$

where $B(x,y)$ *is the beta function.*

Proof. We know the following facts:

$$\Gamma(x) \geq \sqrt{2\pi} x^{x-1/2} e^{-x} \quad \text{for } x > 0,$$
$$\frac{\Gamma(n/k)}{\Gamma((\alpha+n)/k)} = \frac{B(\alpha/k, n/k)}{\Gamma(\alpha/k)} \leq \frac{B(\alpha/k, 1/k)}{\Gamma(\alpha/k)} \quad \text{for } n \geq 1.$$

Since the maximum of $x^{n/k}e^{-dx}$ (with $n \geq 1$) on $x > 0$ is equal to $(n/kd)^{n/k}e^{-n/k}$, we have

$$\begin{aligned} t^n &= e^{dt^k} \times t^n e^{-dt^k} \leq e^{dt^k} \times \max_{x>0}\big(x^{n/k}e^{-dx}\big) \\ &= e^{dt^k} \times \Big(\frac{1}{d}\Big)^{n/k} (n/k)^{n/k} e^{-n/k} \leq e^{dt^k} \Big(\frac{1}{d}\Big)^{n/k} \frac{\sqrt{n/k}}{\sqrt{2\pi}} \Gamma(n/k), \end{aligned}$$

and therefore

$$\begin{aligned} \sum_{n\geq 0} \frac{t^n}{\Gamma((\alpha+n)/k)} &= \frac{1}{\Gamma(\alpha/k)} + \sum_{n\geq 1} \frac{t^n}{\Gamma((\alpha+n)/k)} \\ &\leq \frac{1}{\Gamma(\alpha/k)} + \sum_{n\geq 1} \frac{1}{\Gamma((\alpha+n)/k)} \times e^{dt^k} \Big(\frac{1}{d}\Big)^{n/k} \frac{\sqrt{n/k}}{\sqrt{2\pi}} \Gamma(n/k) \\ &\leq \frac{1}{\Gamma(\alpha/k)} + \sum_{n\geq 1} e^{dt^k} \frac{B(\alpha/k, 1/k)}{\Gamma(\alpha/k)} \Big(\frac{1}{d}\Big)^{n/k} \frac{\sqrt{n/k}}{\sqrt{2\pi}} \\ &\leq \frac{e^{dt^k}}{\Gamma(\alpha/k)} \Big(1 + \frac{B(\alpha/k, 1/k)}{\sqrt{2\pi}} \sum_{n\geq 1} \sqrt{n/k} \Big(\frac{1}{d}\Big)^{n/k}\Big). \end{aligned}$$

This proves (3.17). □

Proof of Theorem 3.9. Take any s satisfying $s \geq \max\{s_a, s_b\}$ and $0 \leq s - 1 < 1/k$, and then define $\kappa > 0$ by $1/\kappa = 1/k - (s-1)$. Let

$$u(t, x) = \sum_{n\geq\mu} u_n(t, x)$$

be the formal solution constructed in Subsection 3.1.

First, let us see the case $s = 1$. In this case, we have $\kappa = k$. By Proposition 3.3, we have

$$\begin{aligned} \sum_{n\geq\mu} |u_n(t, x)| &\leq \sum_{n\geq\mu} \frac{Ch^n}{(|t|^k + 1)^l} \frac{|t|^{n-k}}{\Gamma(n/k)} \exp(c|t|^k) \\ &= \frac{Ch^\mu |t|^{\mu-k}}{(|t|^k+1)^l} \exp(c|t|^k) \sum_{q\geq 0} \frac{(h|t|)^q}{\Gamma((\mu+q)/k)} \quad \text{on } S_I \times D_\rho. \end{aligned}$$

By Lemma 3.10, we know that for any $d > 1$ there is a $C_1 > 0$ such that

$$\sum_{q\geq 0} \frac{(h|t|)^q}{\Gamma((\mu+q)/k)} \leq C_1 \exp(dh^k|t|^k), \quad |t| > 0.$$

Thus, by applying this to the above formula and by setting $M = C_1Ch^\mu > 0$ and $b = c + dh^k > 0$ we have the result (3.16).

Next, let us consider the case $s > 1$ (with $s - 1 < 1/k$). Since

$$n!^{s-1} \le C_1{h_1}^n\Gamma(n(s-1)) \quad n = 1, 2, \dots$$

holds for some $C_1 > 0$ and $h_1 > 0$, by Proposition 3.3, we have

$$\begin{aligned}|u_n(t,x)| &\le \frac{Ch^n}{(|t|^k+1)^l}\frac{C_1{h_1}^n\Gamma(n(s-1))}{\Gamma(n/k)}|t|^{n-k}\exp(c|t|^k)\\ &= \frac{CC_1(hh_1)^n}{(|t|^k+1)^l}\frac{B(n/\kappa, n(s-1))}{\Gamma(n/\kappa)}|t|^{n-k}\exp(c|t|^k)\\ &\le \frac{CC_1(hh_1)^n}{(|t|^k+1)^l}\frac{B(1/\kappa,(s-1))}{\Gamma(n/\kappa)}|t|^{n-k}\exp(c|t|^k) \ \text{ on } S_I \times D_\rho\end{aligned}$$

for any $n \ge \mu$. Therefore, if we set $C_2 = CC_1B(1/\kappa, (s-1))$ and $h_2 = hh_1$ we have

$$\begin{aligned}\sum_{n\ge\mu}|u_n(t,x)| &\le \sum_{n\ge\mu}\frac{C_2{h_2}^n}{(|t|^k+1)^l}\frac{|t|^{n-k}}{\Gamma(n/\kappa)}\exp(c|t|^k)\\ &= \frac{C_2{h_2}^\mu|t|^{\mu-k}}{(|t|^k+1)^l}\exp(c|t|^k)\sum_{q\ge0}\frac{(h_2|t|)^q}{\Gamma((q+\mu)/\kappa)} \quad \text{on } S_I\times D_\rho.\end{aligned}$$

Thus, by using Lemma 3.10 and the condition $\kappa > k$, we can show (3.16) in the same way as in the case $s = 1$. □

3.5. UNIQUENESS OF THE LOCAL SOLUTION

Now, let us show the uniqueness of the local solution of (2.1). To do so, it is enough to prove the result (Theorem 3.11) given below. Recall that for $0 < r < \infty$ we wrote $S_I(r) = \{t \in S_I\,;\, 0 < |t| < r\}$.

Theorem 3.11. *Suppose the conditions* (A_1)–(A_6) *and* (3.1). *Let* $0 < r < \infty$ *and* $R > 0$ *be sufficiently small. If* $u_1(t,x) \in \mathcal{O}(S_I(r)\times D_R)$ *and* $u_2(t,x) \in \mathcal{O}(S_I(r)\times D_R)$ *are two solutions of equation* (2.1) *on* $S_I(r)\times D_R$ *satisfying the estimates* $|u_i(t,x)| \le M_0|t|^{\mu-k}$ *on* $S_I(r)\times D_R$ $(i = 1, 2)$ *for some* $M_0 > 0$, *then we have* $u_1(t,x) = u_2(t,x)$ *on* $S_I(r)\times D_R$.

In this case we will use

$$\phi_n(t;0) = \frac{|t|^{n-k}}{\Gamma(n/k)}, \quad n = 1, 2, \dots .$$

Before the proof of Theorem 3.11, we note that if we consider equation (2.1) on $S_I(r)\times D_R$, by the condition (A_5) we have

$$|f(t,x)| \le F_1\phi_1(t;0) \quad \text{on } S_I(r)\times D_R,$$

$$|a_{i,\alpha}(t,x)| \le A_{i,\alpha,1}\phi_1(t;0) \quad \text{on } S_I(r)\times D_R \quad (i+|\alpha|\le m),$$
$$|b_\nu(t,x)| \le B_{\nu,1}\phi_1(t;0) \quad \text{on } S_I(r)\times D_R \quad (|\nu|\ge 2)$$

for some $F_1>0$, $A_{i,\alpha,1}>0$ and $B_{\nu,1}>0$. Since $r>0$ is assumed to be sufficiently small, by (A_6) we have the condition that the series $\sum_{|\nu|\ge 2} B_{\nu,1}X^{|\nu|}$ is convergent in a neighborhood of $X=0$.

Moreover, by [11, Lemma 7.7], we have the following lemma.

Lemma 3.12. *For any $\mu>0$ there is a constant $\beta>0$ which satisfies the following: if $w(t,x)\in\mathcal{O}(S_I(r)\times D_{R_1})$ for some $R_1>0$ and if the estimate $\|w(t)\|_{R_1}\le A\phi_N(t;0)$ on $S_I(r)$ for some $A>0$ and $N\ge\mu$, we have*

$$\|\mathscr{M}_{i,\alpha}[w](t)\|_{R_1} \le \frac{\beta}{N^{|\alpha|}}A\phi_N(t;0) \quad \text{on } S_I(r) \quad \text{for any } i+|\alpha|\le m.$$

By using these conditions, let us give a proof of Theorem 3.11.

Proof of Theorem 3.11. Let $u_1(t,x)$ and $u_2(t,x)$ be two holomorphic solutions of (2.1) on $S_I(r)\times D_R$ satisfying the estimate $|u_i(t,x)|\le M_0|t|^{\mu-k}$ on $S_I(r)\times D_R$ $(i=1,2)$ for some $M_0>0$.

Set $u(t,x)=u_1(t,x)-u_2(t,x)$. By Lemmas 3.6 and 3.12 we have the following: for any $0<R_1<R$, there is an $M_1>0$ such that

$$\|\mathscr{M}_{i,\alpha}[\partial_x^\alpha u](t)\|_{R_1} \le M_1\phi_1(t;0) \ \text{ on } S_I(r) \quad \text{for any } i+|\alpha|\le m. \tag{3.18}$$

Moreover, we have

$$\begin{aligned} &P(kt^k,x)u \\ &= \sum_{i+|\alpha|\le m} a_{i,\alpha}(t,x) *_k \left(\mathscr{M}_{i,\alpha}[\partial_x^\alpha u]\right) \\ &\quad + \sum_{|\nu|\ge 2} b_\nu(t,x) *_k \left[\prod_{i+|\alpha|\le m}^{*_k} \left(\mathscr{M}_{i,\alpha}[\partial_x^\alpha u_1]\right)^{*_k \nu_{i,\alpha}} - \prod_{i+|\alpha|\le m}^{*_k} \left(\mathscr{M}_{i,\alpha}[\partial_x^\alpha u_2]\right)^{*_k \nu_{i,\alpha}} \right]. \end{aligned}$$

Here we note that we have the expression

$$\begin{aligned} &\prod_{i+|\alpha|\le m}^{*_k} \left(\mathscr{M}_{i,\alpha}[\partial_x^\alpha u_1]\right)^{*_k \nu_{i,\alpha}} - \prod_{i+|\alpha|\le m}^{*_k} \left(\mathscr{M}_{i,\alpha}[\partial_x^\alpha u_2]\right)^{*_k \nu_{i,\alpha}} \\ &\quad = \sum_{i+|\alpha|\le m} c_{\nu,i,\alpha}(t,x) *_k \left(\mathscr{M}_{i,\alpha}[\partial_x^\alpha (u_1-u_2)]\right) \end{aligned}$$

for some holomorphic functions $c_{\nu,i,\alpha}(t,x)\in\mathcal{O}(S_I(r)\times D_{R_1})$ $(i+|\alpha|\le m)$. Let us note a simple calculation:

$$\begin{aligned} &{X_1}^k{Y_1}^m{Z_1}^n - {X_2}^k{Y_2}^m{Z_2}^n \\ &= ({X_1}^k-{X_2}^k){Y_1}^m{Z_1}^n + {X_2}^k({Y_1}^m-{Y_2}^m){Z_1}^n + {X_2}^k{Y_2}^m({Z_1}^n-{Z_2}^n) \end{aligned}$$

$$= ({X_1}^{k-1} + {X_1}^{k-2} X_2 + \ldots + {X_2}^{k-1}) {Y_1}^m {Z_1}^n \times (X_1 - X_2)$$
$$+ {X_2}^k ({Y_1}^{m-1} + {Y_1}^{m-2} Y_2 + \ldots + {Y_2}^{m-1}) {Z_1}^n \times (Y_1 - Y_2)$$
$$+ {X_2}^k {Y_2}^m ({Z_1}^{n-1} + {Z_1}^{n-2} Z_2 + \ldots + {Z_2}^{n-1}) \times (Z_1 - Z_2).$$

By using this argument, we can see that $c_{\nu,i,\alpha}(t,x)$ $(i + |\alpha| \leq m)$ are given by the following: if $\nu_{i,\alpha} = 0$, we have $c_{\nu,i,\alpha}(t,x) = 0$, and if $\nu_{i,\alpha} > 0$, we have

$$c_{\nu,i,\alpha}(t,x) = \prod_{(j,\beta)\prec(i,\alpha)}^{*_k} \left(\mathscr{M}_{j,\beta}[\partial_x^\beta u_2]\right)^{*_k \nu_{j,\beta}}$$
$$*_k \sum_{p+q=\nu_{i,\alpha}-1} \left[\left(\mathscr{M}_{i,\alpha}[\partial_x^\alpha u_1]\right)^{*_k p} *_k \left(\mathscr{M}_{i,\alpha}[\partial_x^\alpha u_2]\right)^{*_k q}\right]$$
$$*_k \prod_{(j,\beta)\succ(i,\alpha)}^{*_k} \left(\mathscr{M}_{j,\beta}[\partial_x^\beta u_1]\right)^{*_k \nu_{j,\beta}},$$

where $\prec$ is any linear order in the set $\{(i,\alpha)\,;\, i+|\alpha| \leq m\}$ (by this order we can write all elements as (i_p, α_p) $(p = 1, 2, \ldots, N)$ so that $(i_1, \alpha_1) \prec (i_2, \alpha_2) \prec \ldots \prec (i_N, \alpha_N)$).

Thus, by setting

$$\gamma_{i,\alpha}(t,x) = a_{i,\alpha}(t,x) + \sum_{|\nu|\geq 2} b_\nu(t,x) *_k c_{\nu,i,\alpha}(t,x), \quad i + |\alpha| \leq m,$$

we see that $\gamma_{i,\alpha}(t,x)$ $(i + |\alpha| \leq m)$ are holomorphic functions on $S_I(r) \times D_{R_1}$ and that $u(t,x)$ satisfies a linear convolution partial differential equation

$$P(kt^k, x)u = \sum_{i+|\alpha|\leq m} \gamma_{i,\alpha}(t,x) *_k \left(\mathscr{M}_{i,\alpha}[\partial_x^\alpha u]\right). \tag{3.19}$$

Since

$$\phi_{|\nu|}(t;0) \leq \frac{\Gamma(1/k)}{\Gamma(|\nu|/k)} r^{|\nu|-1} \phi_1(t;0) \quad \text{on } S_I(r)$$

holds, by Lemma 3.4 and (3.18) we can see that $\gamma_{i,\alpha}(t,x)$ $(i + |\alpha| \leq m)$ satisfy the estimates

$$\|\gamma_{i,\alpha}(t)\|_{R_1} \leq C_{i,\alpha} \phi_1(t;0) \quad \text{on } S_I(r) \quad (i + |\alpha| \leq m)$$

for some $C_{i,\alpha} \geq 0$ $(i + |\alpha| \leq m)$. Let us show the following lemma.

Lemma 3.13. *There is a $K > 0$ such that for any $n = 1, 2, \ldots$ we have*

$$\|\mathscr{M}_{i,\alpha}[\partial_x^\alpha u](t)\|_\rho \leq K^{n-1} \frac{M_1}{(R_1-\rho)^{m(n-1)}} \phi_n(t;0) \quad \textit{on } S_I(r) \tag{3.20$_n$}$$

for any $0 < \rho < R_1$ and $i + |\alpha| \leq m$.

Proof of Lemma 3.13. In the case $n = 1$ this is already proved in (3.18). Let $n \geq 2$ and suppose that $(3.20)_{n-1}$ is already proved. Then by Lemma 3.1, (3.19) and the induction hypothesis we have

$$\|u(t)\|_\rho \leq \frac{1}{\sigma} \sum_{i+|\alpha|\leq m} C_{i,\alpha} K^{n-2} \frac{M_1}{(R_1-\rho)^{m(n-2)}} \phi_n(t;0) \quad \text{on } S_I(r)$$

for any $0 < \rho < R_1$. Therefore, by Lemmas 3.6 and 3.12, we have

$$\|\mathscr{M}_{i,\alpha}[\partial_x^\alpha u](t)\|_\rho \le \frac{\beta}{\sigma} \sum_{i+|\alpha|\le m} C_{i,\alpha} K^{n-2} \frac{M_1(me)^m}{(R_1-\rho)^{m(n-1)}} \phi_n(t;0) \quad \text{on } S_I(r)$$

for any $0 < \rho < R_1$ and $i + |\alpha| \le m$. Thus, if we take $K > 0$ so that

$$K \ge \frac{\beta}{\sigma} \sum_{i+|\alpha|\le m} C_{i,\alpha}(me)^m,$$

we have the result $(3.20)_n$. This proves Lemma 3.13. □

Thus, by letting $n \longrightarrow \infty$ in $(3.20)_n$ (with $(i,\alpha) = (0,0)$) we have $\|u(t)\|_\rho = 0$ for any $0 < \rho < R_1$ and $t \in S_I(r)$, that is, $u(t,x) = 0$ on $S_I(r) \times D_{R_1}$. Since R_1 is taken so that $0 < R_1 < R$, the unique continuation property in x yields $u(t,x) = 0$ on $S_I(r) \times D_R$. This proves Theorem 3.11. □

3.6. COMPLETION OF THE PROOF OF THEOREM 2.2

Let $u(t,x)$ be a holomorphic solution of equation (2.1) on $S_I(\delta) \times D_{R_0}$ for some $\delta > 0$ and $R_0 > 0$, and suppose that it satisfies $|u(t,x)| \le M_0|t|^{\mu-k}$ on $S_I(\delta) \times D_{R_0}$ for some $M_0 > 0$. Let $u^*(t,x)$ be a holomorphic solution of (2.1) on $S_I \times D_R$ constructed in Theorem 3.9. If we consider the equation on $S_I(\delta) \times D_R$, we can apply the uniqueness result in Theorem 3.11. Hence, we have $u(t,x) = u^*(t,x)$ on $S_I(\delta) \times D_R$. This shows that $u^*(t,x)$ is a holomorphic extension of $u(t,x)$ to the domain $S_I \times D_R$. The estimate (2.4) follows from (3.16). This proves Theorem 2.2 under (2.5). □

4. PROOF OF THEOREM 2.2 IN THE GENERAL CASE

In this section we will prove Theorem 2.2 in the general case, that is, under the condition:

$$\lambda_i(0) = 0 \;\text{ or }\; \lambda_i(0) \in \mathbb{C} \setminus \overline{\pi(S_{kI})} \quad \text{for all } i = 1, 2, \ldots, l. \tag{4.1}$$

In order to overcome the difficulty of the case where $\lambda_i(0) = 0$ occurs for some i, we will employ the same method as in Braaksma [3] and Ouchi [9].

We note that if $I = \bigcup_{i=1}^p I_i$ for some open intervals I_i $(i = 1, 2, \ldots, p)$ and if $u(t,x)$ has an analytic extension to $S_{I_i} \times D_R$ for each $i = 1, 2, \ldots, p$, then $u(t,x)$ has an analytic extension to $S_I \times D_R$. This shows that in the proof of Theorem 2.2 we may suppose the condition: $0 < |I| < \pi/2k$.

We write

$$S_I(r] = \{t \in \mathcal{R}(\mathbb{C} \setminus \{0\})\,;\, t \in I, 0 < |t| \le r\},$$
$$L_\theta(r) = \{t \in \mathcal{R}(\mathbb{C} \setminus \{0\})\,;\, \arg t = \theta, 0 < |t| < r\}.$$

Definition 4.1.

(1) We denote by $\mathscr{X}(S_I(r] \times D_R)$ the set of all functions $f(t,x)$ which are continuous on $S_I(r] \times D_R$ $(\subset \mathbb{C}_t \times \mathbb{C}_x^K)$ and holomorphic in $x \in D_R^\circ$ for any fixed $t \in S_I(r]$.
(2) We denote by $\mathscr{X}(L_\theta(r) \times D_R)$ the set of all functions $f(t,x)$ which are continuous on $L_\theta(r) \times D_R$ $(\subset \mathbb{C}_t \times \mathbb{C}_x^K)$ and holomorphic in $x \in D_R^\circ$ for any fixed $t \in L_\theta(r)$.

In the proof of Theorem 2.2 given below, we will start our discussion from the assumption that $u(t,x)$ *is a holomorphic solution of equation* (2.1) *on* $S_I(\delta) \times D_{R_0}$ *for some* $\delta > 0$. From now, we fix $\delta > 0$. Then we take any $r_0 > 0$ such that $0 < r_0 < \delta$ and fix it. Thus,

$$\delta \text{ and } r_0 \text{ are fixed so that } 0 < r_0 < \delta. \tag{4.2}$$

We first note that the meaning of the condition (4.1) lies in the following lemma.

Lemma 4.2.

(1) *If* (4.1) *is satisfied, for* $r_0 > 0$ *in* (4.2) *we can take* $\sigma > 0$ *and* $R_1 > 0$ *so that we have the estimate*

$$|P(kt^k, x)| \geq \sigma(|t|^k + 1)^l \quad \text{on } (S_I \setminus S_I(r_0)) \times D_{R_1}. \tag{4.3}$$

(2) *Therefore, if* $g(t,x) \in \mathscr{X}(S_I \times D_{R_1})$ *satisfies* $g(t,x) = 0$ *on* $S_I(r) \times D_{R_1}$ *for some* $r \geq r_0$, *the equation* $P(kt^k, x)w = g(t,x)$ *has a unique solution* $w(t,x) \in \mathscr{X}(S_I \times D_{R_1})$ *which satisfies* $w(t,x) = 0$ *on* $S_I(r) \times D_{R_1}$. *Moreover, if* $|g(t,x)| \leq A$ *holds on* $S_I \times D_{R_1}$ *we have the estimate*

$$|w(t,x)| \leq \frac{A}{\sigma(|t|^k + 1)^l} \quad \text{on } S_I \times D_{R_1}.$$

4.1. PROOF OF THEOREM 2.2

In this subsection, we will present three propositions and one lemma without proofs, and then we will show that if we admit these result, we can prove Theorem 2.2 in the general case. The proofs of propositions and lemma will be given later.

The first proposition is as follows:

Proposition 4.3 (Extension as a continuous solution in t)**.** *Suppose the conditions* (A_1)–(A_6) *and* (4.1). *Let* $\kappa > 0$ *be the one in* (2.2). *If* $u(t,x) \in \mathscr{X}(S_I(r] \times D_{R_0})$ *is a solution of equation* (2.1) *on* $S_I(r] \times D_{R_0}$ *for some* $r \geq r_0$ *and if it satisfies* $|u(t,x)| \leq M_0 |t|^{\mu-k}$ *on* $S_I(r] \times D_{R_0}$ *for some* $M_0 > 0$, *then* $u(t,x)$ *has an extension* $u^*(t,x) \in \mathscr{X}(S_I \times D_R)$ *on* $S_I \times D_R$ *for some* $R > 0$ *which satisfies the following properties:* $u^*(t,x) = u(t,x)$ *on* $S_I(r] \times D_R$, $u^*(t,x)$ *is a solution of* (2.1) *on* $S_I \times D_R$, *and*

$$|u^*(t,x)| \leq \frac{M}{(|t|^k + 1)^l} |t|^{\mu-k} \exp(b|t|^\kappa) \quad \text{on } S_I \times D_R \tag{4.4}$$

holds for some $M > 0$ *and* $b > 0$.

The next one is a result on the uniqueness of the solution.

Proposition 4.4 (Uniqueness of the local solution)**.** *Suppose the conditions* (A_1)*–*(A_6) *and* (4.1)*. Let* $u_1(t,x) \in \mathscr{X}(L_\theta(r) \times D_R)$ *and* $u_2(t,x) \in \mathscr{X}(L_\theta(r) \times D_R)$ *be two solutions of equation* (2.1) *on* $L_\theta(r) \times D_R$ *for some* $\theta \in I$ *and* $r > r_0$*, and suppose that they satisfy the estimates* $|u_i(t,x)| \le M_0|t|^{\mu-k}$ *on* $L_\theta(r) \times D_R$ $(i = 1,2)$ *for some* $M_0 > 0$*. Then, if* $u_1(t,x) = u_2(t,x)$ *on* $L_\theta(r_1) \times D_R$ *for some* r_1 *with* $r_0 \le r_1 < r$*, we have* $u_1(t,x) = u_2(t,x)$ *on* $L_\theta(r) \times D_R$*.*

The third one is a result on the holomorphic extension in a local region. For t_0 and $r > 0$ we write $\Delta_{t_0}(r) = \{(t^k + t_0^k)^{1/k}\,;\, t \in S_I(r)\}$.

Proposition 4.5 (Holomorphic extension)**.** *Suppose the conditions* (A_1)*–*(A_6) *and* (4.1)*. Let* $u(t,x) \in \mathscr{X}((S_I(r] \cup L_\theta(2^{1/k}r)) \times D_R)$ *be a solution of equation* (2.1) *on* $(S_I(r] \cup L_\theta(2^{1/k}r)) \times D_R$ *for some* $\theta \in I$ *and* $r > r_0$ *which is holomorphic on* $S_I(r) \times D_R^\circ$*, and suppose that* $|u(t,x)| \le M_0|t|^{\mu-k}$ *holds on* $S_I(r] \times D_R$ *for some* $M_0 > 0$*. Set* $t_0 = re^{i\theta}$*, and take any* $r_1 > 0$ *(with* $r_0 \le r_1 < r$*). Then,* $u(t,x)$ *is extended holomorphically up to the domain* $\Delta_{t_0}(r_1) \times D_\rho$ *for some* $0 < \rho < R$ *and its extension is bounded on* $\Delta_{t_0}(r_1) \times D_\rho$*.*

The last one is a general result on the holomorphy of functions.

Lemma 4.6 (On the holomorphy)**.** *Let* S *be an open subset of* $\mathbb{C}_t$*. If* $u(t,x) \in \mathscr{X}(S \times D_R)$ *is holomorphic on* $S \times D_\rho^\circ$ *for some* $0 < \rho < R$*, then* $u(t,x)$ *is holomorphic on* $S \times D_R^\circ$*.*

In the first part of this section we have supposed the condition $0 < |I| < \pi/2k$. By this condition, we have $S_I(r) \cap \Delta_{t_0}(r) = \emptyset$. This fact can be verified by noticing that the condition $S_I(r) \cap \Delta_{t_0}(r) = \emptyset$ is equivalent to the condition $S_{kI}(r^k) \cap (t_0^k + S_{kI}(r^k)) = \emptyset$, and by drawing pictures of $S_{kI}(r^k)$, t_0^k and $t_0^k + S_{kI}(r^k)$.

By using these result, let us give a proof of Theorem 2.2.

Proof of Theorem 2.2. Let $u(t,x)$ be a holomorphic solution of equation (2.1) on $S_I(\delta) \times D_{R_0}$ for some $\delta > 0$, and suppose that $|u(t,x)| \le M_0|t|^{\mu-k}$ holds on $S_I(\delta) \times D_{R_0}$ for some $M_0 > 0$. We may suppose the conditions (4.2) and (4.3).

(1) Take any $r > 0$ (with $r_0 \le r < \delta$); then by Proposition 4.3 we see that $u(t,x)$ (restricted on $S_I(r] \times D_{R_0}$) has an extension $u^*(t,x) \in \mathscr{X}(S_I \times D_R)$ on $S_I \times D_R$ for some $0 < R < R_0$ which satisfies the following properties: $u^*(t,x) = u(t,x)$ on $S_I(r] \times D_R$, $u^*(t,x)$ is a solution of (2.1) on $S_I \times D_R$, and the estimate (4.4) holds for some $M > 0$ and $b > 0$.

(2) Let us consider two solutions $u(t,x)$ and $u^*(t,x)$ on $S_I(\delta) \times D_R$. Then by (1) we have $u^*(t,x) = u(t,x)$ on $S_I(r] \times D_R$, and so by applying Proposition 4.4 we have $u^*(t,x) = u(t,x)$ on $S_I(\delta) \times D_R$. This shows that $u^*(t,x)$ is holomorphic on $S_I(\delta) \times D_R^\circ$.

(3) To show Theorem 2.2 it is enough to prove that this $u^*(t,x)$ is holomorphic on $S_I \times D_R^\circ$; by (2) we already know that $u^*(t,x)$ is holomorphic on $S_I(\delta) \times D_R^\circ$.

(4) Take any $\theta \in I$, $r > 0$, and $r_1 > 0$ (with $r_0 \le r_1 < r < \delta$ and $(r^k + r_1^k)^{1/k} < \delta$), and we consider the function $u^*(t,x)$ on $(S_I(r] \cup L_\theta(2^{1/k}r)) \times D_R$. By (1) we know that this $u^*(t,x)$ is a solution of (2.1) on $(S_I(r] \cup L_\theta(2^{1/k}r)) \times D_R$ and by Proposition 4.5 we

see that $u^*(t,x)$ (restricted on $(S_I(r] \cup L_\theta(2^{1/k}r)) \times D_R$) has a holomorphic extension $u_1(t,x)$ on $\Delta_{t_0}(r_1) \times D_\rho$ for some $0 < \rho < R$ which is bounded on $\Delta_{t_0}(r_1) \times D_\rho$.

Now, we set $U = S_I(\delta) \cap \Delta_{t_0}(r_1)$, and let us consider two functions $u^*(t,x)$ and $u_1(t,x)$ only on $U \times D_\rho$. Since these two functions are holomorphic on $U \times D_\rho$ and since $u^*(t,x) = u_1(t,x)$ on $(U \cap L_\theta(2^{1/k}r)) \times D_\rho$, by the unique continuation property of holomorphic functions we have $u^*(t,x) = u_1(t,x)$ on $U \times D_\rho$. Thus, if we set

$$u^0(t,x) = \begin{cases} u^*(t,x), & \text{if } (t,x) \in S_I(\delta) \times D_\rho, \\ u_1(t,x), & \text{if } (t,x) \in \Delta_{t_0}(r_1) \times D_\rho, \end{cases}$$

we have a holomorphic extension $u^0(t,x)$ of $u^*(t,x)$ (restricted on $S_I(\delta) \times D_\rho$) to the domain $(S_I(\delta) \cup \Delta_{t_0}(r_1)) \times D_\rho$.

(5) Take a sufficiently small $\epsilon > 0$ such that the interval $I_0 = (\theta - \epsilon, \theta + \epsilon)$ satisfies $S_{I_0}((r^k + r_1^k)^{1/k}) \subset S_I(\delta) \cup \Delta_{t_0}(r_1)$, and let us consider two functions $u^*(t,x)$ and $u^0(t,x)$ only on $S_{I_0}((r^k + r_1^k)^{1/k}) \times D_\rho$. Since $u^0(t,x) = u^*(t,x)$ holds on $S_{I_0}(\delta) \times D_\rho$, we see that $u^0(t,x)$ is a holomorphic solution of (2.1) on $S_{I_0}(\delta) \times D_\rho$, and so by the unique continuation property of holomorphic functions we have the result that $u^0(t,x)$ satisfies equation (2.1) also on the domain $S_{I_0}((r^k + r_1^k)^{1/k}) \times D_\rho$. Therefore, by Proposition 4.4 (with R replaced by ρ) we have the conclusion that $u^0(t,x) = u^*(t,x)$ on $S_{I_0}((r^k + r_1^k)^{1/k}) \times D_\rho$. Thus, we have proved that $u^*(t,x)$ is holomorphic on $S_{I_0}((r^k + r_1^k)^{1/k}) \times D_\rho$.

(6) Since $u^*(t,x) \in \mathscr{X}(S_{I_0}((r^k + r_1^k)^{1/k}) \times D_R)$ is known (by (1)), by applying Lemma 4.6 to the conclusion of (5) we see that $u^*(t,x)$ is holomorphic on $S_{I_0}((r^k + r_1^k)^{1/k}) \times D_R^\circ$.

(7) Since $\theta \in I$ and $r_0 \le r_1 < r < \delta$ (with $(r^k + r_1^k)^{1/k} < \delta$) are taken arbitrarily in (4), we can conclude that $u^*(t,x)$ is holomorphic on $S_I(2^{1/k}\delta) \times D_R^\circ$.

(8) If we replace δ by $2^{1/k}\delta$ in (4)-(7), by the same argument as above we can prove that $u^*(t,x)$ is holomorphic on $S_I(2^{2/k}\delta) \times D_R^\circ$. By repeating the same argument, we have the conclusion that $u^*(t,x)$ is holomorphic on $S_I \times D_R^\circ$. This proves Theorem 2.2. □

Thus, to complete the proof of Theorem 2.2 it is sufficient to show Propositions 4.3, 4.4, 4.5 and Lemma 4.6. For $\lambda = \{\lambda_{i,\alpha}\}_{i+|\alpha| \le m} \in \mathbb{N}^N$, we define $|\lambda|$ and $\langle \lambda \rangle_l$ in the same way as in Section 2. The following lemma is used in the discussion below.

Lemma 4.7.

(1) *Let $d > 0$. For any $p = 1, 2, \ldots$ and $|t| > 0$ we have*

$$\phi_{n+p}(t;c) \le C_0 \Big(\frac{\sqrt{2}}{d}\Big)^{p/k} \phi_n(t;c+d) \quad \textit{with } C_0 = \frac{B(1/k,1/k)}{\sqrt{2\pi}}.$$

(2) *Let $a_\lambda(t,x) \in \mathscr{X}(S_I \times D_R)$ $(|\lambda| \ge 1)$ and $w_{i,\alpha}(t,x) \in \mathscr{X}(S_I \times D_R)$ $(i+|\alpha| \le m)$. Suppose that there are $A_\lambda > 0$ $(|\lambda| \ge 1)$, $p_\lambda \in \mathbb{N}^*$ $(|\lambda| \ge 1)$, $M > 0$ and $\mu \in \mathbb{N}^*$ which*

satisfy $\|a_\lambda(t)\|_R \le A_\lambda \phi_{p_\lambda}(t;c)$ *on* S_I $(|\lambda| \ge 1)$, $\|w_{i,\alpha}(t)\|_R \le M\phi_{\mu+k[i+|\alpha|-l]_+}(t;c)$ *on* S_I $(i+|\alpha| \le m)$, *and* $\sum_{|\lambda|\ge 1} A_\lambda t^{p_\lambda} X^{|\lambda|} \in \mathbb{C}\{t,X\}$. *Then, if we set*

$$f(t,x) = \sum_{|\lambda|\ge 1} a_\lambda(t,x) *_k \prod_{i+|\alpha|\le m}^{*_k} \big[w_{i,\alpha}\big]^{*_k \lambda_{i,\alpha}},$$

we have the result that $f(t,x)$ *is well-defined as a function in the class* $\mathscr{X}(S_I \times D_R)$ *and the estimate* $\|f(t)\|_R \le F\phi_\mu(t;c+d)$ *holds on* S_I *for some* $F > 0$ *and* $d > 0$.

Proof. Let us show (1). We note that the maximum of $f(x) = x^a e^{-dx}$ (with $a > 0$) on $x > 0$ is equal to $(a/d)^a e^{-a}$ and so by Stirling's formula we have

$$x^a e^{-dx} \le \Big(\frac{a}{d}\Big)^a e^{-a} \le \Big(\frac{1}{d}\Big)^a \frac{\sqrt{a}\,\Gamma(a)}{\sqrt{2\pi}} \le \Big(\frac{\sqrt{2}}{d}\Big)^a \frac{\Gamma(a)}{\sqrt{2\pi}}, \quad x > 0,$$

where we used the fact that $2^a \ge a$ for $a > 0$. Hence, we have

$$\begin{aligned} \phi_{n+p}(t;c) &= \phi_n(t;c+d) \times |t|^p e^{-d|t|^k} \frac{\Gamma(n/k)}{\Gamma((n+p)/k)} \\ &\le \phi_n(t;c+d) \times \Big(\frac{\sqrt{2}}{d}\Big)^{p/k} \frac{\Gamma(p/k)}{\sqrt{2\pi}} \frac{\Gamma(n/k)}{\Gamma((n+p)/k)} \\ &\le \phi_n(t;c+d)\Big(\frac{\sqrt{2}}{d}\Big)^{p/k} \frac{B(1/k,1/k)}{\sqrt{2\pi}}. \end{aligned}$$

Let us show (2). Let $d > 0$ be sufficiently large. Then we have $\sqrt{2} \le d$. By the usual argument and the result (1), we have

$$\begin{aligned} \|f(t)\|_R &\le \sum_{|\lambda|\ge 1} A_\lambda M^{|\lambda|} \phi_{p_\lambda + k\langle\lambda\rangle_l + \mu|\lambda|}(t;c) \\ &\le \sum_{|\lambda|\ge 1} A_\lambda M^{|\lambda|}\, C_0 \Big(\frac{\sqrt{2}}{d}\Big)^{(p_\lambda + k\langle\lambda\rangle_l + \mu|\lambda| - \mu)/k} \phi_\mu(t;c+d) \\ &\le C_0 \Big(\frac{\sqrt{2}}{d}\Big)^{-\mu/k} \sum_{|\lambda|\ge 1} A_\lambda \Big(\frac{\sqrt{2}}{d}\Big)^{p_\lambda/k} \Big[M\Big(\frac{\sqrt{2}}{d}\Big)^{\mu/k}\Big]^{|\lambda|} \phi_\mu(t;c+d). \end{aligned}$$

In the above we have used the fact $(\sqrt{2}/d)^{\langle\lambda\rangle_l} \le 1$. Since $d > 0$ is sufficiently large, the above series is convergent. This proves (2). □

4.2. PROOF OF PROPOSITION 4.3

Let $u(t,x) \in \mathscr{X}(S_I(r] \times D_{R_0})$ be a solution of equation (2.1) on $S_I(r] \times D_{R_0}$ for some $r \ge r_0$ and suppose that $|u(t,x)| \le M_0|t|^{\mu-k}$ holds on $S_I(r] \times D_{R_0}$ for some $M_0 > 0$. Set

$$u_{\text{ext}}(t,x) = \begin{cases} u(t,x), & \text{if } (t,x) \in S_I(r] \times D_{R_0}, \\ u(rt/|t|,x), & \text{if } (t,x) \in (S_I \setminus S_I(r]) \times D_{R_0}. \end{cases}$$

Then we have $u_{\text{ext}}(t,x) \in \mathscr{X}(S_I \times D_{R_0})$ and $u_{\text{ext}}(t,x) = u(t,x)$ on $S_I(r] \times D_{R_0}$. Since $\|u(t)\|_{R_0} \le M_1 \phi_\mu(t;c)$ holds on $S_I(r]$ for some $M_1 > 0$, by Lemmas 3.5 and 3.6 we have the following: for any $0 < R_1 < R_0$ there is an $M > 0$ such that

$$\|\mathscr{M}_{i,\alpha}[\partial_x^\alpha u_{\text{ext}}](t)\|_{R_1} \le M\phi_{\mu+k[i+|\alpha|-l]_+}(t;c) \quad \text{on } S_I \text{ for any } i+|\alpha| \le m. \tag{4.5}$$

We set

$$\begin{aligned} f_{\text{ext}}(t,x) = &- \sum_{i+|\alpha|\le m} a_{i,\alpha}(t,x) *_k \big(\mathscr{M}_{i,\alpha}[\partial_x^\alpha u_{\text{ext}}]\big) \\ &- \sum_{|\nu|\ge 2} b_\nu(t,x) *_k \prod_{i+|\alpha|\le m}^{*_k} \big(\mathscr{M}_{i,\alpha}[\partial_x^\alpha u_{\text{ext}}]\big)^{*_k \nu_{i,\alpha}} + P(kt^k,x)u_{\text{ext}}. \end{aligned}$$

By Lemma 3.4, (4.5) and Lemma 4.7, we can see that $f_{\text{ext}}(t,x)$ is well-defined as a function in the class $\mathscr{X}(S_I \times D_{R_1})$ and that it satisfies the estimate $\|f_{\text{ext}}(t)\|_{R_1} \le F_1\phi_\mu(t;c+d)$ on S_I for some $F_1 > 0$ and $d > 0$. Since $u_{\text{ext}}(t,x) = u(t,x)$ holds on $S_I(r] \times D_{R_0}$ and since $u(t,x)$ is a solution of (2.1) on $S_I(r] \times D_{R_0}$, we have $f_{\text{ext}}(t,x) = f(t,x)$ on $S_I(r] \times D_{R_1}$: therefore, we see that $f_{\text{ext}}(t,x)$ is holomorphic on $S_I(r) \times D_{R_1}^\circ$.

Now, let us look for an extension $u^*(t,x)$ on $S_I \times D_{R_1}$ as a solution of equation (2.1) in the form:

$$u^*(t,x) = u_{\text{ext}}(t,x) + w(t,x), \quad w(t,x) = 0 \text{ on } S_I(r) \times D_{R_1}.$$

The condition $w(t,x) = 0$ on $S_I(r) \times D_{R_1}$ guarantees that $u^*(t,x)$ is an extension of $u(t,x)$. Since $u^*(t,x)$ must be a solution of (2.1), the unknown function $w(t,x)$ must satisfy the following equation:

$$\begin{aligned} P(kt^k,x)w = f(t,x) - f_{\text{ext}}(t,x) &+ \sum_{i+|\alpha|\le m} a_{i,\alpha}(t,x) *_k \big(\mathscr{M}_{i,\alpha}[\partial_x^\alpha w]\big) \\ &+ \sum_{|\nu|\ge 2} b_\nu(t,x) *_k \Bigg[\prod_{i+|\alpha|\le m}^{*_k} \big(\mathscr{M}_{i,\alpha}[\partial_x^\alpha (u_{\text{ext}}+w)]\big)^{*_k \nu_{i,\alpha}} \\ &\qquad - \prod_{i+|\alpha|\le m}^{*_k} \big(\mathscr{M}_{i,\alpha}[\partial_x^\alpha u_{\text{ext}}]\big)^{*_k \nu_{i,\alpha}}\Bigg]. \end{aligned} \tag{4.6}$$

Lemma 4.8. *Let $X = \{X_{i,\alpha}\}_{i+|\alpha|\le m} \in \mathbb{C}^N$, and let us consider*

$$F(X) = \sum_{|\lambda|\ge 1} f_\lambda X^\lambda \in \mathbb{C}\{X\}$$

with $\lambda = \{\lambda_{i,\alpha}\}_{i+|\alpha|\le m} \in \mathbb{N}^N$, $|\lambda| = \sum_{i+|\alpha|\le m} \lambda_{i,\alpha}$ and $X^\lambda = \prod_{i+|\alpha|\le m} (X_{i,\alpha})^{\lambda_{i,\alpha}}$. Then, we have the formula:

$$F(X+Y) - F(X) = \sum_{|\nu|\ge 1} \Bigg[\sum_{\lambda \succeq \nu} \frac{\lambda!}{\nu!(\lambda-\nu)!} f_\lambda X^{\lambda-\nu}\Bigg] Y^\nu,$$

where $\{\lambda_{i,\alpha}\}_{i+|\alpha|\le m} \succcurlyeq \{\nu_{i,\alpha}\}_{i+|\alpha|\le m}$ *means that* $\lambda_{i,\alpha} \ge \nu_{i,\alpha}$ *holds for all* (i,α) *with* $i+|\alpha|\le m$.

Therefore, by setting

$$\begin{aligned} g(t,x) &= f(t,x) - f_{\mathrm{ext}}(t,x), \\ h_{i,\alpha}(t,x) &= \sum_{|\lambda|\ge 2, \lambda_{i,\alpha}>0} \lambda_{i,\alpha} b_\lambda(t,x) *_k \left[\prod_{(j,\beta)\ne(i,\alpha)}^{*_k} \left(\mathscr{M}_{j,\beta}[\partial_x^\beta u_{\mathrm{ext}}]\right)^{*_k \lambda_{j,\beta}} \right] \\ &\qquad\qquad *_k \left(\mathscr{M}_{i,\alpha}[\partial_x^\alpha u_{\mathrm{ext}}]\right)^{*_k(\lambda_{i,\alpha}-1)}, \\ c_\nu(t,x) &= \sum_{|\lambda|\ge 2, \lambda\succcurlyeq\nu} \frac{\lambda!}{\nu!(\lambda-\nu)!} b_\lambda(t,x) *_k \prod_{i+|\alpha|\le m}^{*_k} \left(\mathscr{M}_{i,\alpha}[\partial_x^\alpha u_{\mathrm{ext}}]\right)^{*_k(\lambda_{i,\alpha}-\nu_{i,\alpha})}, \end{aligned}$$

equation (4.6) is expressed in the form

$$\begin{aligned} P(kt^k,x)w = g(t,x) &+ \sum_{i+|\alpha|\le m} (a_{i,\alpha}(t,x) + h_{i,\alpha}(t,x)) *_k \left(\mathscr{M}_{i,\alpha}[\partial_x^\alpha w]\right) \\ &+ \sum_{|\nu|\ge 2} c_\nu(t,x) *_k \prod_{i+|\alpha|\le m}^{*_k} \left(\mathscr{M}_{i,\alpha}[\partial_x^\alpha w]\right)^{*_k \nu_{i,\alpha}}. \end{aligned} \tag{4.7}$$

This is just the same type of equation as (2.1), but in this case we have the condition $g(t,x)=0$ on $S_I(r]\times D_{R_1}$, and so in the construction of a formal solution on $S_I\times D_{R_1}$ (under the condition $w(t,x)=0$ on $S_I(r)\times D_{R_1}$) we can use (2) of Lemma 4.2.

By the definition, we have $g(t,x)\in \mathscr{X}(S_I\times D_{R_1})$, $g(t,x)=0$ on $S_I(r)\times D_{R_1}$, $h_{i,\alpha}(t,x)\in \mathscr{X}(S_I\times D_{R_1})\cap \mathcal{O}(S_I(r)\times D_{R_1})$ $(i+|\alpha|\le m)$, and $c_\nu(t,x)\in \mathscr{X}(S_I\times D_{R_1})\cap \mathcal{O}(S_I(r)\times D_{R_1})$ $(|\nu|\ge 2)$. Since $g(t,x)=0$ on $S_I(r]\times D_{R_1}$, for any $\mu_1\ge\mu$ we can find a constant $G>0$ such that $\|g(t)\|_{R_1}\le G\phi_{\mu_1}(t;c+d)$ holds on $S_I\times D_{R_1}$. Moreover, if we take $d>0$ sufficiently large, by Lemma 3.4, (4.5) and Lemma 4.7 we can see that

$$\begin{gathered} \|h_{i,\alpha}(t)\|_{R_1} \le H_{i,\alpha}\phi_{\gamma_{i,\alpha}}(t;c+d) \quad \text{on } S_I \quad (i+|\alpha|\le m), \\ \|c_\nu(t)\|_{R_1} \le C_\nu\phi_{\gamma_\nu}(t;c+d) \quad \text{on } S_I \quad (|\nu|\ge 2), \\ \sum_{|\nu|\ge 2} C_\nu t^{\gamma_\nu} X^{|\nu|} \in \mathbb{C}\{t,X\} \end{gathered}$$

hold for some $H_{i,\alpha}>0$ $(i+|\alpha|\le m)$ and $C_\nu>0$ $(|\nu|\ge 2)$, where

$$\begin{aligned} \gamma_{i,\alpha} &= \min\bigl\{q_\lambda + k\langle\lambda\rangle_l - k[i+|\alpha|-l]_+ + \mu(|\lambda|-1)\,; \\ &\qquad\qquad b_\lambda(t,x)\not\equiv 0, |\lambda|\ge 2,\ \lambda_{i,\alpha}>0\bigr\}, \\ \gamma_\nu &= \min\bigl\{q_\lambda + k\langle\lambda-\nu\rangle_l + \mu(|\lambda|-|\nu|)\,; b_\lambda(t,x)\not\equiv 0,\ \lambda\succcurlyeq\nu\bigr\}. \end{aligned}$$

Under these situation, we set

$$\Delta_h = \{(i,\alpha)\in\mathbb{N}\times\mathbb{N}^K\,;\, l+1\le i+|\alpha|\le m, h_{i,\alpha}(t,x)\not\equiv 0\},$$

$$\Delta_c = \{\nu \in \mathbb{N}^N ; |\nu| \ge 2, m_\nu \ge l+1, c_\nu(t,x) \not\equiv 0\},$$
$$s_h = 1 + \max\left[0, \max_{(i,\alpha)\in\Delta_h}\left(\frac{i+|\alpha|-l}{\gamma_{i,\alpha}+k(i+|\alpha|-l)}\right)\right],$$
$$s_c = 1 + \max\left[0, \max_{\nu\in\Delta_c}\left(\frac{m_\nu - l}{\gamma_\nu + k\langle\nu\rangle_l + \mu_1(|\nu|-1)}\right)\right].$$

Then, we have the following lemma.

Lemma 4.9. *As before, we set $s_0 = \max\{s_a, s_b\}$. If μ_1 is sufficiently large, we have $s_0 \ge \max\{s_a, s_h, s_c\}$.*

Proof. If $s_0 = 1$, we may assume that $m \le l$. In this case we have $s_h = 1$ and $s_c = 1$, and so we have the result.

Let us show the case $s_0 > 1$. By the definition, we have $s_0 \ge s_a$. For any $(i,\alpha) \in \Delta_h$ we have $\gamma_{i,\alpha} = q_\lambda + k\langle\lambda\rangle_l - k(i+|\alpha|-l) + \mu(|\lambda|-1)$ for some $\lambda \in \Delta_b$, and so

$$\begin{aligned}\frac{i+|\alpha|-l}{\gamma_{i,\alpha}+k(i+|\alpha|-l)} &= \frac{i+|\alpha|-l}{q_\lambda + k\langle\lambda\rangle_l + \mu(|\lambda|-1)}\\ &\le \frac{m_\lambda - l}{q_\lambda + k\langle\lambda\rangle_l + \mu(|\lambda|-1)} \le s_b - 1 \le s_0 - 1.\end{aligned}$$

This shows that $s_h \le s_0$ holds. Moreover, for any $\nu \in \Delta_c$ we have

$$\frac{m_\nu - l}{\gamma_\nu + k\langle\nu\rangle_l + \mu_1(|\nu|-1)} \le \frac{m-l}{\mu_1} < s_0 - 1$$

if $\mu_1 > (m-l)/(s_0-1)$ holds. Therefore, if $\mu_1 > 0$ is sufficiently large, we have $s_c \le s_0$. This proves Lemma 4.9. □

Thus, by (2) of Lemma 4.2 and by the same argument as in Subsections 3.1–3.4 we can show the following result:

Proposition 4.10. *Under the above situation, equation (4.7) has a solution $w(t,x) \in \mathscr{X}(S_I \times D_R)$ for some $R > 0$ which satisfies $w(t,x) = 0$ on $S_I(r) \times D_R$ and*

$$|w(t,x)| \le \frac{M}{(|t|^k+1)^l}\,|t|^{\mu_1 - k}\exp(b|t|^\kappa) \quad \text{on } S_I \times D_R$$

for some $M > 0$ and $b > 0$.

By setting $u^*(t,x) = u_{\rm ext}(t,x) + w(t,x)$ we have an extension of $u(t,x)$. This proves Proposition 4.3. □

4.3. PROOF OF PROPOSITION 4.4

Let $u_1(t,x) \in \mathscr{X}(L_\theta(r) \times D_R)$ and $u_2(t,x) \in \mathscr{X}(L_\theta(r) \times D_R)$ be two solutions of equation (2.1) on $L_\theta(r) \times D_R$ for some $\theta \in I$ and $r > r_0$ satisfying the estimates $|u_i(t,x)| \le M_0|t|^{\mu-k}$ on $L_\theta(r) \times D_R$ $(i = 1,2)$ for some $M_0 > 0$, and suppose that $u_1(t,x) = u_2(t,x)$ holds on $L_\theta(r_1) \times D_R$ for some r_1 with $r_0 \le r_1 < r$.

We set $u(t,x) = u_1(t,x) - u_2(t,x)$: as we have already seen in Subsection 3.5, $u(t,x)$ satisfies a linear convolution partial differential equation (similar to (3.19)):

$$P(kt^k, x)u = \sum_{i+|\alpha|\le m} \gamma_{i,\alpha}(t,x) *_k \left(\mathscr{M}_{i,\alpha}[\partial_x^\alpha u]\right)$$

for some suitable $\gamma_{i,\alpha}(t,x) \in \mathscr{X}(L_\theta(r) \times D_R^\circ)$. Since $u(t,x) = 0$ holds on $L_\theta(r_1) \times D_R$, we can use (2) of Lemma 4.2. Thus, by the same argument as in the proof of Theorem 3.11 we can show that $u(t,x) = 0$ on $L_\theta(r) \times D_{R_1}$ for any $0 < R_1 < R$. This proves Proposition 4.4. □

4.4. NEW CONVOLUTION ON $S_I(R] \cup \Delta_{T_0}(R)$

Let $r > 0$, $\theta \in I$ and $t_0 = re^{i\theta}$. We denote by $H(S_I(r] \cup \Delta_{t_0}(r))$ the set of all functions $f(t)$ which are continuous on $S_I(r] \cup \Delta_{t_0}(r)$ and holomorphic in $S_I(r) \cup \Delta_{t_0}(r)$. In order to prove the analytic continuation in a local region, Braaksma [3] and Ouchi [9] have used a new convolution $(f \tilde{*}_k g)(t)$ of two functions $f(t)$ and $g(t)$ in $H(S_I(r] \cup \Delta_{t_0}(r))$. Let us recall its definition.

The difficulty lies in the fact that the usual convolution $(f *_k g)(t)$ is well-defined for $t \in S_I(r] \cup L_\theta(2^{1/k}r)$ but not for $t \in \Delta_{t_0}(r) \setminus L_\theta(2^{1/k}r)$. The situation is as follows. For $t \in L_\theta(2^{1/k}r)$ with $|t| > r$, the convolution $(f *_k g)(t)$ is given by

$$(f *_k g)(t) = \int_0^{t_0} f(\tau)g((t^k - \tau^k)^{1/k})d\tau^k + \int_{t_0}^{t} f(\tau)g((t^k - \tau^k)^{1/k})d\tau^k.$$

If we set $x^k = t^k - \tau^k$ in the first term of the right-hand side, by the condition $(t^k - t_0^k)^{1/k} \in L_\theta(r)$ we have

$$\begin{aligned}\int_0^{t_0} f(\tau)g((t^k - \tau^k)^{1/k})d\tau^k &= \int_{(t^k - t_0^k)^{1/k}}^{t} f((t^k - x^k)^{1/k})g(x)dx^k \\ &= \int_{(t^k - t_0^k)^{1/k}}^{t_0} f((t^k - x^k)^{1/k})g(x)dx^k + \int_{t_0}^{t} f((t^k - x^k)^{1/k})g(x)dx^k.\end{aligned}$$

Therefore, for $t \in L_\theta(2^{1/k}r)$ with $|t| > r$, $(f *_k g)(t)$ is written in the form

$$\begin{aligned}(f *_k g)(t) &= \int_{(t^k - t_0^k)^{1/k}}^{t_0} f((t^k - \tau^k)^{1/k})g(\tau)d\tau^k \\ &\quad + \int_{t_0}^{t} f((t^k - \tau^k)^{1/k})g(\tau)d\tau^k + \int_{t_0}^{t} f(\tau)g((t^k - \tau^k)^{1/k})d\tau^k \qquad (4.8) \\ &= I_1 + I_2 + I_3.\end{aligned}$$

We write $L_\theta[t_0,t]=\{z\in L_\theta(2^{1/k}r)\,;\,|t_0|\le|z|\le|t|\}$, etc. Then, in the above integral formula (4.8) we see: in I_1 we have

$$(t^k-\tau^k)^{1/k}\in L_\theta[(t^k-t_0^k)^{1/k},t_0]\quad\text{and}\quad \tau\in L_\theta[(t^k-t_0^k)^{1/k},t_0],$$

in I_2 we have

$$(t^k-\tau^k)^{1/k}\in L_\theta[0,(t^k-t_0^k)^{1/k}]\quad\text{and}\quad \tau\in L_\theta[t_0,t],$$

and in I_3 we have $\tau\in L_\theta[t_0,t]$ and $(t^k-\tau^k)^{1/k}\in L_\theta[0,(t^k-t_0^k)^{1/k}]$.

If we consider the right-hand side of (4.8) for $t\in\Delta_{t_0}(r)$, the variables of the integrants move in the following way: in I_1, the variable of f moves like $t_0\longrightarrow(t^k-t_0)^{1/k}$ in $S_I(r)$ and the variable of g moves like $(t^k-t_0)^{1/k}\longrightarrow t_0$ in $S_I(r)$; in I_2, the variable of f moves like $(t^k-t_0)^{1/k}\longrightarrow 0$ in $S_I(r)$ and the variable of g moves like $t_0\longrightarrow t$ in $\Delta_{t_0}(r)$; in I_3, the variable of f moves like $t_0\longrightarrow t$ in $\Delta_{t_0}(r)$ and the variable of g moves like $(t^k-t_0)^{1/k}\longrightarrow 0$ in $S_I(r)$.

Thus, if we use the formula (4.8) as a new convolution of $f(t)$ and $g(t)$ for $t\in\Delta_{t_0}(r)$ we have a natural generalization of the convolution.

Definition 4.11. For $f(t)\in H(S_I(r]\cup\Delta_{t_0}(r))$ and $g(t)\in H(S_I(r]\cup\Delta_{t_0}(r))$, we define a new convolution $f\tilde{*}_kg$ on $S_I(r]\cup\Delta_{t_0}(r)$ in the following way: if $t\in S_I(r]\cup L_\theta(2^{1/k}r)$, we define the convolution $(f\tilde{*}_kg)(t)$ by the usual formula, and if $t\in\Delta_{t_0}(r)$, we define the convolution $(f\tilde{*}_kg)(t)$ by the right-hand side of (4.8).

In order to estimate the new convolution $(f\tilde{*}_kg)(t)$ on $S_I(r]\cup\Delta_{t_0}(r)$, the following function is very useful. We set

$$h(t)=(|t^k-t_0^k|+|t_0|^k)^{1/k},\quad t\in\Delta_{t_0}(r).$$

Lemma 4.12.

(1) *If $0<|I|<\pi/2k$ and $\theta\in I$ hold, we have*

$$2^{-1/k}h(t)\le|t|\le h(t),\quad t\in\Delta_{t_0}(r).$$

(2) *Let $f(t)\in H(S_I(r]\cup\Delta_{t_0}(r))$ and $g(t)\in H(S_I(r]\cup\Delta_{t_0}(r))$. Then we have $(f\tilde{*}_kg)(t)\in H(S_I(r]\cup\Delta_{t_0}(r))$. If $f(t)=0$ and $g(t)=0$ hold on $S_I(r)$, we have $(f\tilde{*}_kg)(t)=0$ on $S_I(r]\cup\Delta_{t_0}(r)$.*

(3) *Let $f(t)\in H(S_I(r]\cup\Delta_{t_0}(r))$ and $g(t)\in H(S_I(r]\cup\Delta_{t_0}(r))$. Suppose that $g(t)=0$ holds on $S_I(r)$ and that*

$$|f(t)|\le\frac{A}{\Gamma(\alpha/k)}|t|^{\alpha-k}\ \ \text{on } S_I(r],$$
$$|g(t)|\le\frac{B}{\Gamma(\beta/k)}h(t)^{\beta-k}\ \ \text{on } \Delta_{t_0}(r)$$

for some $A>0$, $B>0$, $\alpha>0$ and $\beta>0$: then we have $(f\tilde{*}_kg)(t)=0$ on $S_I(r]$ and

$$|(f\tilde{*}_kg)(t)|\le\frac{AB}{\Gamma((\alpha+\beta)/k)}h(t)^{\alpha+\beta-k}\quad\text{on } \Delta_{t_0}(r). \tag{4.9}$$

Proof. Let us show (1). Let $t \in \Delta_{t_0}(r)$. We have $|t| > r = |t_0|$. Since $t^k - t_0^k \in S_{kI}(r^k)$, we have $|t^k - t_0^k| < r^k$. Therefore, we see that

$$2|t|^k > r^k + r^k > |t_0|^k + |t^k - t_0^k| = h(t)^k.$$

This proves the first inequality. The second inequality comes from

$$|t|^k = |t^k| \leq |t^k - t_0^k| + |t_0|^k = h(t)^k.$$

The part (2) is clear from the definition. (3) is already proved in [9, Lemma 6.4], but for readers' convenience, we give here a proof of (4.9).

Since the condition $g(t) = 0$ on $S_I(r)$ is supposed, by (4.8) we have

$$(f \tilde{*}_k g)(t) = \int_{t_0}^{t} f((t^k - \tau^k)^{1/k}) g(\tau) d\tau^k \quad \text{on } \Delta_{t_0}(r).$$

Let $t \in \Delta_{t_0}(r)$. By setting $x = (\tau^k - t_0^k)^{1/k}$, by integrating in x from 0 to $(t^k - t_0^k)^{1/k}$ and by using the condition

$$h((t_0^k + x^k)^{1/k}) = (|(t_0^k + x^k) - t_0^k| + |t_0|^k)^{1/k} = (|x|^k + |t_0|^k)^{1/k}$$

we have

$$\begin{aligned}
&|(f \tilde{*}_k g)(t)| \\
&= \left| \int_0^{(t^k - t_0^k)^{1/k}} f((t^k - t_0^k - x^k)^{1/k}) g((t_0^k + x^k)^{1/k}) dx^k \right| \\
&\leq \frac{A}{\Gamma(\alpha/k)} \frac{B}{\Gamma(\beta/k)} \int_0^{|t^k - t_0^k|^{1/k}} (|t^k - t_0^k| - \rho^k)^{\alpha/k - 1} (\rho^k + |t_0|^k)^{\beta/k - 1} d\rho^k.
\end{aligned}$$

In addition, by setting $y = \rho^k + |t_0|^k$ and then by setting $y = (|t^k - t_0^k| + |t_0|^k)\eta$ we have

$$\begin{aligned}
&\int_0^{|t^k - t_0^k|^{1/k}} (|t^k - t_0^k| - \rho^k)^{\alpha/k - 1} (\rho^k + |t_0|^k)^{\beta/k - 1} d\rho^k \\
&\quad \leq \int_0^{|t^k - t_0^k| + |t_0|^k} (|t^k - t_0^k| + |t_0|^k - y)^{\alpha/k - 1} y^{\beta/k - 1} dy \\
&\quad = (|t^k - t_0^k| + |t_0|^k)^{\alpha/k + \beta/k - 1} \int_0^1 (1 - \eta)^{\alpha/k - 1} \eta^{\beta/k - 1} d\eta \\
&\quad = h(t)^{\alpha + \beta - k} B(\alpha/k, \beta/k) = h(t)^{\alpha + \beta - k} \frac{\Gamma(\alpha/k)\Gamma(\beta/k)}{\Gamma((\alpha + \beta)/k)}.
\end{aligned}$$

This proves (4.9). □

More generally, if we take $0 < r_1 < r$, we can define the new convolution $(f \tilde{*}_k g)(t)$ on $S_I(r) \cup \Delta_{t_0}(r_1)$ in the same way, and we have the same results as in Lemma 4.12. In addition, we have the following lemma.

Lemma 4.13. *Let $r > 0$, $\theta \in I$, $t_0 = re^{i\theta}$ and $0 < r_1 < r$. Let $f(t) \in H(S_I(r] \cup \Delta_{t_0}(r_1))$ and $g(t) \in H(S_I(r] \cup \Delta_{t_0}(r_1))$. If*

$$|f(t)| \le \frac{A}{\Gamma(\alpha/k)}|t|^{\alpha-k} \ \text{ on } S_I(r], \quad |f(t)| \le \frac{A}{\Gamma(\alpha/k)}h(t)^{\alpha-k} \ \text{ on } \Delta_{t_0}(r_1),$$
$$|g(t)| \le \frac{B}{\Gamma(\beta/k)}|t|^{\beta-k} \ \text{ on } S_I(r], \quad |g(t)| \le \frac{B}{\Gamma(\beta/k)}h(t)^{\beta-k} \ \text{ on } \Delta_{t_0}(r_1)$$

hold for some $A > 0$, $B > 0$, $\alpha > 0$ and $\beta > 0$, then we see that $(f \tilde{*}_k g)(t) \in H(S_I(r] \cup \Delta_{t_0}(r_1))$ and satisfies the estimate

$$|(f \tilde{*}_k g)(t)| \le \begin{cases} \dfrac{AB}{\Gamma((\alpha+\beta)/k)}|t|^{\alpha+\beta-k} & \text{on } S_I(r], \\ \left(2 + \dfrac{r^k + r_1^k}{r^k - r_1^k}\right)\dfrac{AB}{\Gamma((\alpha+\beta)/k)}h(t)^{\alpha+\beta-k} & \text{on } \Delta_{t_0}(r_1). \end{cases} \tag{4.10}$$

Proof. In the case $t \in S_I(r]$, the new convolution is the same as the usual convolution, and so the first inequality of (4.10) follows from Lemma 3.4. Let us show the second inequality of (4.10).

Take any $t \in \Delta_{t_0}(r_1)$ and fix it. We have $|t_0|^k = r^k > r_1^k > |t^k - t_0^k|$, $|t_0|^k - |t^k - t_0^k| > r^k - r_1^k > 0$ and $h(t) \le (r^k + r_1^k)^{1/k}$. By the definition we have

$$\begin{aligned}(f *_k g)(t) &= \int_{(t^k - t_0^k)^{1/k}}^{t_0} f((t^k - \tau^k)^{1/k})g(\tau)d\tau^k \\ &\quad + \int_{t_0}^{t} f((t^k - \tau^k)^{1/k})g(\tau)d\tau^k + \int_{t_0}^{t} f(\tau)g((t^k - \tau^k)^{1/k})d\tau^k \\ &= I_1 + I_2 + I_3.\end{aligned}$$

The parts I_2 and I_3 are estimated in the same way as (4.9) and we have

$$|I_i| \le \frac{AB}{\Gamma((\alpha+\beta)/k)}h(t)^{\alpha+\beta-k} \quad \text{on } \Delta_{t_0}(r_1), \quad i = 2, 3. \tag{4.11}$$

Let us estimate I_1. We take the integration route so that $\tau = (\rho t_0^k + (1-\rho)(t^k - t_0^k))^{1/k}$ with $\rho : 0 \longrightarrow 1$. Then, we have $(t^k - \tau^k)^{1/k} \in S_I(r]$ and $\tau \in S_I(r]$, and

$$|f(t^k - \tau^k)^{1/k})| \le \frac{A|t^k - \tau^k|^{\alpha/k-1}}{\Gamma(\alpha/k)} \le \frac{A}{\Gamma(\alpha/k)}\big((1-\rho)|t_0|^k + \rho|t^k - \tau^k|\big)^{\alpha/k-1},$$
$$|g(\tau)| \le \frac{B|\tau^k|^{\beta/k-1}}{\Gamma(\beta/k)} \le \frac{B}{\Gamma(\beta/k)}\big(\rho|t_0|^k + (1-\rho)|t^k - \tau^k|\big)^{\beta/k-1}.$$

Since $d\tau^k = (t_0^k - (t^k - t_0^k))d\rho$, we have

$$|d\tau^k| = |t_0^k - (t^k - t_0^k)|d\rho \le (|t_0|^k + |t^k - t_0^k|)d\rho \le (r^k + r_1^k)d\rho.$$

Therefore, we have

$$|I_1| \le \frac{AB(r^k + r_1^k)}{\Gamma(\alpha/k)\Gamma(\beta/k)} \int_0^1 \big((1-\rho)|t_0|^k + \rho|t^k - t_0^k|\big)^{\alpha/k-1} \times \big(\rho|t_0|^k + (1-\rho)|t^k - t_0^k|\big)^{\beta/k-1} d\rho.$$

Here, we set $y = \rho|t_0|^k + (1-\rho)|t^k - t_0^k|$. Then we have

$$dy = (|t_0|^k - |t^k - t_0^k|)d\rho \ge (r^k - r_1^k)d\rho,$$

and so

$$\begin{aligned}
|I_1| &\le \frac{AB(r^k + r_1^k)}{\Gamma(\alpha/k)\Gamma(\beta/k)} \int_{|t^k - t_0^k|}^{|t_0|^k} (|t^k - t_0^k| + |t_0|^k - y)^{\alpha/k-1} y^{\beta/k-1} \frac{dy}{(r^k - r_1^k)} \\
&\le \frac{AB(r^k + r_1^k)}{\Gamma(\alpha/k)\Gamma(\beta/k)} \int_0^{|t^k - t_0^k| + |t_0|^k} (|t^k - t_0^k| + |t_0|^k - y)^{\alpha/k-1} y^{\beta/k-1} \frac{dy}{(r^k - r_1^k)} \\
&= \frac{AB(r^k + r_1^k)}{(r^k - r_1^k)\Gamma(\alpha/k)\Gamma(\beta/k)} (|t^k - t_0^k| + |t_0|^k)^{\alpha/k+\beta/k-1} B(\alpha/k, \beta/k) \\
&= \frac{(r^k + r_1^k)}{(r^k - r_1^k)} \frac{AB}{\Gamma((\alpha+\beta)/k)} h(t)^{\alpha+\beta-k}.
\end{aligned} \tag{4.12}$$

By (4.11) and (4.12), we have the second inequality of (4.10). □

Thus, in the case $0 < r_1 < r$ (being fixed), by setting $C_1 = 2 + (r^k + r_1^k)/(r^k - r_1^k)$ and

$$\psi_a(t) = \begin{cases} \dfrac{1}{C_1} \dfrac{|t|^{a-k}}{\Gamma(a/k)} & \text{on } S_I(r], \\[2ex] \dfrac{1}{C_1} \dfrac{h(t)^{a-k}}{\Gamma(a/k)} & \text{on } \Delta_{t_0}(r_1) \end{cases}$$

for $a > 0$, we have the following result.

Corollary 4.14. *Let $r > 0$, $\theta \in I$, $t_0 = re^{i\theta}$ and $0 < r_1 < r$. Let $f(t) \in H(S_I(r] \cup \Delta_{t_0}(r_1))$ and $g(t) \in H(S_I(r] \cup \Delta_{t_0}(r_1))$. If $|f(t)| \le A\psi_a(t)$ and $|g(t)| \le B\psi_b(t)$ hold on $S_I(r] \cup \Delta_{t_0}(r_1)$ for $A > 0$, $a > 0$, $B > 0$ and $b > 0$, then we have $|(f \tilde{*}_k g)(t)| \le AB\psi_{a+b}(t)$ on $S_I(r] \cup \Delta_{t_0}(r_1)$.*

In equation (2.1), estimates in the assumptions are given in the form

$$\|a(t)\|_\rho \leq \frac{A}{\Gamma(n/k)}|t|^{n-k}\exp(c|t|^k) \quad \text{on } S_I.$$

By applying (1) of Lemma 4.12 to this estimate we have $\|a(t)\|_\rho \leq A_1\psi_n(t)$ on $S_I(r] \cup \Delta_{t_0}(r_1)$ with $A_1 = C_1 H A \exp(2cr^k)$ and $H = \max\{1, 2^{1-n/k}\}$. Conversely, the estimate $\|a(t)\|_\rho \leq A_1\psi_n(t)$ on $S_I(r] \cup \Delta_{t_0}(r_1)$ implies the estimate $\|a(t)\|_\rho \leq (H/C_1)A_1|t|^{n-k}/\Gamma(n/k)$ on $S_I(r] \cup \Delta_{t_0}(r_1)$.

Thus, we see that almost all the arguments in the usual case work also for the new convolution if we use $\psi_n(t)$ $(n = 1, 2, \ldots)$ instead of $\phi_n(t; c)$ $(n = 1, 2, \ldots)$.

4.5. PROOF OF PROPOSITION 4.5

Let $r > 0$ and $R > 0$. We denote by $H((S_I(r] \cup \Delta_{t_0}(r)) \times D_R)$ the set of all functions $f(t,x)$ belonging to $\mathscr{X}((S_I(r] \cup \Delta_{t_0}(r)) \times D_R)$ that are holomorphic in $(S_I(r) \cup \Delta_{t_0}(r)) \times D_R^\circ$.

For two functions $f(t,x)$ and $g(t,x)$ in $H((S_I(r] \cup \Delta_{t_0}(r)) \times D_R)$, we define the new convolution $(f \tilde{*}_k g)(t,x)$ with respect to t in the same way as in Definition 4.11, regarding x as a parameter.

In this section, we will prove Proposition 4.5 by considering the following new-convolution equation

$$\begin{aligned} P(kt^k, x)u = f(t,x) &+ \sum_{i+|\alpha| \leq m} a_{i,\alpha}(t,x) \tilde{*}_k (\mathscr{M}_{i,\alpha}[\partial_x^\alpha u]) \\ &+ \sum_{|\nu| \geq 2} b_\nu(t,x) \tilde{*}_k \prod_{i+|\alpha| \leq m}^{\tilde{*}_k} \left(\mathscr{M}_{i,\alpha}[\partial_x^\alpha u]\right)^{\tilde{*}_k \nu_{i,\alpha}} \end{aligned} \tag{4.13}$$

on $(S_I(r] \cup \Delta_{t_0}(r)) \times D_R$. We note that this is the same as (2.1) on $(S_I(r] \cup L_\theta(2^{1/k}r)) \times D_R$, but on $\Delta_{t_0}(r) \times D_R$ we are using the new convolution $\tilde{*}_k$.

Proof of Proposition 4.5. Let $u(t,x) \in \mathscr{X}((S_I(r] \cup L_\theta(2^{1/k}r)) \times D_R)$ be a solution of equation (2.1) on $(S_I(r] \cup L_\theta(2^{1/k}r)) \times D_R$ for some $\theta \in I$ and $r > r_0$ which is holomorphic on $S_I(r) \times D_R^\circ$, and suppose that $|u(t,x)| \leq M_0|t|^{\mu-k}$ holds on $S_I(r] \times D_R$ for some $M_0 > 0$.

We set $t_0 = re^{i\theta}$, take any $r_1 > 0$ (with $r_0 \leq r_1 < r$), and set

$$u_\theta(t,x) = \begin{cases} u(t,x), & \text{if } (t,x) \in S_I(r] \times D_R, \\ u(t_0,x), & \text{if } (t,x) \in \Delta_{t_0}(r_1) \times D_R. \end{cases}$$

We have $u_\theta(t,x) \in H((S_I(r] \cup \Delta_{t_0}(r_1)) \times D_R)$ and $u_\theta(t,x) = u(t,x)$ on $S_I(r] \times D_R$.

Take an $R_1 > 0$ sufficiently small. Let us look for an extension $u^*(t,x)$ on $(S_I(r] \cup \Delta_{t_0}(r_1)) \times D_{R_1}$ as a solution of equation (4.13) in the form:

$$u^*(t,x) = u_\theta(t,x) + w(t,x), \quad w(t,x) = 0 \text{ on } S_I(r) \times D_{R_1}.$$

Then, by the same calculation as in (4.6) and (4.7), equation (4.13) is reduced to the following new-convolution equation with respect to the unknown function w:

$$\begin{aligned} P(kt^k,x)w = g(t,x) &+ \sum_{i+|\alpha|\le m} (a_{i,\alpha}(t,x)+h_{i,\alpha}(t,x))\tilde{*}_k\big(\mathscr{M}_{i,\alpha}[\partial_x^\alpha w]\big) \\ &+ \sum_{|\nu|\ge 2} c_\nu(t,x)\tilde{*}_k \prod_{i+|\alpha|\le m}^{\tilde{*}_k} \big(\mathscr{M}_{i,\alpha}[\partial_x^\alpha w]\big)^{\tilde{*}_k\nu_{i,\alpha}} \end{aligned} \tag{4.14}$$

on $(S_I(r]\cup\Delta_{t_0}(r_1))\times D_{R_1}$, where $g(t,x)\in H((S_I(r]\cup\Delta_{t_0}(r_1))\times D_{R_1})$ with $g(t,x)=0$ on $S_I(r)\times D_{R_1}$, and $h_{i,\alpha}(t,x)$ $(i+|\alpha|\le m)$ and $c_\nu(t,x)$ $(|\nu|\ge 2)$ are given by the same formulas as in (4.7) (with $u_{\rm ext}(t,x)$ replaced by $u_\theta(t,x)$).

Since $w(t,x)=0$ on $S_I(r)\times D_{R_1}$ is supposed, by (2) of Lemma 4.12 we have $(w\tilde{*}_k w)(t)=0$ on $(S_I(r]\cup\Delta_{t_0}(r_1))\times D_{R_1}$. Hence, under the condition $w(t,x)=0$ on $S_I(r)\times D_{R_1}$, equation (4.14) is reduced to the linear equation

$$P(kt^k,x)w = g(t,x) + \sum_{i+|\alpha|\le m} (a_{i,\alpha}(t,x)+h_{i,\alpha}(t,x))\tilde{*}_k\big(\mathscr{M}_{i,\alpha}[\partial_x^\alpha w]\big). \tag{4.15}$$

This equation is much easier than (4.14).

Thus, by the same argument as in the proof of Proposition 4.10 and by using Corollary 4.14 we have the following proposition.

Proposition 4.15. *Under the above situation, equation* (4.15) *has a solution* $w(t,x)\in H((S_I(r]\cup\Delta_{t_0}(r_1))\times D_\rho)$ *for some* $\rho>0$ $(0<\rho<R_1)$ *which satisfies* $w(t,x)=0$ *on* $S_I(r)\times D_\rho$, *and*

$$\|w(t)\|_\rho \le M\psi_{\mu_1}(t) \quad \text{on } S_I(r]\cup\Delta_{t_0}(r_1)$$

for some $\mu_1>0$ *and* $M>0$.

Now, let us complete the proof of Proposition 4.5. We set

$$u^*(t,x)=u_\theta(t,x)+w(t,x) \quad \text{on } (S_I(r]\cup\Delta_{t_0}(r_1))\times D_\rho.$$

Then, we see that $u^*(t,x)\in H((S_I(r]\cup\Delta_{t_0}(r_1))\times D_\rho)$, $u^*(t,x)=u(t,x)$ on $S_I(r)\times D_\rho$, and $u^*(t,x)$ is a solution of equation (4.13). Thus, to complete the proof of Proposition 4.5 it is enough to show that $u^*(t,x)=u(t,x)$ holds on $L_\theta((r^k+r_1^k)^{1/k})\times D_\rho$. This is verified as follows.

Since (4.13) is the same as (2.1) on $L_\theta((r^k+r_1^k)^{1/k})\times D_\rho$, two functions $u^*(t,x)$ and $u(t,x)$ are solutions of (2.1) on $L_\theta((r^k+r_1^k)^{1/k})\times D_\rho$ and they satisfy $u^*(t,x)=u(t,x)$ on $L_\theta(r)\times D_\rho$. Hence, by Proposition 4.4 we have $u^*(t,x)=u(t,x)$ on $L_\theta((r^k+r_1^k)^{1/k})\times D_\rho$. □

4.6. PROOF OF LEMMA 4.6

Let S be an open subset of $\mathbb{C}_t$, and let $u(t,x)\in\mathscr{X}(S\times D_R)$. Suppose that $u(t,x)$ is holomorphic on $S\times D_\rho^\circ$ for some $0<\rho<R$. Let us show that $u(t,x)$ is holomorphic on $S\times D_R^\circ$.

Since $u(t,x) \in \mathscr{X}(S \times D_R)$ is supposed, $u(t,x)$ is holomorphic with respect to $x \in D_R^\circ$, and by Taylor expansion in x we have the expression

$$u(t,x) = \sum_{|\alpha| \geq 0} u_\alpha(t) x^\alpha, \quad t \in S. \tag{4.16}$$

Since $u(t,x)$ is holomorphic on $S \times D_\rho^\circ$, we can regard this as Taylor expansion of the holomorphic function on $S \times D_\rho^\circ$ and we have the condition that $u_\alpha(t)$ ($|\alpha| \geq 0$) are holomorphic functions on S.

Take any $K \Subset S$ and $0 < R_1 < R$; we have $|u(t,x)| \leq M$ on $K \times D_{R_1}$ for some $M > 0$ and by Cauchy's inequality we have the estimates $|u_\alpha(t)| \leq M/R_1^{|\alpha|}$ on K. Then, the series (4.16) is uniformly convergent on any compact subset of $K \times D_{R_1}^\circ$. Since $u_\alpha(t)$ ($|\alpha| \geq 0$) are holomorphic functions on S, this shows that $u(t,x)$ is holomorphic on $K^\circ \times D_{R_1}^\circ$.

Since K and R_1 are taken arbitrarily, we have the result that $u(t,x)$ is holomorphic on $S \times D_R^\circ$. □

5. A GENERALIZATION

In the previous sections, we have proved Theorem 2.2 under the condition that $k > 0$, $\mu > 0$, $p_{i,\alpha} > 0$ ($i + |\alpha| \leq m$) and $q_\nu > 0$ ($|\nu| \geq 2$) are integers. In this section we will generalize Theorem 2.2 to the case where $k > 0$, $\mu > 0$, $p_{i,\alpha} > 0$ ($i + |\alpha| \leq m$) and $q_\nu > 0$ ($|\nu| \geq 2$) are not necessarily integers.

As before, we consider the equation

$$\begin{aligned} P(kt^k, x)u = f(t,x) &+ \sum_{i+|\alpha| \leq m} a_{i,\alpha}(t,x) *_k (\mathscr{M}_{i,\alpha}[\partial_x^\alpha u]) \\ &+ \sum_{|\nu| \geq 2} b_\nu(t,x) *_k \prod_{i+|\alpha| \leq m}^{*_k} \left(\mathscr{M}_{i,\alpha}[\partial_x^\alpha u]\right)^{*_k \nu_{i,\alpha}}. \end{aligned} \tag{5.1}$$

We suppose the conditions (A_2), (A_3), (A_4) and the following:

(A_1^*) $k > 0$ is a real number, and $0 < |I| < 2\pi/k$;

(A_5^*) there are positive numbers $\mu > 0$, $p_{i,\alpha} > 0$ ($i + |\alpha| \leq m$) and $q_\nu > 0$ ($|\nu| \geq 2$) such that the estimates

$$|f(t,x)| \leq \frac{F}{\Gamma(\mu/k)} |t|^{\mu - k} \exp(c|t|^k) \ \text{ on } S_I \times D_{R_0},$$

$$|a_{i,\alpha}(t,x)| \leq \frac{A_{i,\alpha}}{\Gamma(p_{i,\alpha}/k)} |t|^{p_{i,\alpha} - k} \exp(c|t|^k) \ \text{ on } S_I \times D_{R_0} \ \ (i + |\alpha| \leq m),$$

$$|b_\nu(t,x)| \leq \frac{B_\nu}{\Gamma(q_\nu/k)} |t|^{q_\nu - k} \exp(c|t|^k) \ \text{ on } S_I \times D_{R_0} \ \ (|\nu| \geq 2)$$

hold for some $c > 0$, $F \geq 0$, $A_{i,\alpha} \geq 0$ ($i + |\alpha| \leq m$) and $B_\nu \geq 0$ ($|\nu| \geq 2$);

(A_6^*) moreover, there is a $d > 0$ such that $k/d \in \mathbb{N}$, $\mu/d \in \mathbb{N}$, $p_{i,\alpha}/d \in \mathbb{N}$ ($i + |\alpha| \leq m$), $q_\nu/d \in \mathbb{N}$ ($|\nu| \geq 2$), and that the sum

$$\sum_{|\nu|\geq 2} B_\nu t^{q_\nu/d} X^{|\nu|}$$

is convergent in a neighborhood of $(t, X) = (0, 0) \in \mathbb{C}^2$.

Under these assumptions, we define s_a, s_b, s_0 and $\kappa > 0$ by the same formulas as in Section 2. Then, we obtain the following result.

Theorem 5.1. *Suppose the conditions* (A_1^*), (A_2), (A_3), (A_4), (A_5^*) *and* (A_6^*). *Let* $\lambda_1(x), \ldots, \lambda_l(x)$ *be the roots of* $P(\lambda, x) = 0$, *and assume that*

$$\lambda_i(0) = 0 \quad \text{or} \quad \lambda_i(0) \in \mathbb{C} \setminus \overline{\pi(S_{kI})} \quad \text{for } i = 1, 2, \ldots, l.$$

If $u(t, x)$ *is a holomorphic solution of equation* (5.1) *on* $S_I(\delta) \times D_{R_0}$ *for some* $\delta > 0$, *and if it satisfies* $|u(t, x)| \leq M_0 |t|^{\mu-k}$ *on* $S_I(\delta) \times D_{R_0}$ *for some* $M_0 > 0$, *then* $u(t, x)$ *has an analytic continuation* $u^*(t, x)$ *on* $S_I \times D_R$ *for some* $0 < R < R_0$ *such that*

$$|u^*(t, x)| \leq \frac{M}{(|t|^k + 1)^l} |t|^{\mu-k} \exp(b|t|^\kappa) \quad \text{on } S_I \times D_R$$

holds for some $M > 0$ *and* $b > 0$.

As is seen in the proof of Theorem 2.2, the essential part of the proof lies in the construction of a solution on $S_I \times D_R$. In the present case, the following is the key proposition:

Proposition 5.2. *Suppose the condition*

$$\lambda_1(0), \ldots, \lambda_l(0) \in \mathbb{C} \setminus \overline{\pi(S_{kI})}. \tag{5.2}$$

Then, equation (5.1) *has a formal solution*

$$u(t, x) = \sum_{n \geq \mu/d} u_n(t, x)$$

(*we note that* μ/d *is a positive integer*) *which satisfies the following properties:*

(1) $u_n(t, x)$ ($n \geq \mu/d$) *are holomorphic functions on* $S_I \times D_\rho$ *for some* $\rho > 0$;

(2) *there are* $C > 0$ *and* $h > 0$ *such that*

$$|u_n(t, x)| \leq \frac{Ch^n}{(|t|^k + 1)^l} \frac{n!^{d(s-1)}}{\Gamma(dn/k)} |t|^{dn-k} \exp(c|t|^k) \ \text{ on } S_I \times D_\rho$$

holds for any $n \geq \mu/d$ *and* $s \geq \max\{s_a, s_b\}$.

We note that if we set

$$k_1 = k/d, \ \ \mu_1 = \mu/d, \ \ p_{i,\alpha,1} = p_{i,\alpha}/d \ \text{ and } \ q_{\nu,1} = q_\nu/d$$

these k_1, μ_1, $p_{i,\alpha,1}$ $(i+|\alpha|\le m)$ and $q_{\nu,1}$ $(|\nu|\ge 2)$ are positive integers. We set

$$s_{a,1}=1+\max\Big[0,\ \max_{(i,\alpha)\in\Delta_a}\Big(\frac{i+|\alpha|-l}{p_{i,\alpha,1}+k_1(i+|\alpha|-l)}\Big)\Big],$$
$$s_{b,1}=1+\max\Big[0,\ \max_{\nu\in\Delta_b}\Big(\frac{m_\nu-l}{q_{\nu,1}+k_1\langle\nu\rangle_l+\mu_1(|\nu|-1)}\Big)\Big].$$

Then, we have $s_{a,1}-1=d(s_a-1)$ and $s_{b,1}-1=d(s_b-1)$. Therefore, Proposition 5.2 is written as follows.

Proposition 5.3. *Suppose the condition* (5.2)*. Then, equation* (5.1) *has a formal solution*

$$u(t,x)=\sum_{n\ge\mu_1}u_n(t,x) \tag{5.3}$$

which satisfies the following properties:

(1) $u_n(t,x)$ $(n\ge\mu_1)$ *are holomorphic functions on* $S_I(r)\times D_\rho$ *for some* $\rho>0$*;*

(2) *there are* $C>0$ *and* $h>0$ *such that*

$$|u_n(t,x)|\le\frac{Ch^n}{(|t|^k+1)^l}\frac{n!^{s_1-1}}{\Gamma(dn/k)}|t|^{dn-k}\exp(c|t|^k)\ \ \text{on } S_I(r)\times D_\rho$$

holds for any $n\ge\mu_1$ *and* $s_1\ge\max\{s_{a,1},s_{b,1}\}$*.*

Proof. The formal solution (5.3) is determined by a solution of the following recurrent formulas:

$$P(kt^k,x)u_{\mu_1}=f(t,x),$$

and for $n\ge\mu_1+1$

$$P(kt^k,x)u_n=\sum_{i+|\alpha|\le m}a_{i,\alpha}(t,x)*_k\big(\mathscr{M}_{i,\alpha}[\partial_x^\alpha u_{n-p_{i,\alpha,1}-k_1[i+|\alpha|-l]_+}]\big)$$
$$+\sum_{2\le|\nu|\le n-q_{\nu,1}}\ \sum_{\substack{q_{\nu,1}+|n(\nu)|\\+k_1\langle\nu\rangle_l=n}}b_\nu(t,x)*_k\prod_{i+|\alpha|\le m}^{*_k}\prod_{j=1}^{\nu_{i,\alpha}}{}^{*_k}\big(\mathscr{M}_{i,\alpha}[\partial_x^\alpha u_{n_{i,\alpha}(j)}]\big).$$

If we use the functions

$$\phi_n(t;c)=\frac{|t|^{dn-k}}{\Gamma(dn/k)}\exp(c|t|^k),\quad n=1,2,\dots,$$

the part (2) can be proved in the same way as in the proof of Proposition 3.3. □

Thus, by modifying the arguments in Sections 3 and 4 suitably we can give a proof of Theorem 5.1. We may omit the details.

Acknowledgments

The author is partially supported by the Grant-in-Aid for Scientific Research No. 22540206 of Japan Society for the Promotion of Science.

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Received: December 1, 2013.
Revised: December 12, 2014.
Accepted: January 23, 2015.