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SEMIHEREDITARY RINGS AND RELATED TOPICS

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Abstract. A ring A is called right (left) semihereditary if all finitely generated right (left) ideals of A are projective. In this paper we consider non-commutative semihereditary rings and show their connection with non-commutative valuation rings. We also present some criterion for a module to be flat.

1. Introduction

Historically semihereditary rings come from homological algebra and their definition was first appeared in [1].

Definition 1. [1] *A ring A is called right (left) semihereditary if all finitely generated right (left) ideals of A are projective.*

If a ring A is an integral domain, i.e. a commutative ring without divisors of zero, then semihereditary domains coincide with Prüfer domains. Prüfer domains were defined in 1932 by H. Prüfer, and since that time they play a central role in the development of the classical ring theory. Recall that an integral domain is called a Prüfer domain if all its finitely generated ideals are invertible. Since in the case of integral domains any ideal is invertible if and only if it is projective, we obtain that any Prüfer domain is exactly a semihereditary domain. Prüfer domains naturally arise from valuation rings of fields, since for any prime ideal P of a valuation ring A the localization A_P of A is a Prüfer domain. So semihereditary domains can be considered as a global theory for classical valuation rings.

In the non-commutative case there are different generalizations of valuation rings. If we consider invariant valuation rings of division rings which were introduced by Schilling in [6], then we obtain that any invariant valuation ring is a semihereditary ring in the sense of definition 1. So semihereditary rings can be considered as some generalizations of Prüfer domains for non-commutative rings. Another generalization of non-commutative valuation rings were introduced and studied by Dubrovin in [3]. These rings were named Dubrovin valuation rings after him. In this non-commutative valuation theory any Dubrovin valuation ring of a simple Artinian ring Q is exactly a local semihereditary order of Q. So semihereditary orders can be considered as the global theory for Dubrovin valuation rings. Dubrovin valuation rings found a large applications. More information about these rings and semihereditary orders in simple Artinian rings can be found in the book [5].

Semihereditary rings are also interesting from homological point of view, since they belong to the class of rings with weak global dimension ≤ 1.

All rings considered in this paper are assumed to be associative with $1 \neq 0$, and all modules are assumed to be unital. We write $U(A)$ for the group of units of a ring A, and D^* for the multiplicative group of a division ring D. We refer to [4] for general material on theory of rings and modules.

2. Semihereditary rings and valuation rings

For the case of non-commutative rings there are different generalizations for valuation rings. First consider the generalization which was proposed in 1945 by Schilling [6], who extended the concept of a valuation on a field to that on a division ring.

Definition 2. *Let G be a totally ordered group (written additively) with the order relation $\geq$. Add to G a special symbol ∞ such that $x + \infty = \infty + x = \infty$ for all $x \in G$. Let D be a division ring. A valuation on D is a surjective map $v : D \to G \cup \{\infty\}$ which satisfies the following relations*:

1) $v(0) = \infty$,

2) $v(xy) = v(x) + v(y)$;

3) $v(x + y) \geq \min(v(x), v(y))$, *whenever* $x + y \neq 0$,

for any $x, y \in D$.

Then $A = \{x \in D \ : \ v(x) \geq 0\}$ *is a ring which is called the (invariant) valuation ring of D with respect to valuation v, and* $U = \{u \in D^* \ : \ v(u) = 0\}$ *is called the group of valuation units.*

In the general case we obtain the following definition.

Definition 3. [6] *A subring A of a division ring D is called an invariant valuation ring of D if there is a totally ordered group G and a valuation $v : D \to G$ of D such that $A = \{x \in D \ : \ v(x) \geq 0\}$.*

The next proposition gives the basic properties of invariant valuation rings.

Proposition 1. *Let A be an invariant valuation ring of a division ring D with respect to a valuation v. Then*

1. *$aA \subseteq bA$ or $bA \subseteq aA$ for any $a, b \in A$.*
2. *Each ideal of A is two-sided.*
3. *A is a right and a left Ore domain. Therefore it has a left and right division ring of fractions.*
4. *Any finitely generated ideal of A is principal.*

As an immediately consequence of this proposition we obtain the following.

Proposition 2. *Any invariant valuation ring of a division ring D is semihereditary and Bézout ring.*[1]

The following theorem gives the equivalent definitions of a invariant valuation ring.

Theorem 1. *Let A be a ring with a division ring of fractions D which is invariant in D. Then the following statements are equivalent:*

1. *A is an invariant valuation ring of some valuation v on D.*
2. *For any element $x \in D^*$ either $x \in A$ or $x^{-1} \in A$.*
3. *The set of principal ideals of A is linearly ordered by inclusion.*
4. *A is a uniserial ring.*[2]

Definition 4. *A subring A of a division ring D is called a total valuation ring if for each $x \in D^*$ we have $x \in A$ or $x^{-1} \in A$.*

Theorem 1 states that any invariant valuation ring is a total valuation ring, but not conversely. Note that in the case of integral domains the notions of invariant valuation rings and total valuation rings are equivalent to the notion of a classical valuation ring of a field. Theorem 1 also states that any invariant totally valuation ring is uniserial. Warfield [7] showed the connection of total valuation rings with semihereditary rings in the case of local rings.

[1] Recall that a ring A is called a right Bézout ring if any its finitely generated ideal is principal.

[2] Recall that a ring A is called uniserial if all ideals of A are linearly ordered with respect to inclusion.

Theorem 2. *For a local ring A the following properties are equivalent:*

(i) *A is uniserial and semihereditary.*

(ii) *A is a total valuation ring.*

The next type of non-commutative valuation rings was introduced and studied by Dubrovin [3].

Definition 5. *Let S be a simple Artinian ring. A subring A of S, with Jacobson radical $J(A)$, is called a Dubrovin valuation ring if*

1) *$A/J(A)$ is a simple Artinian ring*;

2) *for each $s \in S \setminus A$ there are $a_1, a_2 \in A$ such that $sa_1 \in A \setminus J(A)$ and $a_2 s \in A \setminus J(A)$.*

Note that every Dubrovin valuation ring is a total valuation ring if and only if $A/J(A)$ is a division ring. Hence, if S is a field, then Dubrovin valuation rings of S are exactly the usual valuation rings. The class of Dubrovin valuation rings is much wider than the class of total valuation rings. The following theorem gives the basic characterizations of Dubrovin valuation rings.

Theorem 3. [5] *Let A be a subring of a simple Artinian ring Q. Then the following conditions are equivalent:*

(1) *A is a Dubrovin valuation ring of G.*

(2) *A is a local semihereditary order in Q.*

(3) *A is a local Bézout order in Q.*

3. Semihereditary rings and flat modules

While semisimple rings and hereditary rings are defined uniquely by their projective global dimension, for semihereditary rings we have the following statement which gives the equivalent characterization of semihereditary rings.

Theorem 4. [2] *Let A be a ring. The following conditions are equivalent:*

1. *A is a left semihereditary ring.*
2. w.gl.dim$A \leq 1$ *and A is a right coherent ring.*[3]
3. *Every torsion-less right A-module is flat.*

Note that semihereditary rings are not defined uniquely by the flatness property. There are examples of rings with weak global dimension ≤ 1 which are not semihereditary. Note also that for any ring A, w.gl.dim$A \leq 1$ if and only if every ideal of A is flat. The criteria for modules to be flat are very important. In this section we give one of such criteria.

[3]Recall that a ring A is called right coherent if the direct product of an arbitrary family of copies of A is flat as a right A-module.

Theorem 5. *Let $0 \to X \to P \to M \to 0$ be an exact sequence of right A-modules, where P is projective. Then the following statements are equivalent:*

(1) *M is a flat module.*

(2) *For any $x \in X$ there is a $\theta \in \mathrm{Hom}_A(P, X)$ with $\theta(x) = x$.*

(3) *For any $x_1, x_2, \dots, x_n \in X$ there is a $\theta \in \mathrm{Hom}_A(P, X)$ with $\theta(x_i) = x_i$ for all i.*

Proof.

(1) $\Longrightarrow$ (2). Let P be a projective module, and M a flat module. By the Kaplansky theorem (see e.g. theorem 5.5.1 [4]), P is projective if and only if there is a system of elements $\{p_i \in P \ : i \in I\}$ and a system of homomorphisms $\{\varphi_i\}$, $\varphi_i : P \to A$ such that any element $p \in P$ can be written in the form

$$p = \sum_i p_i(\varphi_i(p)),$$

where only a finite number of elements $\varphi_i(p) \in A$ are not equal to zero.

If $x \in X$, then $x = p_{i_1}a_1 + p_{i_2}a_2 + \ldots + p_{i_m}a_m$, where $a_i = \varphi_{i_1}(x) \in A$. Let $\mathcal{I} = Aa_1 + Aa_2 + \ldots + Aa_m$. Since M is flat, $x \in X \cap P\mathcal{I} = X\mathcal{I}$, by the flatness test (see e.g. proposition 5.4.11 [4]). Therefore $x = \sum x_j c_j$, where $x_j \in X$ and $c_j \in \mathcal{I}$. Now each $c_j = \sum\limits_i b_{ij}a_i$, so $x = \sum\limits_i x'_i a_i$, where $x'_i = \sum\limits_j x_j b_{ij}$. Define $\theta : P \to X$ by $\theta(p_{i_k}) = x'_k$, while θ sends all the other system elements p_i of P into 0. Then

$$\theta(x) = \theta(\sum_{k=1}^{m} p_{i_k}a_k) = \sum_{k=1}^{m}(\theta(p_{i_k})a_k) = \sum_{k=1}^{m} x'_k a_k = x.$$

(2) $\Longrightarrow$ (1). Let $x \in X \cap P\mathcal{I}$, where $\mathcal{I}$ is a left ideal in A. Then $x = p_{i_1}a_1 + p_{i_2}a_2 + \ldots + p_{i_r}a_r$, where $a_i \in A$. Define $\mathcal{I}_x = Aa_1 + Aa_2 + \ldots + Aa_r$, which is a finitely generated left ideal in A. It is clear that $\mathcal{I}_x \subseteq \mathcal{I}$, and so $x \in X\mathcal{I}_x \subset X\mathcal{I}$. Let $\theta \in \mathrm{Hom}_A(P, X)$ with $\theta(x) = x$. Then $x = \theta(p_{i_1})a_1 + \theta(p_{i_2})a_2 + \ldots + \theta(p_{i_r})a_r \in X\mathcal{I}_x$. Therefore $x \in X \cap P\mathcal{I} \subseteq X\mathcal{I}_x \subseteq X\mathcal{I}$. From the flatness test (see e.g. proposition 5.4.11 [4]) it follows that M is flat.

(2) $\Longrightarrow$ (3). This is proved by induction on n. Let $x_1, x_2, \dots, x_n \in X$. If $n = 1$, then the existence of θ follows from (2). Assume that $n > 1$ and (3) holds for all $k < n$. Let $\theta_n : P \to X$ be a homomorphism such that $\theta_n(x_n) = x_n$. Let $y_i = x_i - \theta_n(x_i)$ for $i = 1, 2, \dots, n-1$. By induction hypothesis, there exists a homomorphism θ' such that $\theta'(y_i) = y_i$ for $i = 1, 2, \dots, n-1$. Now define $\theta = \theta' + \theta_n - \theta'\theta_n \in \mathrm{Hom}_A(P, X)$. Then

$$\theta(x_n) = \theta'(x_n) + \theta_n(x_n) - \theta'\theta_n(x_n) = \theta'(x_n) + x_n - \theta' x_n = x_n,$$

$$\theta(x_i) = \theta'(x_i) + \theta_n(x_i) - \theta'\theta_n(x_i) = \theta'(x_i) + (x_i - y_i) - \theta'(x_i - y_i) = \\ = x_i - y_i + \theta'(y_i) = x_i$$

for $i = 1, 2, \dots, n-1$. So θ is a required homomorphism.

(3) $\Longrightarrow$ (2) follows by taking $n = 1$.

From this theorem it immediately follows the theorem which was first proved by Villamayor and was given by Chase in his paper [2].

Theorem 6. [2] *Let $0 \to X \to F \to P \to 0$ be an exact sequence of right A-modules, where F is free with a basis $\{e_i : i \in I\}$. Then the following statements are equivalent:*

(1) *P is a flat module.*

(2) *For any $x \in X$ there is a $\theta \in \mathrm{Hom}_A(F, X)$ with $\theta(x) = x$.*

(3) *For any $x_1, x_2, \dots, x_n \in X$ there is a $\theta \in \mathrm{Hom}_A(F, X)$ with $\theta(x_i) = x_i$ for all i.*

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