Jan Długosz University in Częstochowa
Scientific Issues, Mathematics XV, Częstochowa 2010

THE LOGIC DUAL TO SOBOCIŃSKI'S n–VALUED LOGIC

Anetta Górnicka

Institute of Mathematics and Computer Science
Jan Długosz University of Częstochowa
al. Armii Krajowej 13/15, 42-200 Częstochowa, Poland
e-mail: a.gornicka@ajd.czest.pl

Abstract. In this paper, we describe the logic dual to n–valued Sobociński logic. According to the idea presented by Malinowski and Spasowski [1], we introduce the consequence dual to the consequence of n–valued Sobociński logic in two ways: by a logical matrix and by a set of rules of inference. Then we prove that both approaches are equivalent and the consequence is dual in Wójcicki sense (see [3]).

1. Introduction

By a *language of a propositional logic* (*propositional calculus*) we mean an absolutely free algebra $J = (S, \mathbb{F})$, where S is the set of all formulas built in the standard way on a countable set of propositional variables $p_1, p_2, \ldots$ using functors from the set $\mathbb{F}$.

Let $\mathbf{C}$ denote the family of all consequences in S and let $Cn \in \mathbf{C}$. The consequence dCn dual to the consequence Cn is defined as follows:

Definition 1.

$$\alpha \in dCn(X) \Leftrightarrow \exists_Y \Big(Y \subseteq X \wedge card(Y) < \aleph_0 \wedge \bigcap_{\beta \in Y} Cn(\{\beta\}) \subseteq Cn(\{\alpha\}) \Big)$$

for all formulas $\alpha, \beta \in S$ and every $X \subseteq S$.

The definition of a dual consequence applied here was given by Wójcicki [3].

Let $J = (S, \{\Rightarrow, \neg\})$ be the language of Sobociński's n–valued logic described in [2].

Definition 2. *n–valued implicational-negational Sobociński propositional calculus is determined by the following matrix:*

$$\mathfrak{M}_{Sob} = (\{0, 1, 2, \ldots, n-1\}, \{1, 2, \ldots, n-1\}, \{\Rightarrow, \neg\}), \quad n \geq 3.$$

Here the only nondesignated value is 0.
Functions $\Rightarrow, \neg$ are defined as follows:

$$x \Rightarrow y = \begin{cases} y & \text{if} \quad x \neq y, \\ n-1 & \text{if} \quad x = y, \end{cases}$$

$$\neg x = \begin{cases} x+1 & \text{if} \quad x < n-1, \\ 0 & \text{if} \quad x = n-1, \end{cases}$$

for any $x, y \in \{0, 1, \ldots, n-1\}$.

Let us consider the following matrix, which will be called dual to the matrix $\mathfrak{M}_{Sob}$:

$$\mathfrak{M}^d_{Sob} = (\{0, 1, 2, \ldots, n-1\}, \{0\}, \{\Rightarrow, \neg\}), \quad n \geq 3,$$

where functions $\Rightarrow$ and $\neg$ are defined in the same way as in the matrix $\mathfrak{M}_{Sob}$.

Definition 3.

1. $\neg^* \alpha \stackrel{df}{=} (\alpha \Rightarrow \neg(\alpha \Rightarrow \alpha))$.

2. $\alpha \vee^* \beta \stackrel{df}{=} (\neg^* \alpha \Rightarrow \beta)$.

We call the functors $\neg^*$ and $\vee^*$ the strong negation and the strong disjunction, respectively.

It is easy to observe that a function $\neg^*$ defined by

$$\neg^*(x) = \begin{cases} n-1 & \text{if} \quad x = 0, \\ 0, & \text{otherwise}, \end{cases}$$

corresponds in the matrix $\mathfrak{M}_{Sob}$ to the functor $\neg^*$.
Similarly, a function $\vee^*$ defined by

$$x \vee^* y = \begin{cases} y & \text{if} \quad y \geq 1, \\ 0 & \text{if} \quad x = 0 \text{ and } y = 0, \\ n-1 & \text{if} \quad x \geq 1 \text{ and } y = 0, \end{cases}$$

corresponds in the matrix $\mathfrak{M}_{Sob}$ to the functor $\vee^*$.

Lemma 1. *For arbitrary formulas* $\alpha, \beta \in S$ *and for any homomorphism* $h : J \to (\{0,1,2,\dots,n-1\}, \{\Rightarrow, \neg^*, \vee^*\})$ *the following statements are true:*

1. *if* $h(\alpha \Rightarrow \beta), h(\alpha) \in \{1,2,\dots,n-1\}$, *then* $h(\beta) \in \{1,2,\dots,n-1\}$,
2. $h(\alpha \Rightarrow \beta) = 0$ *iff* $h(\alpha) \in \{1,2,\dots,n-1\}$ *and* $h(\beta) = 0$,
3. $h(\alpha) \in \{1,2,\dots,n-1\}$ *iff* $h(\neg^*\alpha) = 0$,
4. $h(\alpha \vee^* \beta) \in \{1,2,\dots,n-1\}$ *iff* $h(\alpha) \in \{1,2\dots,n-1\}$ *or* $h(\beta) \in \{1,2,\dots,n-1\}$.

Let us consider two inference rules:

$$r_{mp} : \frac{\alpha \Rightarrow \beta, \alpha}{\beta}, \qquad r^d_{mp} : \frac{\neg^*(\alpha \Rightarrow \beta), \beta}{\alpha}.$$

Let $R = \{r_{mp}\}, R^d = \{r^d_{mp}\}$.

Denote by Hom the set of all homomorphisms from $(S, \{\Rightarrow, \neg\})$ into $(\{0,1,\dots,n-1\}, \{\Rightarrow, \neg\})$ and let $X \subseteq S$. We define the matrix consequence $C_{\mathfrak{M}}(X)$, the content $E(\mathfrak{M})$ of the matrix $\mathfrak{M}$ and the consequence $C_R(X)$ based on inference rules from the set X in the standard way:

Definition 4.

1. $C_{\mathfrak{M}_{Sob}}(X) =$ $= \{\alpha \in S : \forall_{h \in Hom}(h(X) \subseteq \{1,\dots,n-1\} \Rightarrow h(\alpha) \in \{1,\dots,n-1\})\}$.
2. $C_{\mathfrak{M}^d_{Sob}}(X) = \{\alpha \in S : \forall_{h \in Hom}(h(X) \subseteq \{0\} \Rightarrow h(\alpha) = 0)\}$.
3. $E(\mathfrak{M}_{Sob}) = \{\alpha \in S : \forall_{h \in Hom} h(\alpha) \in \{1,2,\dots,n-1\}\}$.
4. $E(\mathfrak{M}^d_{Sob}) = \{\alpha \in S : \forall_{h \in Hom} h(\alpha) = 0\}$.
5. $C_R(X)$ *is the least set* Y, *which is closed under the rule* r_{mp} *and which satisfies* $E(\mathfrak{M}_{Sob}) \cup X \subseteq Y$.
6. $C_{R^d}(X)$ *is the least set* Y, *which is closed under the rule* r^d_{mp} *and which satisfies* $E(\mathfrak{M}^d_{Sob}) \cup X \subseteq Y$.

2. Some properties of $C_{\mathfrak{M}_{Sob}}, C_{\mathfrak{M}^d_{Sob}}, C_R$ and C_{R^d}

Since modus ponens is the primitive rule of $C_R(X)$ and, as can be easily seen, $\alpha \Rightarrow \alpha, \alpha \Rightarrow (\beta \Rightarrow \alpha), (\alpha \Rightarrow (\beta \Rightarrow \gamma)) \Rightarrow ((\alpha \Rightarrow \beta) \Rightarrow (\alpha \Rightarrow \gamma)) \in E(\mathfrak{M}_{Sob})$, then the classical deduction theorem holds:

Lemma 2. *For arbitrary $\alpha, \beta \in S$ and $X \subseteq S$*

$$\beta \in C_R(X \cup \{\alpha\}) \textit{ iff } \alpha \Rightarrow \beta \in C_R(X).$$

Proof. Let us assume that the sequence $\alpha_1, \dots, \alpha_n$ is the proof based on the set $X \cup \{\alpha\}$ of a formula β. We prove, by induction, that for any $1 \leqslant k \leqslant n$ it holds

$$\alpha \Rightarrow \alpha_k \in C_R(X).$$

Let $k = 1$. Then $\alpha_1 = \alpha$ or $\alpha_1 \in X$.

If $\alpha_1 = \alpha$, then since $\alpha \Rightarrow \alpha \in E(\mathfrak{M}_{Sob})$, we get $\alpha \Rightarrow \alpha_1 \in C_R(X)$.

If $\alpha_1 \in X$, then noticing that $\alpha_1 \Rightarrow (\alpha \Rightarrow \alpha_1) \in E(\mathfrak{M}_{Sob})$, we can see that the sequence $\alpha_1 \Rightarrow (\alpha \Rightarrow \alpha_1), \alpha_1, \alpha \Rightarrow \alpha_1$ is the proof based on X of the formula $\alpha \Rightarrow \alpha_1$.

Assume now that $k > 1$ and for any $i < k, \alpha \Rightarrow \alpha_i \in C_R(X)$.

If $\alpha_k \in X \cup \{\alpha\}$, then the proof is analogous as in the case $k = 1$.

Thus, let α_k results by r_{mp} from α_i, α_j for some $i, j < k$.
Therefore $\alpha_j = \alpha_i \Rightarrow \alpha_k$ and $\alpha \Rightarrow \alpha_i$, $\alpha \Rightarrow (\alpha_i \Rightarrow \alpha_k) \in C_R(X)$. Suppose $\beta_0, \dots, \beta_{n-1}, \alpha \Rightarrow \alpha_i$ and $\gamma_0, \dots, \gamma_{m-1}, \alpha \Rightarrow (\alpha_i \Rightarrow \alpha_k)$ are proofs of $\alpha \Rightarrow \alpha_i$ and $\alpha \Rightarrow \alpha_j$, respectively. Then the sequence
$\beta_0, \dots, \beta_{n-1}, \gamma_0, \dots, \gamma_{m-1}, (\alpha \Rightarrow (\alpha_i \Rightarrow \alpha_k)) \Rightarrow ((\alpha \Rightarrow \alpha_i) \Rightarrow (\alpha \Rightarrow \alpha_k))$,
$\alpha \Rightarrow (\alpha_i \Rightarrow \alpha_k), (\alpha \Rightarrow \alpha_i) \Rightarrow (\alpha \Rightarrow \alpha_k), \alpha \Rightarrow \alpha_i, \alpha \Rightarrow \alpha_k$ is a proof of $\alpha \Rightarrow \alpha_k$, because $(\alpha \Rightarrow (\beta \Rightarrow \gamma)) \Rightarrow ((\alpha \Rightarrow \beta) \Rightarrow (\alpha \Rightarrow \gamma)) \in E(\mathfrak{M}_{Sob})$.

In the end, let us assume that the sequence $\alpha_1, \dots, \alpha_n$ is the proof based on X of the formula $\alpha \Rightarrow \beta$. Then $\alpha_n = \alpha \Rightarrow \beta$. It is easy to observe that the sequence $\alpha_1, \dots, \alpha_n, \alpha, \beta$ is the proof based on $X \cup \{\alpha\}$ of the formula β. □

The next Lemma follows directly from definitions and Lemma 1.

Lemma 3. *For arbitrary $\alpha, \beta \in S$ and $X \subseteq S$*

1. $\beta \in C_{\mathfrak{M}_{Sob}}(X \cup \{\alpha\})$ *iff* $\alpha \Rightarrow \beta \in C_{\mathfrak{M}_{Sob}}(X)$.
2. $\alpha \in C_{\mathfrak{M}_{Sob}}(\{\beta\})$ *iff* $\beta \in C_{\mathfrak{M}^d_{Sob}}(\{\alpha\})$.
3. $\alpha \in C_{\mathfrak{M}^d_{Sob}}(\{\beta\})$ *iff* $\neg^*(\alpha \Rightarrow \beta) \in C_{\mathfrak{M}^d_{Sob}}(\emptyset)$.
4. *The consequences $C_{\mathfrak{M}_{Sob}}, C_{\mathfrak{M}^d_{Sob}}, C_R$ and C_{R^d} are finitary.*

Lemma 4.

1. *The rule r_{mp} is an admissible rule of the consequence $C_{\mathfrak{M}_{Sob}}$.*
2. *The rule r^d_{mp} is an admissible rule of the consequence $C_{\mathfrak{M}^d_{Sob}}$.*

Proof.

1. By Lemma 1, for any homomorphism $h \in Hom$ such that $h(\alpha \Rightarrow \beta), h(\alpha) \in \{1, \ldots, n-1\}$ we have $h(\beta) \in \{1, \ldots, n-1\}$. This means that $\beta \in C_{\mathfrak{M}_{Sob}}(\{\alpha \Rightarrow \beta, \alpha\})$ and then modus ponens is an admissible rule in $C_{\mathfrak{M}_{Sob}}$.

2. The proof can be carried out on the basis of Definition 4 and Lemma 1.

□

Lemma 5.

1. $C_{\mathfrak{M}^d_{Sob}}(\emptyset) = C_{R^d}(\emptyset) = E(\mathfrak{M}^d_{Sob})$.

2. $C_{\mathfrak{M}_{Sob}}(\emptyset) = C_R(\emptyset) = E(\mathfrak{M}_{Sob})$.

3. $C_{\mathfrak{M}_{Sob}} = C_R$.

Proof. Equalities *1.* and *2.* follow directly from definitions. The proof of the equality *3.* runs as follows:

Let $X \subseteq S$. To prove the inclusion $C_{\mathfrak{M}_{Sob}}(X) \subseteq C_R(X)$ assume that $\alpha \in C_{\mathfrak{M}_{Sob}}(X)$. Due to the finitariness of the matrix consequence $C_{\mathfrak{M}_{Sob}}$ there exists a finite set $X_0 \subseteq X$ such that $\alpha \in C_{\mathfrak{M}_{Sob}}(X_0)$.

If $X_0 = \emptyset$, then using equality *2.*, we infer that $\alpha \in C_R(X_0)$ and therefore $\alpha \in C_R(X)$.

Let $X_0 = \{\alpha_1, \ldots, \alpha_m\}$.
By Lemma 3, we get $\alpha_1 \Rightarrow (\ldots \Rightarrow (\alpha_m \Rightarrow \alpha)\ldots) \in C_{\mathfrak{M}_{Sob}}(\emptyset)$. Then, by equality *2.* and Lemma 2, we have that $\alpha \in C_R(\{\alpha_1, \ldots, \alpha_m\})$. As $X_0 \subseteq X$, we see that $\alpha \in C_R(X)$.

To prove the inclusion $C_R(X) \subseteq C_{\mathfrak{M}_{Sob}}(X)$, we apply Lemma 2, Lemma 3 and the fact that C_R is finitary. □

Let us define recursively a generalized strong disjunction by

Definition 5.

1. $\vee^*(\alpha) = \alpha$,

2. $\vee^*(\alpha, \beta) = \alpha \vee^* \beta$,

3. $\vee^*(\alpha_1, \ldots, \alpha_{n+1}) = \vee^*(\vee^*(\alpha_1, \ldots, \alpha_n), \alpha_{n+1}), \quad n \geqslant 2$.

Lemma 6. *For any natural number $m \geqslant 1$:*

$$C_{\mathfrak{M}^d_{Sob}}(\{\vee^*(\alpha_1, \ldots, \alpha_m)\}) = C_{\mathfrak{M}^d_{Sob}}(\{\alpha_1, \ldots, \alpha_m\}).$$

Proof. We are going to show that for any formula $\alpha \in S$,

$$\alpha \in C_{\mathfrak{M}^d_{Sob}}(\{\vee^*(\alpha_1, \dots, \alpha_m)\}) \text{ iff } \alpha \in C_{\mathfrak{M}^d_{Sob}}(\{\alpha_1, \dots, \alpha_m\}).$$

By Lemma 3, we have the following chain of equivalent statements:

$$\begin{aligned} \alpha \in C_{\mathfrak{M}^d_{Sob}}(\{\vee^*(\alpha_1, \dots, \alpha_m)\}) &\text{ iff } \vee^*(\alpha_1, \dots, \alpha_m) \in C_{\mathfrak{M}_{Sob}}(\{\alpha\}) \\ &\text{ iff } \alpha \Rightarrow \vee^*(\alpha_1, \dots, \alpha_m) \in C_{\mathfrak{M}_{Sob}}(\emptyset). \end{aligned}$$

The equivalence $\alpha \Rightarrow \vee^*(\alpha_1, \dots, \alpha_m) \in C_{\mathfrak{M}_{Sob}}(\emptyset)$ iff $\alpha \in C_{\mathfrak{M}^d_{Sob}}(\{\alpha_1, \dots, \alpha_m\})$ can be justified in the following way:

„$\Rightarrow$". Suppose that there exists a homomorphism $h_0 \in Hom$ such that $h_0(\{\alpha_1, \dots, \alpha_m\}) \subseteq \{0\}$ and $h_0(\alpha) \in \{1, \dots, n-1\}$. Then, by Lemma 1, we get $h_0(\alpha \Rightarrow \vee^*(\alpha_1, \dots, \alpha_m)) = 0$.

„$\Leftarrow$". Let $\alpha \in C_{\mathfrak{M}^d_{Sob}}(\{\alpha_1, \dots, \alpha_m\})$ and let us suppose that there exists a homomorphism h_1 such that $h_1(\alpha \Rightarrow \vee^*(\alpha_1, \dots, \alpha_m)) = 0$. By Lemma 1, we have $h_1(\alpha) \in \{1, \dots, n-1\}$ and $h_1(\vee^*(\alpha_1, \dots, \alpha_m)) = 0$. According to Lemma 1, we obtain $h_1(\{\alpha_1, \dots, \alpha_m\}) \subseteq \{0\}$, so $h_1(\alpha) = 0$. This contradicts our assumption. □

Lemma 7. *For any natural number* $m \geqslant 1$*:*

$$C_{R^d}(\{\vee^*(\alpha_1, \dots, \alpha_m)\}) \subseteq C_{R^d}(\{\alpha_1, \dots, \alpha_m\}).$$

Proof. The proof is inductive on m.

Let us observe that $\neg^*(\neg^*(\alpha_1 \vee^* \alpha_2 \Rightarrow \alpha_1) \Rightarrow \alpha_2) \in E(\mathfrak{M}^d_{Sob})$. By Lemma 5 and Definition 4, we have $\alpha_1 \vee^* \alpha_2 \in C_{R^d}(\{\alpha_1, \alpha_2\})$.
Thus $C_{R^d}(\{\alpha_1 \vee^* \alpha_2\}) \subseteq C_{R^d}(\{\alpha_1, \alpha_2\})$.

Assume that $C_{R^d}(\{\vee^*(\alpha_1, \dots, \alpha_k)\}) \subseteq C_{R^d}(\{\alpha_1, \dots, \alpha_k\})$ for some $k \geq 2$. We show that

$$C_{R^d}(\vee^*(\alpha_1, \dots, \alpha_{k+1})\}) \subseteq C_{R^d}(\{\alpha_1, \dots, \alpha_{k+1}\}).$$

Indeed, $C_{R^d}(\{\vee^*(\alpha_1, \dots, \alpha_{k+1})\}) = C_{R^d}(\{\vee^*(\vee^*(\alpha_1, \dots, \alpha_k), \alpha_{k+1})\}) \subseteq$
$\subseteq C_{R^d}(\{\vee^*(\alpha_1, \dots, \alpha_k), \alpha_{k+1}\}) = C_{R^d}(\{\vee^*(\alpha_1, \dots, \alpha_k)\} \cup \{\alpha_{k+1}\}) =$
$= C_{R^d}(C_{R^d}(\{\vee^*(\alpha_1, \dots, \alpha_k)\}) \cup \{\alpha_{k+1}\}) \subseteq C_{R^d}(C_{R^d}(\{\alpha_1, \dots, \alpha_k\}) \cup \{\alpha_{k+1}\}) =$
$= C_{R^d}(\{\alpha_1, \dots, \alpha_k\} \cup \{\alpha_{k+1}\}) = C_{R^d}(\{\alpha_1, \dots, \alpha_{k+1}\}).$ □

Lemma 8. *For arbitrary formulas* $\alpha, \alpha_1, \dots, \alpha_m \in S$
$\alpha \in C_{\mathfrak{M}_{Sob}}(\{\vee^*(\alpha_1, \dots, \alpha_m)\})$ *iff* $\alpha \in C_{\mathfrak{M}_{Sob}}(\{\alpha_1\}) \cap \ldots \cap C_{\mathfrak{M}_{Sob}}(\{\alpha_m\})$.

Proof. It is a direct consequence of Lemma 1 and Definition 4. □

Lemma 9.

1. $C_{\mathfrak{M}_{Sob}}(\{\alpha\}) = S \Leftrightarrow \alpha \in C_{\mathfrak{M}^d_{Sob}}(\emptyset)$.
2. $C_{\mathfrak{M}^d_{Sob}}(\{\alpha\}) = S \Leftrightarrow \alpha \in C_{\mathfrak{M}_{Sob}}(\emptyset)$.

Proof.

1. „$\Rightarrow$". Let us assume that $C_{\mathfrak{M}_{Sob}}(\{\alpha\}) = S$. Since $\neg^*(p \Rightarrow p) \in C_{\mathfrak{M}_{Sob}}(\{\alpha\})$, then applying Lemma 3, we get $\alpha \in C_{\mathfrak{M}^d_{Sob}}(\{\neg^*(p \Rightarrow p)\})$.
   But $\neg^*(p \Rightarrow p) \in C_{\mathfrak{M}^d_{Sob}}(\emptyset)$, so $\alpha \in C_{\mathfrak{M}^d_{Sob}}(\emptyset)$.

   „$\Leftarrow$". Let us assume that $\alpha \in C_{\mathfrak{M}^d_{Sob}}(\emptyset)$. By Lemma 1 and Definition 4, we get $h(\alpha \Rightarrow \gamma) \in \{1, \ldots, n-1\}$ for every homomorphism h and any formula $\gamma \in S$. By Definition 4 and Lemma 3, we obtain that $\gamma \in C_{\mathfrak{M}_{Sob}}(\{\alpha\})$ for any formula $\gamma \in S$, so $S \subseteq C_{\mathfrak{M}_{Sob}}(\{\alpha\})$. As the opposite inclusion trivially holds, we obtain $C_{\mathfrak{M}_{Sob}}(\{\alpha\}) = S$.
2. The proof is analogous as above. □

3. Main result

Now, we consider the consequences dual in the sense of Definition 1 to the consequences C_R and $C_{\mathfrak{M}_{Sob}}$ and their relation to $C_{\mathfrak{M}^d_{Sob}}$ and C_{R^d}.

Theorem 1.

$$C_{R^d} = C_{\mathfrak{M}^d_{Sob}} = dC_{\mathfrak{M}_{Sob}} = dC_R.$$

Proof. 1° $C_{R^d} = C_{\mathfrak{M}^d_{Sob}}$.

By Lemma 5, we know that $C_{R^d}(\emptyset) = C_{\mathfrak{M}^d_{Sob}}(\emptyset)$ and since, by Lemma 4, the rule r^d_{mp} is an admissible rule of the consequence $C_{\mathfrak{M}^d_{Sob}}$, we get $C_{R^d}(X) \subseteq C_{\mathfrak{M}^d_{Sob}}(X)$ for every $X \subseteq S$, which means that $C_{R^d} \leq C_{\mathfrak{M}^d_{Sob}}$.

Now, let $\alpha \in C_{\mathfrak{M}^d_{Sob}}(X)$. Since the consequence $C_{\mathfrak{M}^d_{Sob}}$ is finitary, there exists a finite set X_0 such that $X_0 \subseteq X$ and $\alpha \in C_{\mathfrak{M}^d_{Sob}}(X_0)$.

If $X_0 = \emptyset$, then by Lemma 5 we get $\alpha \in C_{R^d}(X)$.

Assume then that $X_0 = \{\alpha_1, \ldots, \alpha_m\}$.
Applying Lemma 6, we have $\alpha \in C_{\mathfrak{M}^d_{Sob}}(\{\vee^*(\alpha_1, \ldots, \alpha_m)\})$. In turn, Lemma 3 yields that $\neg^*(\alpha \Rightarrow \vee^*(\alpha_1, \ldots, \alpha_m)) \in C_{\mathfrak{M}^d_{Sob}}(\emptyset)$. Therefore, by Lemma 5, we obtain that $\neg^*(\alpha \Rightarrow \vee^*(\alpha_1, \ldots, \alpha_m)) \in C_{R^d}(\emptyset)$.

Hence, $\alpha \in C_{R^d}(\{\vee^*(\alpha_1, \ldots, \alpha_m)\}) \subseteq C_{R^d}(\{\alpha_1, \ldots, \alpha_m\})$ and then $\alpha \in C_{R^d}(X)$.

Thus we have shown that $C_{\mathfrak{M}^d_{Sob}} \leq C_{R^d}$.

2° $C_{\mathfrak{M}^d_{Sob}} = dC_{\mathfrak{M}_{Sob}}$.

Let $\alpha \in C_{\mathfrak{M}^d_{Sob}}(X)$. Then, by finitariness of $C_{\mathfrak{M}^d_{Sob}}$, we deduce that $\alpha \in C_{\mathfrak{M}^d_{Sob}}(X_1)$ for a finite set $X_1 \subseteq X$.

If $X_1 = \emptyset$, then by Lemma 9
$C_{\mathfrak{M}_{Sob}}(\{\alpha\}) = S$. Hence $\bigcap_{\beta\in\emptyset} C_{\mathfrak{M}_{Sob}}(\{\beta\}) \subseteq C_{\mathfrak{M}_{Sob}}(\{\alpha\})$, i.e. $\alpha \in dC_{\mathfrak{M}_{Sob}}(X)$.

If $X_1 = \{\alpha_1, \dots, \alpha_m\}$, then $\alpha \in C_{\mathfrak{M}^d_{Sob}}(\{\alpha_1, \dots, \alpha_m\})$. Applying Lemmas 6 and 3, we obtain that $\vee^*(\alpha_1, \dots, \alpha_m) \in C_{\mathfrak{M}_{Sob}}(\{\alpha\})$. From this and Lemma 8, we have $C_{\mathfrak{M}_{Sob}}(\{\alpha_1\}) \cap \dots \cap C_{\mathfrak{M}_{Sob}}(\{\alpha_m\}) \subseteq C_{\mathfrak{M}_{Sob}}(\{\alpha\})$. Thus $\alpha \in dC_{\mathfrak{M}_{Sob}}(X)$ by Definition 1. We have just shown that $C_{\mathfrak{M}^d_{Sob}} \leq dC_{\mathfrak{M}_{Sob}}$.

Suppose now that $\alpha \in dC_{\mathfrak{M}_{Sob}}(X)$. By Definition 1, there exists a finite set $Y \subseteq X$ such that $\bigcap_{\beta\in Y} C_{\mathfrak{M}_{Sob}}(\{\beta\}) \subseteq C_{\mathfrak{M}_{Sob}}(\{\alpha\})$.

If $Y = \emptyset$, then from the fact that $\bigcap_{\beta\in\emptyset} C_{\mathfrak{M}_{Sob}}(\{\beta\}) = S$ and Lemma 9, we obtain that $\alpha \in C_{\mathfrak{M}^d_{Sob}}(X)$.

Therefore, let us assume that $Y = \{\beta_1, \dots, \beta_m\}$.
Thus $C_{\mathfrak{M}_{Sob}}(\{\beta_1\}) \cap \dots \cap C_{\mathfrak{M}_{Sob}}(\{\beta_m\}) \subseteq C_{\mathfrak{M}_{Sob}}(\{\alpha\})$. By Lemma 8, we have that $C_{\mathfrak{M}_{Sob}}(\{\vee^*(\beta_1, \dots, \beta_m)\}) \subseteq C_{\mathfrak{M}_{Sob}}(\{\alpha\})$, i.e.,
$\vee^*(\beta_1, \dots, \beta_m) \in C_{\mathfrak{M}_{Sob}}(\{\alpha\})$.

Applying Lemma 3, we conclude that $\alpha \in C_{{\mathfrak{M}_{Sob}}^d}(\{\vee^*(\beta_1, \dots, \beta_m)\})$. Then, according to Lemma 6, we obtain that $\alpha \in C_{\mathfrak{M}^d_{Sob}}(\{\beta_1, \dots, \beta_m\})$. Hence $\alpha \in C_{\mathfrak{M}^d_{Sob}}(X)$ because $Y = \{\beta_1, \dots, \beta_m\} \subseteq X$. This proves that $dC_{\mathfrak{M}_{Sob}}(X) \subseteq C_{\mathfrak{M}^d_{Sob}}(X)$, so $dC_{\mathfrak{M}_{Sob}} \leq C_{\mathfrak{M}^d_{Sob}}$.

3° The equality $dC_{\mathfrak{M}_{Sob}} = dC_R$ follows directly from Lemma 5. □

Therefore, the sentential logic (S, C_{R^d}) can be regarded as a logic dual to the Sobociński's n–valued logic (S, C_R). Moreover, it is characterized by the matrix ${\mathfrak{M}_{Sob}}^d$.

References

[1] G. Malinowski, M. Spasowski. Dual counterparts of Łukasiewicz's sentential calculi. *Studia Logica*, **33** (2), 153–162, 1974.

[2] B. Sobociński. Axiomatization of certain many-valued systems of the theory of deduction. *Roczniki prac naukowych zrzeszenia asystentów Uniwersystetu Józefa Piłsudskiego w Warszawie*, No. 1, 399–419, 1936.

[3] R. Wójcicki. Dual counterparts of consequence operations. *Bull. Sect. Logic*, **2** (1), 201–214, 1973.