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Bounded linear stability region in Linear Matrix Inequalities for a multivariable object

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Abstract
This paper is devoted to the concept of stability in linear matrix inequalities. Especially to the analysis of a bounded stability region for a multivariable 3DOF system. The study is focused on two different, user defined, bounded LMI regions where poles are calculated with the use on an algorithm and placed in the left half plane of complex variable plane *s*. It is shown that in the studied case the shape of the region is irrelevant compared to the location of region bounds. And more specifically only the active bounds where the poles are located on it's limits are significant to controller operation.

Introduction

This paper contains basic concepts of robust control and more precisell LMI, Linear Matrix Inequalities, which have different than ususall definitions of stability. In detail, the scope of this paper is the study of methods for selecting regions and bounds in stability of Linear Matrix Inequalities. In the existing literature about LMI regions and additional bounds [1, 2, 3] the author has not found information on how to find the correct region for a specific controlled object. As an attempt to answer that question several simulations have been made for a multivariable object. The object was a model of a training ship "Blue Lady" [4]. The study was focused on what influence regions and bounds selection has on the control of three velocieties, essential for trajectory control of a ship during sailing. The evaluation criteria was based on the impact of regions and bounds on the dynamic parameters of the controlled object as in [5]. The rest of the paper is constructed as follows: chapter two introduces basic concepts of stability in linear matrix inequalities, followed by chapter three which describes selected bounded regions of linear stability. Chapter four presents the basic concepts of Linear matrix inequality, followed by chapter five which presents a short description of a multivariable 3DOF system which is a "Blue Lady" ship model. Chapter six is the case study with simulations for two selected bounded stability regions and their influence on control performance. The filan chapter, seven, contains conclusions.

Stability in LMI

In automation stability is one of the most basic and most frequently addressed concepts. Regardless of how automation theorists and practitioners define stability – one thing is certain – without bringing your controlled system to stability there is no automation. The main aim of control is keeping your controlled system in such working conditions that the output signal corresponds to the input signal without human intervention, despite the presence of interference. Before the concept of stability for linear matrix inequalities is explained some basic definitions of control theory will be brought forward first. For the concept of stability in state variable models, where it is assumed that unit step functions do not exist and only initial values are non-zero ($x(0) = x_0$) the state variable equation is [6]:

$$\frac{\mathrm{d}\,x(t)}{\mathrm{d}\,t} = Ax(t) \Leftrightarrow \dot{x} = Ax \qquad (1)$$

where: $x(t)$ – state variable vector, $A \in R^{nxn}$ – linear system parameters matrix.

Based on definition No. 45 from [6] it was determined that if for point x_r, it is true that $Ax_r = 0$, than for every $t \geq 0$ that point is called the equilibrium point of the system described in equation 1. Having defined the equilibrium point of the system it is necessary to determine its stability thus the reference to definition No. 47 from [6].

We call the equilibrium point x_r stable if for every positive number ε such a number η can be found that the system phase trajectory starting in point x_0 which is inside a sphere with a radius of η, will stay inside a sphere, with a radius of ε, for every instant $t \geq 0$. Based on the above, if we have an autonomous linear system (without any unit step functions) we can determine an autonomous linear asymptotically stable system for which all eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ of system matrix A have a negative real parts $\text{Re}_{\lambda k} < 0$ for $i = 0,...,n$. Defining of Laypunov function is bound with the deduction stating that a single-valued scalar function $V(x)$ of state vector x which is continuous with first derivatives towards state variables $x_1, x_2, ..., x_n$ is called a Laypunov function in Ω area if:

– that function is positive definite in Ω area, which means that $V(x) \succ 0$ for $x \neq 0$ and $V(0) = 0$;
– time derivative t of $V(x)$ function is negative definite in Ω area (for $x \neq 0$ and $\dot{V} = 0$ only for $x = 0$)

or negative semi-definite in Ω area ($\dot{V}(x) \leq 0$ for any $x \in \Omega$). Based on the above theorem a positive definite Laypunow function has the form of:

$$V(x) = x^T P x \tag{2}$$

Combining conditions from basic control concepts for linear matrix inequalities with the Laypunov function 2 to determine the stability condition it must be noted that:

$$\frac{\mathrm{d}V(x)}{\mathrm{d}t} = \dot{x}^T P x + x^T P \dot{x} \tag{3}$$

Replacing $\dot{x} = Ax$ and substituting it to the equation we can receive:

$$x^T (A^T P + PA) x \tag{4}$$

It turns out that the necessary and sufficient condition for the linear system to be asymptotically stable is, based on the Laypunov function (whose time derivative is negative for every $x \neq 0$), finding a positive definite symmetric matrix P (unknown variable), which is shown below:

$$A^T P + PA \prec 0 \tag{5}$$

The above inequality means that the problem of finding a symmetric positive definite matrix P is called the feasibility problem.

Stability of the system with additional bounds

If during linear system $\dot{x} = Ax$ study a positive definite symmetric matrix P has been found which means that $P = P^T \succ 0$ (based on Laypunov inequality, see 5) and based on Schur's compliment we receive:

$$\begin{bmatrix} -A^T P - PA & 0 \\ 0 & P \end{bmatrix} \succ 0 \tag{6}$$

Fulfilling condition 6 determines that eigenvalues of matrix A are placed in the left half plane of complex variable plane s. Implementing additional bounds to the shape of the region has a key influence on the dynamic properties of the control system working in a closed loop with the specified controlled object. It is important that the bound is a convex set. Let us define a stability region as a subset C_{stab} with the following two properties:

$$\begin{cases} \lambda \in C_{\text{stab}} \Rightarrow \bar{\lambda} \in C_{\text{stab}} \\ C_{\text{stab}} = \text{is convex} \end{cases} \tag{7}$$

Assuming that area C relates to the whole set of complex numbers, meaning that $s \in C$ and $\bar{s}$ is a conjugate number to s. Below some examples of bounded stability regions are shown [1]:

a) $C_{\text{stab1}} = C^{-1} \Leftrightarrow s + \bar{s} \prec 0$ – left half plane of complex variable plane s (Fig. 1a);
b) $C_{\text{stab2}} = \{s \in C_{\text{stab}} \| s + q | < r\}$ – using Schur's compliment we receive $\begin{bmatrix} -r & s+q \\ s+q & -r \end{bmatrix} \prec 0$ left half plane of complex variable plane s with a circle that has a centre at q and a radius of r (Fig. 1b).

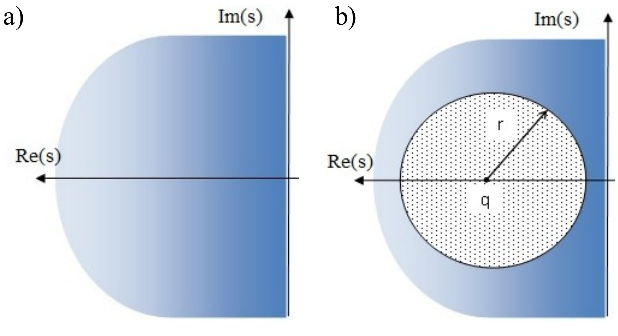

Fig. 1. a) Left half plane of complex variable plane s; b) Left half plane with a circle, where r – radius of the circle, q – center of the circle

c) $C_{\text{stab3}} = \{s \in C_{\text{stab}} | \alpha_1 < \text{Re}(s) < \alpha_2\}$ – vertical stripe described by α_1 i α_2 using Schur's compliment we receive
$\begin{bmatrix} (s+\bar{s}) - 2\alpha_2 & 0 \\ 0 & -(s+\bar{s}) + 2\alpha_1 \end{bmatrix} \prec 0$ (Fig. 2a);

d) $C_{\text{stab4}} = \{s \in C_{\text{stab}} | \text{Re}(s)\tan\phi < |-\text{Im}(s)|$ – sector with an angle of flare $\phi \in (0, \pi/2)$ using Schur's compliment we receive
$\begin{bmatrix} (s+\bar{s})\sin(\phi) & -(s+\bar{s})\cos(\phi) \\ (s-\bar{s})\cos(\phi) & (s+\bar{s})\sin(\phi) \end{bmatrix} \prec 0$ (Fig. 2b).

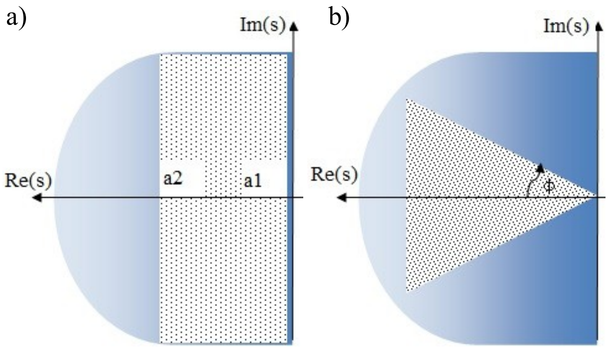

Fig. 2. a) Vertical stripe described by α_1 and α_2; b) Left half plane with an angle of flare φ sector

The above conditions can be brought down to the following notation:

$$C_{stab} = \{s \in C | P + Qs + Q^T s \prec 0\} \tag{8}$$

where: $P = P^T$.

In this paper main focus has been given to comparing two regions, one of which was shown on figure 1, and the other one is a combination of regions shown on figures 2a and 2b. Regions defined in [1, 3, 7, 8] have been tested, which will be described later in this paper.

Linear Matrix Inequalities LMI – the principles

Linear Matrix Inequalities LMI are described by canonical form [1, 8, 9]:

$$F(x) := F_0 + \sum_{i=1}^{m} F_i x_i \tag{9}$$

If for a sequence of true and symmetrical matrices:

$$F_0, F_1, \ldots, F_m \tag{10}$$

a matrix has the following relation $F_i = F_i^T \in R^{nxn}$ for $i = 0,\ldots,m$ and a vector is $x = [x_1, x_2, \ldots x_m]^T \in R^m$.

The LMI is related to the feasibility problem which comes down to searching for the answer to the question whether there exists a solution x to the LMI problem in its overall form shown below:

$$A(x) \prec 0 \tag{11}$$

Where A is the state matrix of the control system. In order to create an LMI for a control system for the object is necessary to check if the eigenvalues of the matrix A of the controlled closed loop system are placed in the left half plane of the complex variable plane s (subsection 3). Next the feasibility problem and stability can be checked with the Laypunov function (2). After checking the above conditions dynamic properties of the control system can be designed by pole placement in a specific part of the complex variable plane s. A defined plane for pole placement was designated C_{stab} [subsection Stability of the system with additional bounds]. A multivariable control system MIMO (Multiple Input Multiple Output) for controlled system G with a transfer matrix is shown in figure 3.

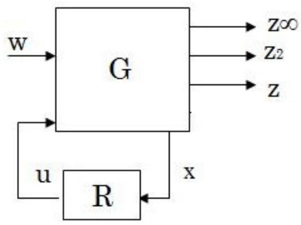

Fig. 3. Structure of a system for tracking set value to synthesize a multivariable controller, G – object, R – controller

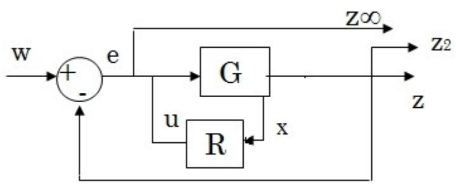

Fig. 4. Structure of a system for tracking set value to synthesize a multivariable regulator, G – object, R – control

For which the space state equalities are as follows:

$$\begin{cases} \dot{x} &= Ax + B_u u + B_w w \\ z &= Cx + D_{zu} u + D_{zw} w \\ z_2 &= C_2 x + D_{2u} u + D_{2w} w \\ z_\infty &= C_\infty x + D_{\infty u} u + D_{\infty w} w \end{cases} \tag{12}$$

where: A – state matrix with dimensions $n \times n$, B – input matrix with dimensions $n \times r$, **C** – output matrix with dimensions $m \times n$, D – direct feedback matrix with dimensions $m \times r$, x – state of the system with a vector dimension n, w – input signal with a vector dimension r, z – output signal, with a vector dimension m, measured by sensors.

A closed loop system shown in figure 3 or 4 can be described with the below space state equations, assuming that:

$$u = Rx, A_{cl} = A + B_u R, B_{cl} = B_w, x_{cl} = x \quad (13)$$

$$\begin{cases} \dot{x}_{cl} & = & A_{cl} x_{cl} & + & B_{cl} w \\ z & = & (C_z & + & D_{zu} R) x_{cl} & + & D_{zw} w \\ z_2 & = & (C_2 & + & D_{2u} R) x_{cl} & + & D_{2w} w \\ z_\infty & = & (C_\infty & + & D_{\infty u} R) x_{cl} & + & D_{\infty w} w \end{cases} \quad (14)$$

where: cl – closed, marks vectors and matrices used in the description of the closed loop system. Matrices in (14) depend on the structure of the controler, used norms and controlled object matrix. After pole placement in the left half plane of the complex variable plane s further controller synthesis requires a description according to a defined standard such as H_∞ which for LMI is defined as:

$$\begin{bmatrix} (A + B_u R) X_\infty + \\ + X_\infty (A + B_u R)^T & B_w & X_\infty (C_\infty + D_{\infty u} R)^T \\ B_w^T & -\gamma I & D_{\infty w}^T \\ (C_\infty + D_{\infty u} R) X_\infty & D_{\infty w} & -\gamma I \end{bmatrix} \prec 0 \quad (15)$$

The norm for G system matrix is lesser than the scalar variable gamma γ [8] if and only if, and there exists a matrix $X_\infty \succ 0$. After calculations, the value of variable gamma is an approximation of the upper limit of the standard. The H_∞ norm, in general, allows to design a controller which will bring control signals down to zero as fast as possible. Assuming some constant values for γ, based on the analysis of the Pareto curve [7], we can approximate values of H_2 standard which is shown below:

$$\begin{bmatrix} Q & (C_2 + D_{zu} R) X_2 \\ X_2 (C_2 + D_{zu} R) & X_2 \end{bmatrix} \succ 0 \quad (16)$$

where: Q – square matrix. The number of signals included in the norm affects the size of the matrix. Trace of a matrix $Tr(Q)$ is a top estimation of the H_2 standard if: $X = X_\infty = X_2$ and $Y = RX$, assuming, based on the Laypunov condition, that matrix X is positive definite we can determine the following relation $R = Y X^{-1}$. The value of H_2 standard allows to minimize the energy of control signal. LMI conditions defined above which are:

- checking if the eigenvalues of matrix A are placed in the left half plane of complex variable plane s checking if a symmetric matrix of P type exists for matrix A;
- determining the area to place poles in a designer defined region determining H_∞ standard;
- determining H_2 standard;
- determining a suboptimal solution for the specific controlled object are the successive stages of space state controller synthesis for a multivariable object, which will be explained later in this paper.

Multidimensional object "Blue Lady" ship model

Controller synthesis using linear matrix inequalities is performed for automatic control of a multiva-riable system which is the "Blue Lady" ship model. The physical ship model named "Blue Lady" is used by the Foundation for Safety of Navigation and Environment Protection at the Silm lake near Ilawa in Poland for training of navigators. It is one of the series of 7 various training ships exploited on the lake. The ship "Blue Lady" is an isomorphous model of a VLCC (Very Large Crude Carrier) tanker, built from epoxide resin laminate in 1:24 scale. It is equipped with battery-fed electric drives (main propeller, blade rudder, four thrusters) and the control steering post at the stern for two persons. The silhouette of the ship is presented in figure 5.

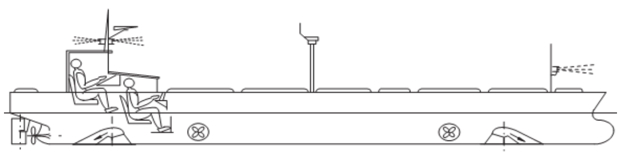

Fig. 5. Silhouette of "Blue Lady" with cockpit arrangement and GPS antenna

More details about presented vessel one can find in [10]. Finally, the multivariable linear state model of the system has the following form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} a_{uu} & 0 & 0 \\ 0 & a_{vv} & a_{rv} \\ a_{ur} & a_{vr} & a_{rr} \end{bmatrix} * \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} b_{uu} & 0 & 0 \\ 0 & b_{vv} & b_{rv} \\ b_{ur} & b_{vr} & b_{rr} \end{bmatrix} * \begin{bmatrix} \tau_x \\ \tau_y \\ \tau_p \end{bmatrix} \quad (17)$$

$$\begin{bmatrix} u \\ v \\ r \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad (18)$$

where 0 values denoted canceled channels. The coeffcient values of the model parameters were obtained as average values from all identification experiments. State model coefficients were as follows [4]. Controlled object matrix in the form presented in (17) and (18) can be inserted to LMI matrix description from (13) and (14). A block diagram of the controlled system has been presented in figure 6.

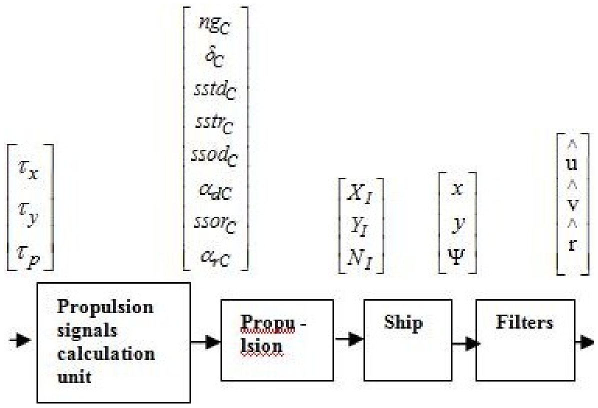

Fig. 6. Block diagram of the controlled system for identification process u, v, r – ships velocities, x, y, ψ – ships position and course, $[ng_c, \delta_c]^T$ vectors – command signals for propulsion and steering equipment, $[X_I, Y_I, N_I]^T$ vectors – forces and moments generated by propulsion and steering equipment

Training ship "Blue Lady", being the controlled system in the considered case, has three input signals $[\tau_x, \tau_y, \tau_p]$ where τ_x – required force (thrust) in the ships longitudinal axis, τ_y – required force (thrust) in the ships lateral axis, τ_p – required turning moment. Taking into consideration the number and type of propellers eight command signals for propulsion and steering equipment are implemented $[ng_c, \delta_c, ..., \alpha_c]$. Next $[X_1, Y_1, N_1]$ are forces and moments created by propulsion and steering equipment, and the three output signals are position coordinates $x(t)$, $y(t)$, and the heading $\psi(t)$. It turned out during identification process, that three signal channels demonstrated weak correlation between output and input signals: $\tau_x \Rightarrow \hat{u}, \tau_y \Rightarrow \hat{v}, \tau_p \Rightarrow \hat{r}$.

Case study

Linear matrix inequalities, their properties and successive stages allow to determine dynamic parameters for a space state controller for a multivariable controlled object. In this paper one of the steps has been shown, which is defining the region in the left half plane of the complex variable plane s, to achieve control system stability. In LMI zeros and poles can be placed in a user defined region. For the input and calculation of the above parameters with the LMI method the LMI Control Toolbox in Matlab, together with additional toolboxes SeDuMi (Self – Dual – Minimization) and YALMIP (Yet Another LMI Preprocessor), was used [11, 12]. Using the above tools computer simulations have been performed that show the influence of region selection on control system performance. Detailed description of the simulation station can be found in [13]. The algorithm implemented in Matlab places zeros and poles in an optimal way in the area defined by the user. So the user does not decide on the position of zeros and poles of the LMI controller directly but only defines the bounds of the required region. Nevertheless region selection is done on empiric trial and error basis, and for the moment, to the knowledge of the author, no mathematical methods for region shape or size calculation exist to achieve optimal controller operation. Below results of simulations are presented that show region selection influence on control quality for longitudinal velocity (u), lateral velocity (v) and angular velocity (r) of "Blue Lady" ship model. The first region was defined as a sector with two vertical stripes where $\alpha_1 = -0.54$, $\alpha_2 = -5.54$ and the angle $\phi = \pi/3$.

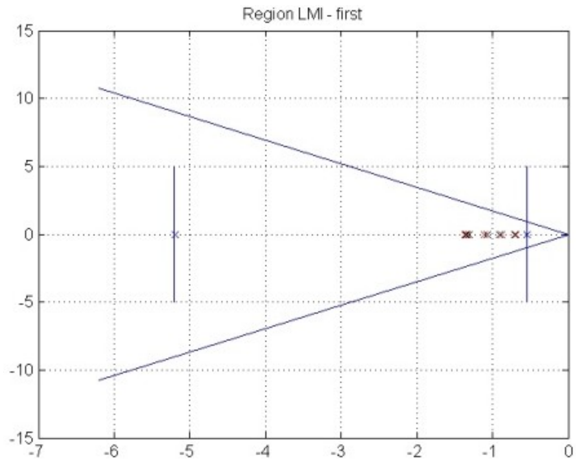

Fig. 7. First LMI region $\alpha_1 = -0.54$, $\alpha_2 = -5.54$ and angle $\phi = \pi/3$

With the above defined region further steps of LMI controller synthesis for multivariable "Blue Lady" ship model have been performed. The first simulation shows only one constant value which is longitudinal velocity $u = 0.1$ m/s, the dynamics of the controller after the maneuver has been performed can be seen on figure 8a. Additionally simulations for the same region have been made for three constant values of $u = 0.1$ m/s, $v = 0.08$ m/s and $r = -0.3$ rad/s, which can be seen on figure 8b. Conducted experiments show that the multivariable controller, for longitudinal, lateral and angular velocities whose poles are placed in the region defined as (Fig. 7) is stable and the required velocity values are obtained after a time of no more than 200 s.

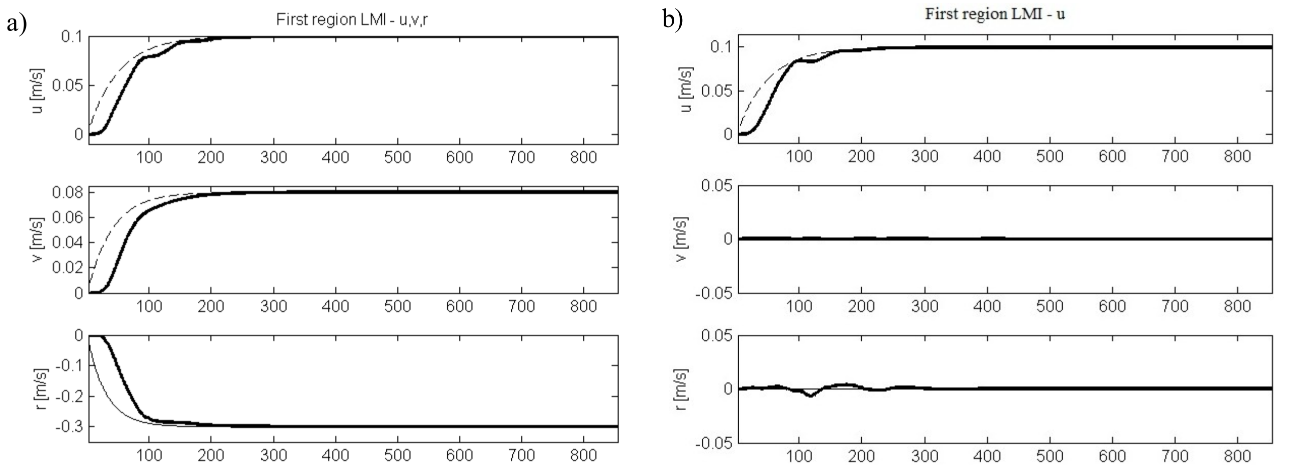

Fig. 8. a) given value $u = 0.1$ m/s – thin line, controlled value – thick line, remaining velocities = 0; b) given values $u = 0.1$ m/s, $v = 0.08$ m/s and $r = -0.3$ rad/s – thin line, controlled values – thick line

And for the above defined region shown in figure 9 a state space controller has been synthesized whose dynamics for the same set values as in the previous simulation, has been shown in figure 10.

The controller whose poles are placed in the region shown in figure 7 works the same way as the one with poles placed in the region shown in figure 9. It is also stable with the required velocity values being obtained after a time of no more than 250 s.

Next parameters have been calculated by trial and error for a region shown in figure 9 where $r = 3$ and $q = -3.54$.

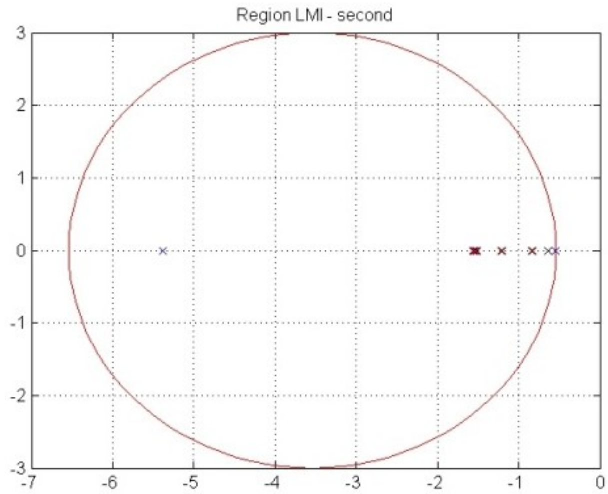

Fig. 9. Second LMI region: $r = 3$, $q = -3.54$

Conclusions

Operation of controllers, whose poles are placed in regions shown in figures 7 and 9, is the same because their parameters and calculated algorithmically are the same. It is clearly visible that in the case of a multivariable controller that is controlling ship velocities the key is defining region vertical bounds. It is related to the fact that imaginary parts of poles calculated algorithmically in the s plane are always equal to zero. And as such both in the case of a region defined as a section with two vertical bars and as a sphere the user has only defined the range, in which real parts of the poles can

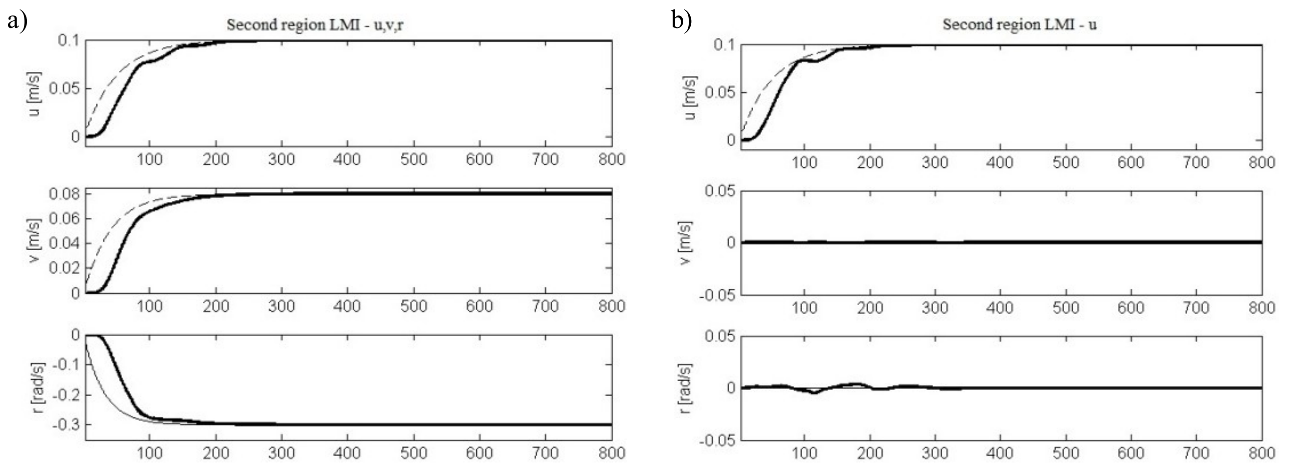

Fig. 10. a) given value $u = 0.1$ m/s – thin line, controlled value – thick line, remaining velocities = 0; b) given values $u = 0.1$ m/s, $v = 0.08$ m/s and $r = -0.3$ rad/s – thin line, controlled values – thick line

change. This indicates that in the presented LMI controller the selection of regions between the one shown in figure 7 and the one shown in figure 9 has no direct influence on the control of specific parameters like damping time, sample time or overshoot. Changing region bounds, or in other words the allowable range of real parts of the poles, on the other hand has a significant influence on control performance in a closed loop system. Since the regions are selected by trial and error it is necessary to conduct several simulations. Only then the user can decide which region corresponds to all controller parameters from initial design criteria. When selecting a region is essential to observe what form do the algorithmically calculated poles take and which of the bounds is the active one, meaning for which of the bounds do the poles take place on its limit. The above simulations indicate that modifying the bounds that do not have any influence on the algorithmically calculated pole values, the non active bounds does not have any influence on the operation of the control system. For different multivariable controlled objects modifying active bounds on the other hand has a significant influence on the way how the controller behaves. It needs to be mentioned that for diffrent controlled objects zeros and poles will be located diffrently. So a modification of other bounds (eg. angle of flare) than in this paper might be key to achieving correct results.

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