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MATHEMATICAL MODEL ANALYSIS OF SAMPLE FROM POLYURETHANE VIBROINSULATION MAT

Summary: The paper presents further developments in mathematical modelling of vibroinsulation mats. There was proposed the introduction of new parameters for characterizing the properties of the mat. An attempt to analyze the model in terms of its dynamic properties resulting from the determination of frequency transfer function was made. In conclusion the methodology of the studies necessary to quantify the material constants model was described.

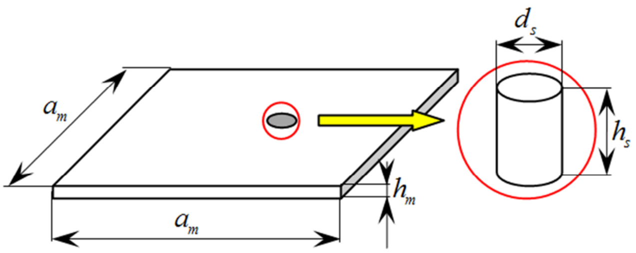
Key words: vibroinsulation mat, loss modulus, storage modulus, mathematical model

1. INTRODUCTION

Vibroinsulation mats used for loaded track railways must meet appropriate standards for static and dynamic properties. The tests are carried out in laboratory using a testing machine, working at specified frequencies. The test specimens have a prismatic shape of plan dimensions 500 × 500 mm and a height corresponding to the thickness of the final product.

Due to lack of access to this type of equipment, author decided to adopt an instrument DMA 242D Netzsh for that purpose. In this case, the test sample takes the form of a cylinder, height up to 6 mm.

Figure 1 shows symbolically the dimensions and geometry of the produced vibroinsulation mats and analyzed sample cut from the specimen.



$$A_m = a_m \cdot a_m$$
$$A_s = \frac{\pi d_s^2}{4}$$
$$A_m \gg A_s$$
$$h_m \approx h_s$$

Fig. 1. Manufactured mat and a cut out (manufactured) sample, A_m – area of polyurethane mas, A_s – area of cylindrical sample

Rys. 1. Wyprodukowana mata oraz wycięta (wyprodukowana) próbka, A_m – powierzchnia maty poliuretanowej, A_s – powierzchnia próbki cylindrycznej

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It is necessary to develop methods for transfer of results obtained by testing samples of small size, disproportionate to the final product, which can be used process the results obtained from the available DMA device. The first step in this direction is to develop a mathematical model of tested sample.

2. MATHEMATICAL MODEL ANALYSIS

a. Analysis of limitations – the simplifying postulates

Using of classical similarity method is not possible in the present case because the basic condition for their application is geometric similarity of the final product and the sample. As shown in Figure 1, this condition cannot be achieved due to two reasons: difference of shape (prismatic and cylindrical) and impossibility to provide the scale of the similarity between two corresponding linear dimensions. The shape and dimensions of the test samples are imposed by available measuring equipment. Testing device is DMA 242 D produced by NETZSCH company. The maximum sample radius results from the limited hardware possibilities, accepted standards and the following relationship:

$$F_{\max} = \sigma_{\max} \cdot A_s \tag{1}$$

where:

$F_{\max}$ – maximum force possible to realize on the DMA 242D, in this case 7,272 N,

$\sigma_{\max}$ – maximum stress range $N \cdot m^{-2}$, according to DIN 45673-5:2010-08,

A_s – sample area, mm^2.

From equation (1) the maximum radius of the cylindrical sample can be determined:

$$r = \sqrt{\frac{2 \cdot F_{\max}}{\pi \cdot \sigma_{\max}}} \approx 13,6\,\mathrm{mm} \tag{2}$$

Due to the complexity of created mathematical model [1], it was important to determine and apply a series of simplifications, which allow specify the constituent parameters, based on the subsequent experimental studies. Vibroinsulation mat is porous, roughly one half of the pores is open and the other one closed. The mat is coated on its whole surface by a layer of a clearly increased density, called "skin".

It was assumed that the structure of the material is isotropy and homogeneity. Its material properties strongly depend on temperature, and therefore temperature has to be stabilized during experimental studies. In the first approximation any relationships take the linear character. The stress distribution in the sample is only uniaxial.

b. The solution of the equations describing the model

As noted in the abstract the mathematical model of sample was shown in paper [1].

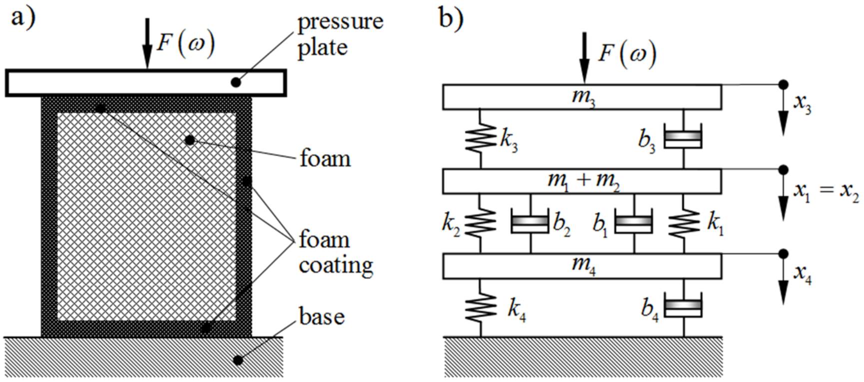

Fig. 2. Sample tested: a) cross-section of the actual sample, b) created physical model
Rys. 2. Badana próbka: a) przekrój próbki rzeczywistej, b) utworzony model fizyczny

Based on Figure 2, it is possible to write the equations of motion [1]:

$$\begin{aligned}
&m_3\ddot{x}_3 + b_3(\dot{x}_3-\dot{x}_1)+k_3(x_3-x_1)=F(\omega)\\
&(m_1+m_2)\ddot{x}_1+(b_1+b_2)(\dot{x}_1-\dot{x}_4)+(k_1+k_2)(x_1-x_4)=b_3(\dot{x}_3-\dot{x}_1)+k_3(x_3-x_1)\\
&m_4\ddot{x}_4+b_4\dot{x}_4+k_4x_4=(b_1+b_2)(\dot{x}_1-\dot{x}_4)+(k_1+k_2)(x_1-x_4)
\end{aligned} \tag{3}$$

where:

x_i – displacement of the upper layer of the sample element, m,
m_i – mass, kg,
b_i – damping coefficient, Ns·m^{-2},
k_i – stiffness, N·m^{-2},
$F(\omega)$ – driving force, according to DIN 45673-5, N,
$i=1,2,3,4$ the index that means respective: fundamental (porous) part of the sample, upper, side and lower skin layer.

It was assumed that the displacements are calculated from the static stable balance point. This assumption allow an adoption of zero initial conditions $x_i(0)=0$ and $\dot{x}_i(0)=0$. After applying to the (3) Laplace transform and rearranging data formula was given by:

$$\begin{aligned}
&m_3s^2X_3+(b_3s+k_3)(X_3-X_1)=F(s)\\
&(m_1+m_2)s^2X_1+\left[(b_1+b_2)s+(k_1+k_2)\right](X_1-X_4)=(b_3s+k_3)(X_3-X_1)\\
&(m_4s^2+b_4s+k_4)X_4=\left[(b_1+b_2)s+(k_1+k_2)\right](X_1-X_4)
\end{aligned} \tag{4}$$

where:

$X_i = X(s)$ means the Laplace transformed outputs (displacements), m,

$F(s)$ – the Laplace transformed input (driving force), N,

s – Laplace operator, $[s^{-1}]$, other symbols as in formula (3).

The system defined by equations (3) can be written in matrix form

$$\begin{bmatrix} -(b_3 s + k_3) & m_3 s^2 + b_3 s + k_3 & 0 \\ (m_1 + m_2)s^2 + (b_1 + b_2 + b_3)s + k_1 + k_2 + k_3 & -(b_3 s + k_3) & -[(b_1 + b_2)s + k_1 + k_2] \\ -[(b_1 + b_2)s + (k_1 + k_2)] & 0 & m_4 s^2 + (b_1 + b_2 + b_4)s + k_1 + k_2 + k_4 \end{bmatrix} \begin{bmatrix} X_1 \\ X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} F(s) \\ 0 \\ 0 \end{bmatrix} \tag{5a}$$

or in short form:

$$\mathbf{A}_m \mathbf{X} = \mathbf{F} \tag{5b}$$

where:

$$\mathbf{A}_m = \begin{bmatrix} -(b_3 s + k_3) & m_3 s^2 + b_3 s + k_3 & 0 \\ (m_1 + m_2)s^2 + (b_1 + b_2 + b_3)s + k_1 + k_2 + k_3 & -(b_3 s + k_3) & -[(b_1 + b_2)s + k_1 + k_2] \\ -[(b_1 + b_2)s + (k_1 + k_2)] & 0 & m_4 s^2 + (b_1 + b_2 + b_4)s + k_1 + k_2 + k_4 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & 0 & a_{33} \end{bmatrix}$$

is a matrix of coefficients resulting from the material properties of particular elements of this sample

$\mathbf{X} = \begin{bmatrix} X_1 \\ X_3 \\ X_4 \end{bmatrix}$ is output quantity vector (in our case the images of deviations of the upper surface of the particular elements from stable balance

$\mathbf{F} = \begin{bmatrix} F(s) \\ 0 \\ 0 \end{bmatrix}$ is a vector of inputs, reduced in the present case to one driving force.

The value we are interested in: $X(s) = X_3(s)$, can be calculated from the equation

$$X(s) = X_3(s) = \frac{|\mathbf{A}_{m3}|}{|\mathbf{A}_m|} \tag{6}$$

where:

$|\mathbf{A}_m|$ is the main determinant of a matrix derived earlier $\mathbf{A}_m$,

$$|\mathbf{A}_{m3}| = \begin{vmatrix} -(b_3 s + k_3) & F(s) & 0 \\ (m_1+m_2)s^2 + (b_1+b_2+b_3)s + k_1+k_2+k_3 & 0 & -[(b_1+b_2)s+k_1+k_2] \\ -[(b_1+b_2)s+k_1+k_2] & 0 & m_4 s^2 + (b_1+b_2+b_4)s + k_1+k_2+k_4 \end{vmatrix} = \begin{vmatrix} a_{11} & F(s) & 0 \\ a_{21} & 0 & a_{23} \\ a_{31} & 0 & a_{33} \end{vmatrix}$$

is a auxiliary determinant associated with the variable $X(s) = X_3(s)$.

Therefore, the main determinant takes the form:

$$\begin{aligned} |\mathbf{A}_m| &= |a_{11}a_{22}a_{33} + a_{31}a_{12}a_{23} - a_{21}a_{12}a_{33}| = \\ &= (b_3 s + k_3)^2 \left[m_4 s^2 + (b_1+b_2+b_4)s + k_1+k_2+k_4 \right] + \\ &+ \left[(b_1+b_2)s + (k_1+k_2) \right]^2 \left(m_3 s^2 + b_3 s + k_3 \right) + \\ &- \left[(m_1+m_2)s^2 + (b_1+b_2+b_3)s + k_1+k_2+k_3 \right] \left(m_3 s^2 + b_3 s + k_3 \right) \\ &\left[m_4 s^2 + (b_1+b_2+b_4)s + k_1+k_2+k_4 \right] \end{aligned} \tag{7}$$

However, determinant $|\mathbf{A}_{m3}|$ has form:

$$\begin{aligned} |\mathbf{A}_{m3}| &= (a_{31}a_{23} - a_{21}a_{33}) F(s) = \\ &\Big\{ \left[(b_1+b_2)s + k_1+k_2 \right]^2 - \left[(m_1+m_2)s^2 + (b_1+b_2+b_3)s + k_1+k_2+k_3 \right] \\ &\left[m_4 s^2 + (b_1+b_2+b_4)s + k_1+k_2+k_4 \right] \Big\} F(s) \end{aligned} \tag{8}$$

The transfer function is defined as the ratio of the Laplace transform of the output $X(s)$ to the Laplace transform of the input $F(s)$ is given by (9):

$$G(s) = \frac{X(s)}{F(s)} \tag{9}$$

After substituting $s = i\omega$ for the Laplace operator, where ω is angular frequency of driving force $F(\omega)$), we get a spectral transfer function $G(i\omega)$, which can be shown as $G(i\omega)=\mathrm{Re}(G(\omega))+i\,\mathrm{Im}(G(\omega))$. The real part of $\mathrm{Re}(G(\omega))=G'$ is an equivalent of storage modulus E' and describes the ability to store potential energy and release it upon deformation, while the imaginary part of $\mathrm{Im}(G(\omega))=G''$ corresponds to loss modulus E'' proportional to energy dissipation in the form of heat upon deformation. The modules E' and E'' are known from dynamic mechanical analysis.

3. DECOMPOSITION OF THE SAMPLE

The basic problem is to determine the parameters m_i, b_i, k_i for $i=1,2,3,4$ While parameters m_i don't present difficulties, because $m_1=\rho_f V_1$ and $m_i=\rho_s V_i \quad i=2,3,4$, where ρ_f is density of foam and ρ_s is density of the skin, other parameters are difficult to identify because they depend on dimensions and shape of sample. But we can see that there are only two types of material: the porous foam and the skin.

As seen in Fig. 3, the sample components analyzed separately have different shape. Below is shown the proposal to introduce new universal properties: characteristic damping coefficient $\hat{b}$ and characteristic stiffness $\hat{k}$ related to the unit volume of material. Then the damping coefficient and stiffness for each element are defined by formulas:

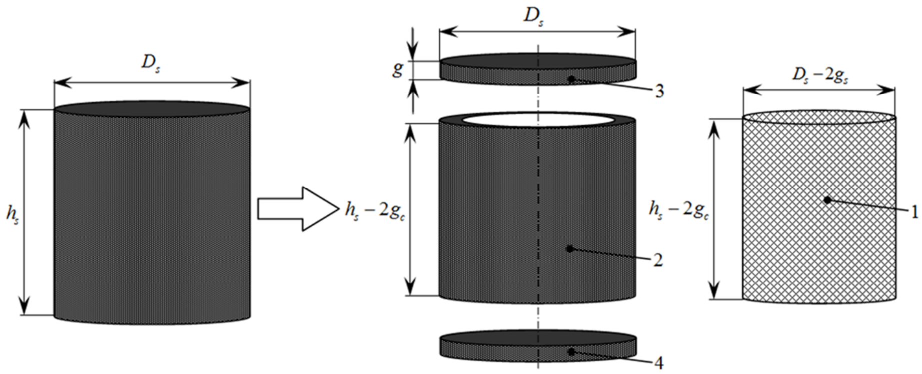

Fig. 3. Possible decomposition of the sample to the basic elements
Rys. 3. Możliwa dekompozycja próbki na podstawowe elementy

$$b_i=\hat{b}\frac{\hat{A}_i}{\hat{h}_t} \quad \mathrm{N\cdot s\cdot m^{-1}} \tag{10}$$

$$k_i=\hat{k}\frac{\hat{A}_i}{\hat{h}_t} \quad \mathrm{N\cdot m^{-1}} \tag{11}$$

where in formulas (10) and (11):

b_i – damping coefficient of *i-th* element of the sample, N·s·m^{-1},

k_i – stiffness of *i-th* element of the sample, N·m^{-1},

$\hat{h}_i = \frac{h}{h_0}$ $h_0 = 1\text{m}$ – relative height *i-th* element of the sample, –,

$\hat{A}_i = \frac{A}{A_0}$ $A_0 = 1\text{m}^2$ – relative area *i-th* element of the sample, –,

h – real height of *i-th* element of the sample, measured along the axis of driving force, m,

A – real surface perpendicular to the direction of the driving force of *i-th* element of the sample, m^2.

Possibilities of manufacturing different samples are presented in Fig. 4. Distribution of the skin on tested sample depends on whether it is cut or manufactured. Different samples allow to determine the material properties by the experimental methods for porous foam and skin.

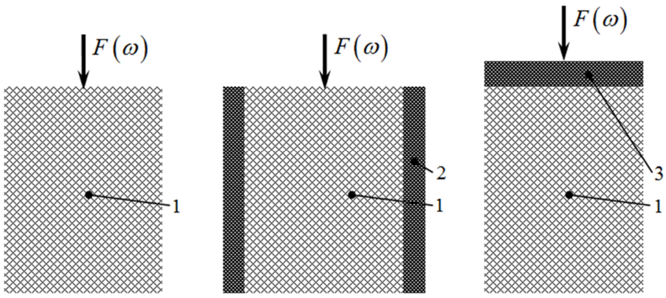

Fig. 4. The possibilities of skin position in the sample
Rys. 4. Możliwości rozmieszczenia lica próbki

Equations (3) for simplified samples presented in Fig. 4 are reduced equations as written below, respectively:

$$m_1\ddot{x}_1 + b_1\dot{x}_1 + k_1x_1 = F(\omega t) \tag{12}$$

$$(m_1 + m_2)\ddot{x}_1 + (b_1 + b_2)\dot{x}_1 + (k_1 + k_2)x_1 = F(\omega t) \tag{13}$$

$$\begin{aligned} m_3\ddot{x}_3 + b_3(\dot{x}_3 - \dot{x}_1) + k_3(x_3 - x_1) &= F(\omega t) \\ m_1\ddot{x}_1 + b_1\dot{x}_1 + k_1x_1 &= b_3(\dot{x}_3 - \dot{x}_1) + k_3(x_3 - x_1) \end{aligned} \tag{14}$$

$$\begin{aligned} m_3(\ddot{x}_3 - \ddot{x}_1) + b_3(\dot{x}_3 - \dot{x}_1) + k_3(x_3 - x_1) &= F(\omega t) \\ (m_1 + m_2)\ddot{x}_1 + (b_1 + b_2)\dot{x}_1 + (k_1 + k_2)x_1 &= b_3(\dot{x}_3 - \dot{x}_1) + k_3(x_3 - x_1) \end{aligned} \tag{15}$$

In general, to determine the four parameters $\hat{k}_1$, $\hat{k}_2 = \hat{k}_3 = \hat{k}_4$, $\hat{b}_1$, and $\hat{b}_2 = \hat{b}_3 = \hat{b}_4$ it is sufficient to study only two types of samples because the obtained transfer functions are complex numbers and each of them enables determination of the two properties. Studies of the other two types of samples will be used to verify the assumptions.

4. CONCLUSIONS

The obtained mathematical model of the sample is linear. At the present stage of research it is difficult to evaluate the degree of its real nonlinearity. It seems that at first should be tested the simplest type of samples (consisting only of specific part of mats – porous, without skin) resultant from decomposition of the basic sample. For the model of this sample defined analytically it will be possible to determine experimentally the parameters b_1 and k_1 for different angular frequency ω defined in the standard. Differences from the constants, parameters b_1, and k_1 will be a measure of the nonlinearity of the model (its inaccuracy) and allow evaluation error during test.

In the next steps of research should be verified the correctness of the assumptions made in the formulas (10) and (11). In the studies will be used samples with different cross sections and heights. On them will be applied the force with constant circular frequency.

The model presented in this paper is the first linear approximation, but allows understanding of the future the research program and familiarization with the essence of the problem.

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ANALIZA MODELU MATEMATYCZNEGO PRÓBKI MATY WIBROIZOLACYJNEJ

Streszczenie: W pracy zaprezentowano kolejny etap rozwoju modelu matematycznego poliuretanowej maty wibroizolacyjnej. Zaproponowano wprowadzenie nowych parametrów chrakteryzujących własności maty. Podjęto próbę analizy modelu ze względu na jego własności dynamiczne wynikające z określenia transmitancji częstotliwościowej. W podsumowaniu opisano metodykę badań koniecznych do ilościowego określenia stałych materiałowych modelu.

Słowa kluczowe: mata wibroizolacyjna, moduł stratności, moduł zachowawczy, model matematyczny