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FRACTIONAL NON CONFORMABLE HERMITE-HADAMARD INEQUALITIES FOR GENERALIZED ϕ-CONVEX FUNCTIONS

Abstract. In this paper, we present new inequalities of the Hermite–Hadamard type for generalized ϕ-convex functions, within the framework of non conformable fractional integrals.

Key words: fractional derivatives and integrals, fractional integral inequalities, generalized convex mappings.

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1. Introduction

The classic Mittag–Leffler function, it plays an active role in fractional calculus, is defined by

$$\mathbf{E}_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+\alpha k)}, \quad \alpha \in C, \quad Re(\alpha) > 0, \quad z \in \mathbb{C},$$

where Γ is the well known Gamma function. The original function $E_{\alpha,1}(z) = E_\alpha(z)$ was defined and studied by Mittag-Leffler in the year 1903, that is, a uniparameter function, see [9, 10]. It is a direct generalization of the exponential function. Wiman proposed and studied a generalization of the role of Mittag–Leffler, who we'll call it the Mittag–Leffler function with two parameters $E_{\alpha,\beta}(z)$, (see [16]), Agarwal in 1953 and Humbert and Agarwal in 1953, also made contributions to the final formalization of this function, see also [3].

Now we define generalized ϕ-convex sets and functions as follows:

Definition 1. *A non empty set $\mathcal{K}$ is called generalized ϕ-convex, if*

$$u + \imath \mathbf{E}_\alpha(v-u) \in \mathcal{K}$$

holds for all $u, v \in \mathcal{K}$ and $\imath \in [0,1]$.

Definition 2. *A function $h : \mathcal{K} \subseteq \mathbb{R} \to \mathbb{R}$ is said to be generalized ϕ-convex on a generalized ϕ-convex set $\mathcal{K}$, if the inequality*

$$h(u + \imath \mathbf{E}_\alpha(v-u)) \leq \imath h(v) + (1-\imath)h(u)$$

holds for all $u, v \in \mathcal{K}$ and $\imath \in [0,1]$ with $\mathbf{E}_\alpha(\cdot)$ the Mittag-Leffler function. We say that h is ϕ-concave if $-h$ is ϕ-convex.

Remark 1. 1). If $\mathcal{F}$ is an increasing function, and considering that $e^z \geq z$ it is clear that, if $\mathcal{F}$ is ϕ-convex then it is convex.

2). $h(z) = z^{-1/2}$ is a ϕ-convex and it is not convex.

3). This result is still valid in the case of a general concave function.

Now, we give the definition of the non conformable fractional derivative with its important properties which are useful in order to obtain our main results, which is explained in the following definition:

Definition 3. *Given a function $h : [0, +\infty) \to \mathbb{R}$. Then, the non conformable fractional derivative of h of order υ at $\imath$ is defined by*

$$N_4^\upsilon(h)(\imath) = \lim_{\varepsilon \to 0} \frac{h\left(\imath + \varepsilon e^{\imath^\upsilon}\right) - h(\imath)}{\varepsilon}, \quad \upsilon \in (0,1), \quad \imath > 0.$$

If h is υ–differentiable in some $(0, \upsilon)$, $\upsilon > 0$, $\lim_{\imath \to 0^+} h^{(\upsilon)}(\imath)$ exist, then define

$$h^{(\upsilon)}(0) = \lim_{\imath \to 0^+} h^{(\upsilon)}(\imath).$$

Additionally, note that if h is differentiable, then

$$N_4^\upsilon(h)(\imath) = e^{\imath^\upsilon} h'(\imath), \quad \text{where } h'(\imath) = \lim_{\epsilon \to 0} \frac{h(\imath + \epsilon) - h(\imath)}{\epsilon}. \tag{1}$$

Remark 2. The adjective conformable may or may not be appropriate here, since this was initially referred to as a conformable fractional derivative $D_\upsilon h(\imath)$, when $\upsilon \to 1$ fulfill $D_\upsilon h(\imath) \to h'(\imath)$; i.e., the conformable derivative preserves the angle of the tangent line to the curve, while in the above definition, taking into account (1), this angle is not conserved (more details and examples on non conformable derivative can be found at [4]). On the other hand, [1] presents a classification of most derivatives called fractional, very useful for gathering more information.

Following the ideas presented in [4, 12], we can easily prove the next result.

Theorem 1. *Let $\upsilon \in (0,1]$ and h, g be υ–differentiable at a point $\imath > 0$. Then*

- $N_4^{\upsilon}(uh+vg) = uN_4^{\upsilon}(h) + vN_4^{\upsilon}(g)$ *for all* $u, v \in \mathbb{R}$,
- $N_4^{\upsilon}(hg) = N_4^{\upsilon}(g) + gN_4^{\upsilon}(h)$,
- $N_4^{\upsilon}\left(\frac{h}{g}\right) = \frac{hN_4^{\upsilon}(g) - gN_4^{\upsilon}(h)}{g^2}$,
- $N_4^{\upsilon}(c) = 0$ *for all constant function* $h(\imath) = c$,
- $N_4^{\upsilon}\left(-\frac{1}{\upsilon}\Gamma\left(\frac{1}{\upsilon}, \imath^{\upsilon}\right)\right) = 1$.

Now, we give the definition of non conformable fractional integral.

Definition 4. *Let* $\upsilon \in (0,1]$ *and* $0 \leq u < v$. *We say that a function* $\mathcal{F} : [\wp_1, \wp_2] \to \mathbb{R}$ *is* υ*–fractional integrable on* $[\wp_1, \wp_2]$, *where* $\mathcal{F} \in \mathbf{L}_{\upsilon}[\wp_1, \wp_2]$, *if the integral*

$$_{N_4}J_u^{\upsilon}\mathcal{F}(x) = \int_u^x \frac{\mathcal{F}(\imath)}{e^{\imath^{\upsilon}}} d\imath$$

exists and is finite.

From the above, we can define the following integrals, which will play an important role in our work.

Definition 5. *Suppose that* $h \in \mathbf{L}_{\upsilon}[\wp_1, \wp_2]$. *The quasi left and right* N*–fractional integrals are defined by*

$$_{N_4}J_{\wp_1+}^{\upsilon}\mathcal{F}(x) = \int_{\wp_1}^x (x-\imath)^{-\upsilon} e^{-\left(\frac{x-\imath}{\wp_2-\wp_1}\right)^{\upsilon}} \mathcal{F}(\imath) d\imath, \quad x > \wp_1$$

and

$$_{N_4}J_{\wp_2-}^{\upsilon}\mathcal{F}(x) = \int_x^{\wp_2} (\imath - x)^{-\upsilon} e^{-\left(\frac{\imath - x}{\wp_2-\wp_1}\right)^{\upsilon}} \mathcal{F}(\imath) d\imath, \quad \wp_2 > x,$$

respectively.

Remark 3. To facilitate the reading of the work, we present some examples of non conformable integrals of certain elementary functions.

1.

$$N_4 \ J_0^{\frac{1}{2}}\left[x^p\right] = \int_0^1 \frac{x^p}{e^{\sqrt{x}}} \, dx = -2\left[\Gamma(2(p+1),1) - \Gamma(2(p+1),1)\right].$$

2.

$$N_4 \ J_{1+}^{\frac{2}{3}}\left[\sin^2 x\right] = \int_1^2 (2-x)^{-\frac{2}{3}} \ e^{-(2-x)^{\frac{2}{3}}} \ sin^2 x \, dx = 1.98194115.$$

3.

$$N_4 \ J_{0+}^{\frac{1}{5}}\left[\sin 2x\right] = \int_0^{\frac{1}{2}} \left(\frac{1}{2} - x\right)^{-\frac{1}{5}} \ e^{-\sqrt[5]{1-2x}} sin 2x \, dx = 0.17758203.$$

4.

$$N_4\ J^{\frac{1}{5}}_{4+}\left[\ x^2\right] = \int_4^5 \ (5-x)^{-\frac{1}{5}}\ e^{-\sqrt[5]{5-x}} x^2\, dx = 12.11343529.$$

Inequalities of the Hermite-Hadamard type play a prominent role in the theory of convex functions, see [5, 2]. If $\mathcal{F} : I \to \mathbb{R}$ is a convex function on the interval I, then for any $\wp_1, \wp_2 \in I$ with $\wp_1 < \wp_2$, we have the following double inequality:

$$\mathcal{F}\left(\frac{\wp_1+\wp_2}{2}\right) \leq \frac{1}{\wp_2-\wp_1}\int_{\wp_1}^{\wp_2} \mathcal{F}(\imath)d\imath \leq \frac{\mathcal{F}(\wp_1)+\mathcal{F}(\wp_2)}{2}. \tag{2}$$

In recent years, this inequality has been generalized to conformable fractional integrals, some results for convex functions can be consulted in [6, 7, 13, 8, 14, 15, 17, 18].

The aim of our article is to establish some new inequalities connected with the Hermite-Hadamard inequalities via quasi left and right N-fractional integral in Definition 4.

2. Main results

We will start with the following identity that will be useful in sequel.

Lemma 1. *Let $\upsilon \in (0,1)$, $\mathcal{F} : [\wp_1, \wp_2] \to [0, +\infty)$ be a differentiable function defined on $[\wp_1, \wp_2]$, with $0 < \wp_1 < \wp_2$. If $\mathcal{F}' \in \mathbf{L}_\upsilon [\wp_1, \wp_2]$, then*

$$\frac{(\upsilon-1)}{(\wp_2-\wp_1)^{2-\upsilon}}\left[{}_{N_4}J^{\upsilon}_{\wp_2^-}\mathcal{F}(\wp_1) + {}_{N_4}J^{\upsilon}_{\wp_1}\mathcal{F}(\wp_2)\right] + \frac{(e-1)(\mathcal{F}(\wp_2)+\mathcal{F}(\wp_1))}{e(\wp_2-\wp_1)}$$
$$= \int_0^1 \left[e^{-(1-\imath)^\upsilon} - e^{-\imath^\upsilon}\right]\mathcal{F}'(\wp_1\imath + (1-\imath)\wp_2)d\imath.$$

Proof. We can write I as follows:

$$\begin{aligned} I &= \int_0^1 \left[e^{-(1-\imath)^\upsilon} - e^{-\imath^\upsilon}\right]\mathcal{F}'(\wp_1\imath + (1-\imath)\wp_2)d\imath \\ &= \int_0^1 e^{-(1-\imath)^\upsilon}\mathcal{F}'(\wp_1\imath + (1-\imath)\wp_2)\, d\imath - \int_0^1 e^{-\imath^\upsilon}\mathcal{F}'(\wp_1\imath + (1-\imath)\wp_2)\, d\imath. \end{aligned}$$

Integrating by parts the first integral, we obtain

$$\begin{aligned}
&\int_0^1 e^{-(1-\imath)^{\upsilon}}\mathcal{F}'(\wp_1\imath+(1-\imath)\wp_2)\,d\imath\\
&\quad=\frac{1}{\wp_2-\wp_1}\left(\mathcal{F}(\wp_1)-e^{-1}\mathcal{F}(\wp_2)\right)\\
&\quad\quad-\frac{(1-\upsilon)}{(\wp_2-\wp_1)}\int_0^1 e^{-(1-\imath)^{\upsilon}}\mathcal{F}(\wp_1\imath+(1-\imath)\wp_2)d\imath\\
&\quad=\frac{1}{\wp_2-\wp_1}\left(\mathcal{F}(\wp_1)-e^{-1}\mathcal{F}(\wp_2)\right)\\
&\quad\quad+\frac{(1-\upsilon)}{(\wp_2-\wp_1)^{2-\upsilon}}\int_0^1 (x-\wp_1)^{-\upsilon}e^{-\left(\frac{x-\wp_1}{\wp_2-\wp_1}\right)^{\upsilon}}\mathcal{F}(x)dx\\
&\quad=\frac{1}{\wp_2-\wp_1}\left(\mathcal{F}(\wp_1)-e^{-1}\mathcal{F}(\wp_2)\right)+\frac{(1-\upsilon)}{(\wp_2-\wp_1)^{2-\upsilon}}\quad {}_{N_4}J^{\upsilon}_{\wp_2^-}\mathcal{F}(\wp_1).
\end{aligned}$$

Of the second, we have

$$\begin{aligned}
\int_0^1 e^{-\imath^{\upsilon}}\mathcal{F}'(\wp_1\imath+(1-\imath)\wp_2)\,d\imath\;=\;&\frac{1}{\wp_2-\wp_1}\left(\mathcal{F}(\wp_2)-e^{-1}\mathcal{F}(\wp_1)\right)\\
&+\frac{(1-\upsilon)}{(\wp_2-\wp_1)^{2-\upsilon}}\quad {}_{N_4}J^{\upsilon}_{\wp_1^+}\mathcal{F}(\wp_2).
\end{aligned}$$

From these results we obtain the required equality. ■

Corollary 1. *Under the conditions of the previous lemma, if $\mathcal{F}'$ is an increasing and generalized ϕ–convex function we have*

$$\begin{aligned}
&\frac{(\upsilon-1)}{(\wp_2-\wp_1)^{2-\upsilon}}\left[{}_{N_4}J^{\upsilon}_{\wp_2^-}\mathcal{F}(\wp_1)+{}_{N_4}J^{\upsilon}_{\wp_1}\mathcal{F}(\wp_2)\right]+\frac{(e-1)(\mathcal{F}(\wp_2)+\mathcal{F}(\wp_1))}{e(\wp_2-\wp_1)}\\
&\quad\leq\int_0^1\left[e^{-(1-\imath)^{\upsilon}}-e^{-\imath^{\upsilon}}\right]\mathcal{F}'(\wp_2+\imath\mathbf{E}_{\alpha}(\wp_1-\wp_2))d\imath.
\end{aligned}$$

Theorem 2. *Let $\upsilon\in(0,1)$ and $\mathcal{F}:[\wp_1,\wp_2]\to[0,+\infty)$ be a differentiable function. If $\mathcal{F}'\in\mathbf{L}_{\upsilon}[\wp_1,\wp_2]$ and increasing function, then*

$$\frac{(1-\upsilon)}{(\wp_2-\wp_1)^{2-\upsilon}}\left[{}_{N_4}J^{\upsilon}_{\wp_2^-}\mathcal{F}(\wp_1)+{}_{N_4}J^{\upsilon}_{\wp_1}\mathcal{F}(\wp_2)\right]\leq\frac{(e-1)(\mathcal{F}(\wp_2)+\mathcal{F}(\wp_1))}{e(\wp_2-\wp_1)}.$$

Proof.

$$\begin{aligned}
&\int_0^1\left[e^{-(1-\imath)^{\upsilon}}-e^{-\imath^{\upsilon}}\right]\mathcal{F}'(\wp_2+\imath\mathbf{E}_{\alpha}(\wp_1-\wp_2))d\imath\\
&\quad=\int_0^{1/2}\left[e^{-(1-\imath)^{\upsilon}}-e^{-\imath^{\upsilon}}\right]\mathcal{F}'(\wp_2+\imath\mathbf{E}_{\alpha}(\wp_1-\wp_2))d\imath
\end{aligned}$$

$$+\int_{1/2}^{1}\left[e^{-(1-\imath)^{\upsilon}}-e^{-\imath^{\upsilon}}\right]\mathcal{F}'(\wp_2+\imath\mathbf{E}_{\alpha}(\wp_1-\wp_2))d\imath$$

$$=\int_{0}^{1/2}\left[e^{-(1-\imath)^{\upsilon}}-e^{-\imath^{\upsilon}}\right]\mathcal{F}'(\wp_2+\imath\mathbf{E}_{\alpha}(\wp_1-\wp_2))d\imath$$

$$+\int_{1/2}^{1}\left[e^{-(\imath)^{\upsilon}}-e^{-(1-\imath)^{\upsilon}}\right]\mathcal{F}'(\wp_1+\imath\mathbf{E}_{\alpha}(\wp_2-\wp_1))d\imath$$

$$=\int_{0}^{1/2}\left[e^{-(1-\imath)^{\upsilon}}-e^{-\imath^{\upsilon}}\right]$$

$$\times\left[\mathcal{F}'(\wp_2+\imath\mathbf{E}_{\alpha}(\wp_1-\wp_2))-\mathcal{F}'(\wp_1+\imath\mathbf{E}_{\alpha}(\wp_2-\wp_1))\right]d\imath.$$

Since the integrand is nonnegative, we obtain the desired inequality. ■

Corollary 2. *Under the assumptions of Theorem 2, taking $\mathcal{F}\leq K$, we get*

$$\frac{(1-\upsilon)}{(\wp_2-\wp_1)^{2-\upsilon}}\left[{}_{N_4}J^{\upsilon}_{\wp_2-}\mathcal{F}(\wp_1)+{}_{N_4}J^{\upsilon}_{\wp_1}\mathcal{F}(\wp_2)\right]\leq\frac{2K(e-1)}{e(\wp_2-\wp_1)}. \tag{3}$$

Theorem 3. *Let $\upsilon\in(0,1)$ and $\mathcal{F}:[\wp_1,\wp_2]\to[0,+\infty)$ be a differentiable function defined on $[\wp_1,\wp_2]$, with $0<\wp_1<\wp_2$. If $\mathcal{F}'\in\mathbf{L}_{\upsilon}[\wp_1,\wp_2]$, increasing and $|\mathcal{F}'|$ is a generalized ϕ–convex function, then*

$$\frac{(\upsilon-1)}{(\wp_2-\wp_1)^{2-\upsilon}}\left[{}_{N_4}J^{\upsilon}_{\wp_2-}\mathcal{F}(\wp_1)+{}_{N_4}J^{\upsilon}_{\wp_1}\mathcal{F}(\wp_2)\right]+\frac{(e-1)(\mathcal{F}(\wp_2)+\mathcal{F}(\wp_1))}{e(\wp_2-\wp_1)}$$
$$\leq\left(\Gamma\left(\frac{1}{\upsilon},0\right)-\Gamma\left(\frac{1}{\upsilon},1\right)\right)\left(\frac{|\mathcal{F}'(\wp_1)|+|\mathcal{F}'(\wp_2)|}{\upsilon}\right),$$

where $\Gamma(\wp_1,x)$ is the classical gamma function.

Proof. From Corollary 1, we have

$$\frac{(\upsilon-1)}{(\wp_2-\wp_1)^{2-\upsilon}}\left[{}_{N_4}J^{\upsilon}_{\wp_2-}\mathcal{F}(\wp_1)+{}_{N_4}J^{\upsilon}_{\wp_1}\mathcal{F}(\wp_2)\right]$$
$$+\frac{(e-1)(\mathcal{F}(\wp_2)+\mathcal{F}(\wp_1))}{e(\wp_2-\wp_1)}$$
$$\leq\int_{0}^{1}\left[e^{-(1-\imath)^{\upsilon}}-e^{-\imath^{\upsilon}}\right]\left|\mathcal{F}'(\wp_2+\imath\mathbf{E}_{\alpha}(\wp_1-\wp_2))\right|d\imath$$
$$\leq\int_{0}^{1}e^{-(1-\imath)^{\upsilon}}\left|\mathcal{F}'(\wp_2+\imath\mathbf{E}_{\alpha}(\wp_1-\wp_2))\right|d\imath$$
$$+\int_{0}^{1}e^{-\imath^{\upsilon}}\left|\mathcal{F}'(\wp_2+\imath\mathbf{E}_{\alpha}(\wp_1-\wp_2))\right|d\imath$$

$$\leq \int_0^1 e^{-(1-\imath)^{\upsilon}} \left(\imath \left|\mathcal{F}'(\wp_1)\right| + (1-\imath)\left|\mathcal{F}'(\wp_2)\right|\right) d\imath$$
$$+ \int_0^1 e^{-\imath^{\upsilon}} \left(\imath \left|\mathcal{F}'(\wp_1)\right| + (1-\imath)\left|\mathcal{F}'(\wp_2)\right|\right) d\imath.$$

After calculating both integrals, we obtain the desired result. ■

Corollary 3. *Under the assumptions of Theorem 3, taking* $|\mathcal{F}'| \leq K$, *we get*

$$(4)\quad \frac{(\upsilon-1)}{(\wp_2-\wp_1)^{2-\upsilon}}\left[{}_{N_4}J^{\upsilon}_{\wp_2^-}\mathcal{F}(\wp_1) + {}_{N_4}J^{\upsilon}_{\wp_1}\mathcal{F}(\wp_2)\right] + \frac{(e-1)(\mathcal{F}(\wp_2)+\mathcal{F}(\wp_1))}{e(\wp_2-\wp_1)}$$
$$\leq \frac{2K}{\upsilon}\left(\Gamma\left(\frac{1}{\upsilon},0\right) - \Gamma\left(\frac{1}{\upsilon},1\right)\right).$$

If the function $\mathcal{F}$ is generalized ϕ-convex, then the following result holds:

Theorem 4. *Let* $\upsilon \in (0,1)$ *and* $\mathcal{F} : [\wp_1, \wp_2] \to [0, +\infty)$ *be a generalized* ϕ*–convex and increasing function defined on* $[\wp_1, \wp_2]$, *with* $0 < \wp_1 < \wp_2$, *then*

$$(5)\quad \left(\frac{\upsilon}{\wp_2-\wp_1}\right) {}_{N_4}J^{\upsilon}_{\wp_1}\mathcal{F}(\wp_2)$$
$$\leq \min\left\{\mathcal{F}(\wp_2)(A-B) - \mathcal{F}(\wp_1)C, \mathcal{F}(\wp_1)(A-B) - \mathcal{F}(\wp_2)C\right\},$$

where

$$A = \Gamma\left(\frac{2}{\upsilon},1\right) - \Gamma\left(\frac{1}{\upsilon},1\right), \quad B = \Gamma\left(\frac{2}{\upsilon},0\right) - \Gamma\left(\frac{1}{\upsilon},0\right)$$

and

$$C = \Gamma\left(\frac{2}{\upsilon},1\right) - \Gamma\left(\frac{2}{\upsilon},0\right).$$

Proof. Taking into account that, on $[\wp_1, \wp_2]$, we have

$$(6)\quad \imath^{\upsilon} = \left(\frac{\imath}{\imath-\wp_1}\right)^{\upsilon}(\imath-\wp_1)^{\upsilon} \geq \left(\frac{\wp_1}{\wp_2-\wp_1}\right)^{\upsilon}(\imath-\wp_1)^{\upsilon},$$

$$(7)\quad \imath^{\upsilon} = \left(\frac{\imath}{\wp_2-\imath}\right)^{\upsilon}(\wp_2-\imath)^{\upsilon} \geq \left(\frac{\wp_1}{\wp_2-\wp_1}\right)^{\upsilon}(\wp_2-\imath)^{\upsilon}.$$

From (6) and (7), we obtain, respectively,

$$(8)\quad e^{-\imath^{\upsilon}} \leq e^{-\left(\frac{\wp_1}{\wp_2-\wp_1}\right)^{\upsilon}(\imath-\wp_1)^{\upsilon}},$$

$$e^{-\imath^{\upsilon}} \leq e^{-\left(\frac{\wp_1}{\wp_2-\wp_1}\right)^{\upsilon}(\wp_2-\imath)^{\upsilon}}. \tag{9}$$

From (8), we get

$$\begin{aligned}
{}_{N_4}J^{\upsilon}_{\wp_1}\mathcal{F}(\wp_2) &\leq (\wp_2-\wp_1)\int_{\wp_1}^{\wp_2}\mathcal{F}(\imath)e^{-\wp_1{}^{\upsilon}\left(\frac{\imath-\wp_1}{\wp_2-\wp_1}\right)^{\upsilon}}d\imath \\
&\leq (\wp_2-\wp_1)\int_0^1\mathcal{F}(\wp_1+(\wp_2-\wp_1)z)e^{-\wp_1{}^{\upsilon}z^{\upsilon}}dz \\
&\leq (\wp_2-\wp_1)\int_0^1\mathcal{F}(\wp_1+z\mathbf{E}_{\alpha}(\wp_2-\wp_1))e^{-z^{\upsilon}}dz \\
&\leq (\wp_2-\wp_1)\int_0^1\left[\mathcal{F}(\wp_1)z+(1-z)\mathcal{F}(\wp_2)\right]e^{-z^{\upsilon}}dz \\
&\leq \{\mathcal{F}(\wp_2)(A-B)-\mathcal{F}(\wp_1)C\}
\end{aligned}$$

Similarly, from (9), we have

$${}_{N_4}J^{\upsilon}_{\wp_1}\mathcal{F}(\wp_2) \leq \mathcal{F}(\wp_1)(A-B)-\mathcal{F}(\wp_2)C.$$

From the last inequality we obtain the desired result. ■

Corollary 4. *Under the assumptions of Theorem 4, taking $|\mathcal{F}'| \leq K$, we get*

$$\left(\frac{\upsilon}{\wp_2-\wp_1}\right)\,{}_{N_4}J^{\upsilon}_{\wp_1}\mathcal{F}(\wp_2) \leq K\left(\Gamma\left(\frac{1}{\upsilon},0\right)-\Gamma\left(\frac{1}{\upsilon},1\right)\right).$$

Theorem 5. *Let $\upsilon \in (0,1)$ and $\mathcal{F} : [\wp_1, \wp_2] \to [0, +\infty)$ be a generalized ϕ–convex and increasing function defined on $[\wp_1, \wp_2]$, with $0 < \wp_1 < \wp_2$, then*

$$\begin{aligned}
{}_{N_4}J^{\upsilon}_{\wp_1}\mathcal{F}(\wp_2) \leq\ & \frac{1}{\upsilon(\wp_2-\wp_1)^3} \\
& \times \left\{H(\mathcal{F}(\wp_2)-\mathcal{F}(\wp_1))-G(\wp_2\mathcal{F}(\wp_2)-\wp_1\mathcal{F}(\wp_1))\right\},
\end{aligned} \tag{10}$$

where

$$H=\Gamma\left(\frac{2}{\upsilon},\wp_2\right)-\Gamma\left(\frac{2}{\upsilon},\wp_1{}^{\upsilon}\right), \quad G=\Gamma\left(\frac{1}{\upsilon},\wp_2{}^{\upsilon}\right)-\Gamma\left(\frac{1}{\upsilon},\wp_1{}^{\upsilon}\right).$$

Proof. The change of variable $\imath = \wp_2 s + \wp_1(1-s)$ and the ϕ-convexity of $\mathcal{F}$, give

$$\begin{aligned}
\int_{\wp_1}^{\wp_2}\mathcal{F}(\imath)e^{-\imath^{\upsilon}}d\imath &= \frac{1}{(\wp_2-\wp_1)}\int_0^1\mathcal{F}\left(\wp_2 s+\wp_1(1-s)\right)e^{-(\wp_2 s+\wp_1(1-s))^{\upsilon}}ds \\
&\leq \frac{1}{(\wp_2-\wp_1)}\int_0^1\mathcal{F}\left(\wp_1+s\mathbf{E}_{\alpha}(\wp_2-\wp_1)\right)e^{-(\wp_2 s+\wp_1(1-s))^{\upsilon}}ds \\
&\leq \frac{1}{(\wp_2-\wp_1)}\int_0^1\left[\mathcal{F}(\wp_1)s+\mathcal{F}(\wp_2)(1-s)\right]e^{-(\wp_2 s+\wp_1(1-s))^{\upsilon}}ds.
\end{aligned}$$

Integrating the last expression, we obtain

$$\begin{aligned}\int_0^1 &[\mathcal{F}(\wp_1)s+\mathcal{F}(\wp_2)(1-s)]\, e^{-(\wp_2 s+\wp_1(1-s))^\upsilon} ds\\ &=\frac{1}{\upsilon(\wp_2-\wp_1)^2}\left[\left(\Gamma\left(\frac{2}{\upsilon},{\wp_2}^\upsilon\right)-\Gamma\left(\frac{2}{\upsilon},{\wp_1}^\upsilon\right)\right)(\mathcal{F}(\wp_2)-\mathcal{F}(\wp_1))\right.\\ &\quad\left.-\left(\Gamma\left(\frac{1}{\upsilon},{\wp_2}^\upsilon\right)-\Gamma\left(\frac{1}{\upsilon},{\wp_1}^\upsilon\right)\right)(\wp_2\mathcal{F}(\wp_2)-\wp_1\mathcal{F}(\wp_1))\right],\end{aligned}$$

where we get the desired inequality. ■

Corollary 5. *Under the assumptions of Theorem 5, taking $\mathcal{F}\le K$, we get*

$$ {}_{N_4}J^\upsilon_{\wp_1}\mathcal{F}(\wp_2)\le\frac{1}{\upsilon(\wp_2-\wp_1)^3}\le KG(\wp_1-\wp_2). \tag{11}$$

The inequality (10) can be refined, if we use the notion of generalized ϕ-convexity directly, as the following result shows.

Theorem 6. *Let $\upsilon\in(0,1)$ and $\mathcal{F}:[\wp_1,\wp_2]\to[0,+\infty)$ be a generalized ϕ–convex and increasing function defined on $[\wp_1,\wp_2]$, with $0<\wp_1<\wp_2$, then*

$$\begin{aligned} {}_{N_4}J^\upsilon_{\wp_1}\mathcal{F}(\wp_2)\le&\frac{1}{\upsilon(\wp_2-\wp_1)^2\mathbf{E}_\alpha(\wp_2-\wp_1)}\\ &\times\left[\mathcal{F}(\wp_1)\left[\left[\Gamma\left(\frac{2}{\upsilon},\left(\frac{(\wp_2-\wp_1)^2}{\mathbf{E}_\alpha(\wp_2-\wp_1)}+\wp_1\right)^\upsilon\right)-\Gamma\left(\frac{2}{\upsilon},{\wp_1}^\upsilon\right)\right]\right.\right.\\ &-\left.\wp_2\left[\Gamma\left(\frac{1}{\upsilon},\left(\frac{(\wp_2-\wp_1)^2}{\mathbf{E}_\alpha(\wp_2-\wp_1)}+\wp_1\right)^\upsilon\right)-\Gamma\left(\frac{1}{\upsilon},{\wp_1}^\upsilon\right)\right]\right]\\ &+\mathcal{F}(\wp_2)\left[\left[\Gamma\left(\frac{2}{\upsilon},\left(\frac{(\wp_2-\wp_1)^2}{\mathbf{E}_\alpha(\wp_2-\wp_1)}+\wp_1\right)^\upsilon\right)-\Gamma\left(\frac{2}{\upsilon},{\wp_1}^\upsilon\right)\right]\right.\\ &+\left.\left.\wp_1\left[\Gamma\left(\frac{1}{\upsilon},\left(\frac{(\wp_2-\wp_1)^2}{\mathbf{E}_\alpha(\wp_2-\wp_1)}+\wp_1\right)^\upsilon\right)-\Gamma\left(\frac{1}{\upsilon},{\wp_1}^\upsilon\right)\right]\right]\right].\end{aligned}$$

Proof. The change of variable $\imath=\wp_2 s+\wp_1(1-s)$, the ϕ–convexity of $\mathcal{F}$ and integrating, we have

$$\begin{aligned}\int_{\wp_1}^{\wp_2}\mathcal{F}(\imath)e^{-\imath^\upsilon}d\imath=&\frac{1}{\mathbf{E}_\alpha(\wp_2-\wp_1)}\\ &\times\int_0^{\frac{\wp_2-\wp_1}{\mathbf{E}_\alpha(\wp_2-\wp_1)}}\mathcal{F}(\wp_1+s\mathbf{E}_\alpha(\wp_2-\wp_1))\, e^{-(\wp_1+s\mathbf{E}_\alpha(\wp_2-\wp_1))^\upsilon}ds\\ \le&\frac{1}{\mathbf{E}_\alpha(\wp_2-\wp_1)}\int_0^{\frac{\wp_2-\wp_1}{\mathbf{E}_\alpha(\wp_2-\wp_1)}}[\mathcal{F}(\wp_2)s+\mathcal{F}(\wp_1)(1-s)]\, e^{-(\wp_2 s+\wp_1(1-s))^\upsilon}ds\end{aligned}$$

$$= \frac{1}{\upsilon(\wp_2 - \wp_1)^2 \mathbf{E}_\alpha(\wp_2 - \wp_1)} \left\{ \mathcal{F}(\wp_1)(P - \wp_2 Q) + \mathcal{F}(\wp_2)(P + \wp_1 Q) \right\},$$

where

$$P = \left[\Gamma\left(\frac{2}{\upsilon}, \left(\frac{(\wp_2 - \wp_1)^2}{\mathbf{E}_\alpha(\wp_2 - \wp_1)} + \wp_1 \right)^{\upsilon} \right) - \Gamma\left(\frac{2}{\upsilon}, {\wp_1}^{\upsilon} \right) \right]$$

and

$$Q = \left[\Gamma\left(\frac{1}{\upsilon}, \left(\frac{(\wp_2 - \wp_1)^2}{\mathbf{E}_\alpha(\wp_2 - \wp_1)} + \wp_1 \right)^{\upsilon} \right) - \Gamma\left(\frac{1}{\upsilon}, {\wp_1}^{\upsilon} \right) \right].$$

This completes the proof. ■

Remark 4. Taking suitable values of $\upsilon \in (0, 1)$ and different values of α that using in Mittag–Leffler function in all proved results of this paper, we get different new interesting inequalities. The details are left to the interested reader.

3. Conclusion

In a similar way interested reader can obtain new results for generalized ϕ–convex functions by using different operators such as the k–Riemann-Liouville fractional integral, Katugampola fractional integrals, the conformable fractional integral, Hadamard fractional integrals, etc., and these results can be applied in different areas of pure and applied sciences.

In recent years, with the appearance of differential and integral fractional operators, various criticisms and questions have arisen. We could argue for various reasons but we would like to use some phrases from [11]: "In my opinion, it is not correct to assume that a given fractional calculus operator can describe very well the dynamics of all types of complex phenomena" and "The advantage of the fractional calculus is that we do not have a single fractional order operator, but we have classes of operators which are valid for the specific types of real data" is more than demonstrated by the History of Mathematics, throughout the development of our science, which the researchers have not stopped, have continued searching for new tools. On the other hand, in the Conclusions the author stresses "In my opinion, the correct formulations of the fractional modelling will play a fundamental role in clarifying the importance of fractional calculus operators with or without semigroup property", in other words, the real world it does not adapt to our tools, we must create the tools to study it better!

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