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CONTACT PROBLEM OF CONDUCTING AND HEATED PUNCH ON A MULTIFIELD FOUNDATION

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The solution for a multifield material subjected to temperature loading in a circular region is presented in an explicit analytical form. The study concerns the steady – state thermal loading infinite region (heated embedded inclusion), half – space region and two – constituent magneto – electro – thermo – elastic material region. The new mono – harmonic potential functions, obtained by the author, are used in the analysis of punch problem. The more interested case in which the contact region is annular is analyzed. By using the methods of triple integral equations and series solution technique the solution for an indentured multifield substrate over an annular contact region is given. The sensitivity analysis of obtained indentation parameters shows some interesting points. In particular, it shows that the increasing of the applied electric and magnetic potentials reduces the indentation depth in multifield materials.

Key words: magneto – electro – thermo – elastic medium, conducting and heated punch, contact problem, exact solution.

1. Introduction

Solutions for various cases of contact problems can be found in the books by Gladwell (1980), Johnson (1985), Hills *et al*. (1993), Raous *et al*. (1995) and Rogowski (2006; 2006). However, in these books elastic fields themselves are considered and distinctive mathematical methods are used and elaborated. However, there is no parallel work in the domain of magneto – electro – thermo – elasticity. In the recent paper by Chen *et al*. (2010) the authors obtained the coupled fields for indentation of a multiferroic composite half – space for three common indenters: flat – ended, conical and spherical. The complete contact problem is considered under the assumption of circular contact region. Various important discussions related to indentation of piezoelectric materials, e.g., on the piezoresponse force microscopy (Kalinin *et al*., 2004) can be directly borrowed and applied to piezomagnetic materials. It is reasonable to assume that the extension of the findings to multifield composite materials is valid. The effective solution to the contact problem of multifield foundation and truncated conical punch or punch with a concave base was obtained and published recently by the author (Rogowski and Kaliński, 2012; Rogowski, 2012). In this paper, five potential mono – harmonic functions, obtained by the author, are utilized to solve the punch problem in which the contact region is always annular. The outer circumference of the annulus coincides with the edge of the punch. The inner circumference will shrink with an increasing load. The inner radius is not known a priori and is obtained from the conditions of equal thermal displacement and indentation mechanical depth of the punch on this circumference. On this boundary the phenomenon of adhesive contact is observed. The problem is solved by triple integral equations technique. The relationships between the force, electric charge, magnetic flux, temperature, indentation depth of the punch and electric and magnetic potentials on the boundary are derived.

2. Axisymmetric solutions of a multifield body

The generalized multifield solution considered here is independent of the variable θ in the cylindrical coordinate system (r,θ,z). Therefore, the mechanical displacement u_r and u_z, the electric potential ϕ and magnetic potential ψ, the mechanical stresses σ_r, σ_θ, σ_z and σ_{rz}, the electric displacements D_r and D_z, the magnetic inductions B_r and B_z can be generally expressed as (Rogowski, 2014).

$$u_r(r,z)=\sum_{i=0}^{4} a_{1i}\lambda_i \frac{\partial\varphi_i}{\partial r}, \qquad u_z(r,z)=\sum_{i=0}^{4}\frac{\partial\varphi_i}{\partial z_i},$$

$$\phi(r,z)=-\sum_{i=0}^{4} a_{3i}\frac{\partial\phi_i}{\partial z_i}, \qquad \psi(r,z)=-\sum_{i=0}^{4} a_{4i}\frac{\partial\phi_i}{\partial z_i},$$

$$\sigma_r=-\sum_{i=0}^{4} a_{5i}\lambda_i\frac{\partial^2\phi_i}{\partial z_i^2}-(c_{11}-c_{12})\frac{u_r}{r}-\beta_1 T,$$

$$\sigma_\theta=-\sum_{i=0}^{4} a_{5i}\lambda_i\frac{\partial^2\phi_i}{\partial z_i^2}-(c_{11}-c_{12})\frac{\partial u_r}{\partial r}-\beta_1 T, \tag{2.1}$$

$$\sigma_z=\sum_{i=0}^{4}\frac{a_{5i}}{\lambda_i}\frac{\partial^2\varphi_i}{\partial z_i^2}+\lambda_0^{-2}\beta_1 T, \qquad \sigma_{rz}=\sum_{i=0}^{4} a_{5i}\frac{\partial^2\varphi_i}{\partial r\partial z_i}+\lambda_0^{-1}\beta_1 a_{00}\frac{\partial^2\varphi_0}{\partial r\partial z_0},$$

$$D_r=\sum_{i=0}^{4} a_{6i}\lambda_i^2\frac{\partial^2\varphi_i}{\partial r\partial z_i}, \qquad D_z=\sum_{i=0}^{4} a_{6i}\lambda_i\frac{\partial^2\varphi_i}{\partial z_i^2},$$

$$B_r=\sum_{i=0}^{4} a_{7i}\lambda_i^2\frac{\partial^2\phi_i}{\partial r\partial z_i}, \qquad B_z=\sum_{i=0}^{4} a_{7i}\lambda_i\frac{\partial^2\phi_i}{\partial z_i^2}$$

In addition

$$E_r=-\frac{\partial\varphi}{\partial r}, \qquad E_z=-\frac{\partial\varphi}{\partial z}, \qquad H_r=-\frac{\partial\psi}{\partial r}, \qquad H_z=-\frac{\partial\psi}{\partial z}, \tag{2.2}$$

are the components of electric and magnetic field vectors.

In the fundamental solution $\varphi_i(r,z_i)$ are the harmonic functions of the variables r and $z_i=\lambda_i z$, that is

$$\frac{\partial^2\varphi_i}{\partial r^2}+\frac{1}{r}\frac{\partial\varphi_i}{\partial r}+\frac{\partial^2\varphi_i}{\partial z_i^2}=0, \quad i=0,1,2,3,4. \tag{2.3}$$

The temperature in a steady – state and uncoupled thermoelastic problem is governed by

$$\lambda_r\left(\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)+\lambda_z\frac{\partial^2 T}{\partial z^2}=0 \tag{2.4}$$

where λ_r, λ_z are the heat conduction coefficients in unit W/Km and T is described by $\varphi_0(r,z_0)$ as follows

$$T(r,z_0)=a_{00}\frac{\partial^2\varphi_0}{\partial z_0^2}, \qquad z_0=\lambda_0 z, \qquad \lambda_0=\sqrt{\frac{\lambda_r}{\lambda_z}}\,. \tag{2.5}$$

The quantities λ_i $(i=1,2,3,4)$ are eigenvalues of multifield material defined as the roots of equation

$$a\lambda^8+b\lambda^6+c\lambda^4+d\lambda^2+e=0 \tag{2.6}$$

where a, b, c, d and e are given in the Appendix A by Eqs A1. The roots of Eq.(2.6) are presented by formulae A4. The other material parameters are given by

$$\begin{Bmatrix}a_{1i}\\a_{3i}\\a_{4i}\end{Bmatrix}=\begin{bmatrix}\lambda_i^2(e_{31}+e_{15}) & -(\varepsilon_{33}\lambda_i^2-\varepsilon_{11}) & -(d_{33}\lambda_i^2-d_{11})\\ \lambda_i^2(q_{31}+q_{15}) & -(d_{33}\lambda_i^2-d_{11}) & -(\mu_{33}\lambda_i^2-\mu_{11})\\ c_{11}+c_{13}\lambda_i^2 & e_{31}+e_{33}\lambda_i^2 & q_{31}+q_{33}\lambda_i^2\end{bmatrix}^{-1}\begin{Bmatrix}e_{33}\lambda_i^2-e_{15}+p_3\lambda_i a_{00}\delta_{i0}\\ q_{33}\lambda_i^2-q_{15}+\gamma_3\lambda_i a_{00}\delta_{i0}\\ c_{33}\lambda_i^2+c_{13}-\left(\beta_3+\beta_1\lambda_i^{-2}\right)a_{00}\lambda_i\delta_{i0}\end{Bmatrix},$$

$$a_{5i}=c_{11}a_{1i}-c_{13}+e_{31}a_{3i}+q_{31}a_{4i}\,,$$

$$\lambda_i^2 a_{6i}=\lambda_i^2 e_{15}a_{1i}+e_{15}+\varepsilon_{11}a_{3i}+d_{11}a_{4i}\,,$$

$$\lambda_i^2 a_{7i}=\lambda_i^2 q_{15}a_{1i}+q_{15}+\mu_{11}a_{4i}+d_{11}a_{3i}\,,$$

$$a_{00}=\frac{a\lambda_0^8+b\lambda_0^6+c\lambda_0^4+d\lambda_0^2+e}{a_2\lambda_0^6+b_2\lambda_0^4+c_2\lambda_0^2+d_2} \tag{2.7}$$

where in the last equation the denominator is exactly given by the determinant of the matrix in Eq.(2.7). The unit of $[a_{00}]=10^6 K$. In addition, the thermal moduli β_1, β_3, pyroelectric parameter p_3 and pyromagnetic parameter γ_3 are defined as follows

$$\begin{gathered}\beta_1=(c_{11}+c_{12})\alpha_r+c_{13}\alpha_z, \qquad \beta_3=2c_{13}\alpha_r+c_{33}\alpha_z,\\ p_3=2e_{11}\alpha_r+e_{33}\alpha_z, \qquad \gamma_3=2q_{31}\alpha_r+q_{33}\alpha_z\end{gathered} \tag{2.8}$$

where α_r and α_z are the coefficients of thermal expansion $[\alpha]=10^6 1/K$. The units of these parameters are

$$[\beta_1,\beta_3]=10^5\,N/m^2K, \quad [p_3]=10^{-6}\,C/m^2K, \quad [\gamma_3]=10^{-4}\,N/AmK\,. \tag{2.9}$$

The multifield material is characterized by twenty one material constants. There are: five elastic constants c_{11}, c_{12}, c_{13}, c_{33}, c_{44} (in units GPa), three piezoelectric constants e_{31}, e_{15}, e_{33} (in units C/m^2), three piezomagnetic constants q_{31}, q_{15}, q_{33} (in units N/Am), two magneto – electric constants d_{11}, d_{33} (in units $10^{-9}C/Am$), two dielectric permittivities ε_{11}, ε_{33} (in units $10^{-10}C/Vm$), two magnetic permeabilities μ_{11}, μ_{33} (in units $10^{-6}N/A^2$), two coefficients of thermal expansion α_r, α_z (in units $10^{-6}1/K$) and two coefficients of heat conduction λ_r, λ_z (in unit W/Km). Any solution other than explicit analytical ones is impractical in the context of multifield material.

3. The temperature changes

We seek the harmonic function $\varphi_0(r,z_0)$ in the form

$$\varphi_0(r,z_0)=\frac{1}{a_{00}}\int_0^a\left[\frac{1}{2}\xi_0\eta_0\ln\frac{1+\eta_0}{1-\eta_0}-\xi_0-\tan^{-1}\frac{1}{\xi_0}\right]x\varphi(x)dx \tag{3.1}$$

where $\varphi(x)$ to be determined from the thermal field boundary conditions.

The spheroidal coordinates ξ_i and η_i are related to cylindrical coordinates r, $\lambda_i z$ by equations

$$r^2=x^2\left(1+\xi_i^2\right)\left(1-\eta_i^2\right),\qquad \lambda_i z=x\xi_i\eta_i,\qquad \xi_i\ge 0,\qquad |\eta_i|\le 1 \tag{3.2}$$

and are associated with λ_i (here $i=0$) and with $0\le x\le a$. The derivatives may be easily calculated, that is

$$\begin{aligned}
&\frac{\partial\phi_0}{\partial r}=-\frac{1}{a_{00}}\frac{1}{r}\int_0^a x\xi_0(1-\eta_0)\phi(x)dx,\\
&\frac{\partial\varphi_0}{\partial z}=\frac{\lambda_0}{2a_{00}}\int_0^a ln\frac{1+\eta_0}{1-\eta_0}\varphi(x)dx,\\
&\frac{\partial^2\phi_0}{\partial r\partial z}=\frac{\lambda_0}{a_{00}}\frac{1}{r}\int_0^a\left[1-\frac{\partial}{\partial x}(x\eta_0)\right]\phi(x)dx,\\
&\frac{\partial^2\phi_0}{\partial z^2}=\frac{\lambda_0^2}{a_{00}}\int_0^a\frac{\xi_0}{\xi_0^2+\eta_0^2}\frac{\phi(x)}{x}dx,\\
&\frac{\partial^2\phi_0}{\partial r^2}=\frac{1}{a_{00}}\int_0^a\left[\frac{x}{r^2}\xi_0(1-\eta_0)-\frac{\xi_0}{x\left(\xi_0^2+\eta_0^2\right)}\right]\phi(x)dx,\\
&\frac{\partial^3\varphi_0}{\partial z^3}=-\frac{\lambda_0^3}{a_{00}}\int_0^a\frac{\partial}{\partial x}\left(\frac{\eta_0}{x\left(\xi_0^2+\eta_0^2\right)}\right)\varphi(x)dx.
\end{aligned} \tag{3.3}$$

The quantities $Q = -\lambda_z \, \partial T/\partial z$, ϕ, ψ and u_z associated with the thermoelastic potential ϕ_0 vanish for $z = 0, r \geq a$, since $\eta = 0$ for $z = 0, r \geq a$.

The temperature and heat flux are

$$T(r,z) = \int_0^a \frac{\xi_0}{\xi_0^2 + \eta_0^2} \frac{\phi(x)}{x} dx, \quad Q(r,z) = \sqrt{\lambda_r \lambda_z} \int_0^a \frac{\partial}{\partial x} \left(\frac{\eta_0}{x\left(\xi_0^2 + \eta_0^2\right)} \right) \phi(x) dx . \tag{3.4}$$

The condition for the prescribed temperature $T(r,0) = f(r)$ for $r \leq a$ gives

$$\int_0^r \frac{\varphi(x) dx}{\sqrt{r^2 - x^2}} = f(r). \tag{3.5}$$

This is Abel’s integral equation with the solution

$$\phi(x) = \frac{2}{\pi} \frac{d}{dx} \int_0^x \frac{r f(r) dr}{\sqrt{x^2 - r^2}} . \tag{3.6}$$

If we assume the distribution of temperature to be cylindrical constant or revolution conical or revolution parabolic, that is if

$$f(r) = T_0 \left(1, 1 - \frac{r}{a}, 1 - \left(\frac{r}{a} \right)^2 \right), \quad r \leq a , \tag{3.7}$$

the solution $\phi(x)$ will be

$$\phi(x) = \frac{2}{\pi} T_0 \left(1, 1 - \frac{\pi x}{2a}, 1 - 2\left(\frac{x}{a} \right)^2 \right), \tag{3.8}$$

respectively.

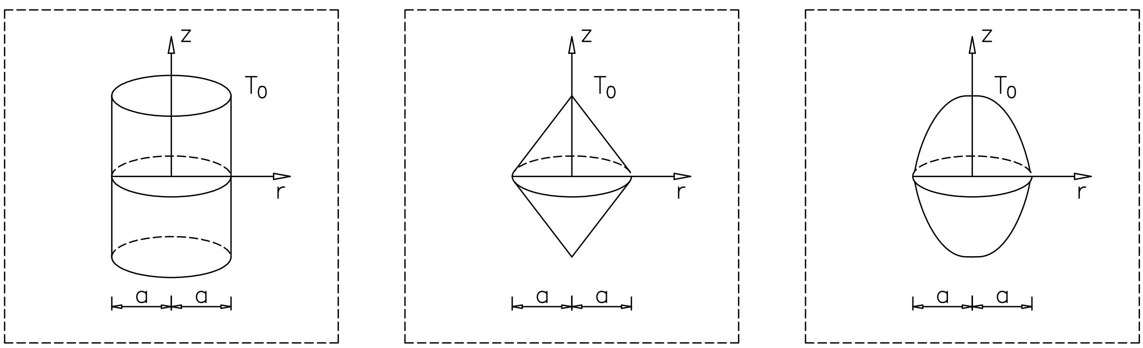

Fig.1. Multifield material with temperature change in the circular region $r \leq a$ as to be constant or revolution conical and revolution parabolic.

For constant temperature T_0 we have (here ξ_0 and η_0 are for $x = a$)

$$\phi_0(r,z) = \frac{2}{\pi} T_0 a^2 a_{00} \left[\frac{\lambda_0 z}{a} \left(\frac{1}{2} \ln \frac{1+\eta_0}{1-\eta_0} - \frac{1}{\eta_0} \right) - \frac{1}{2} \left(\tan^{-1} \frac{1}{\xi_0} - \tan^{-1} \frac{\lambda_0 z}{a} \right) + \right.$$

$$\left. - \frac{\lambda_0^2 z^2}{a^2} \left(\tan^{-1} \xi_0 + \frac{1}{\xi_0} - \tan^{-1} \frac{\lambda_0 z}{a} - \frac{a}{\lambda_0 z} \right) - \frac{r^2}{4a^2} \left(\tan^{-1} \frac{1}{\xi_0} - \frac{\xi_0}{1+\xi_0^2} \right) \right],$$

$$u_r^{(0)}(r,z) = -\frac{T_0 r a_{10} \lambda_0}{\pi a_{00}} \left(\frac{\pi}{2} - \tan^{-1} \xi_0 + \frac{\xi_0}{1+\xi_0} \frac{1-\eta_0}{1+\eta_0} \right),$$

$$\begin{bmatrix} u_z^{(0)}(r,z) \\ \phi^{(0)}(r,z) \\ \psi^{(0)}(r,z) \end{bmatrix} = \frac{2T_0 a}{\pi} \begin{bmatrix} 1 \\ -a_{30} \\ -a_{40} \end{bmatrix} \frac{1}{a_{00}} \left[\frac{1}{2} ln \frac{1+\eta_0}{1-\eta_0} - \eta_0 \left(1 - \xi_0 \tan^{-1} \frac{1}{\xi_0} \right) \right],$$

$$\begin{bmatrix} \sigma_{zr}^{(0)}(r,z) \\ E_r^{(0)}(r,z) \\ H_r^{(0)}(r,z) \\ D_r^{(0)}(r,z) \\ B_r^{(0)}(r,z) \end{bmatrix} = \frac{2T_0}{\pi a_{00}} \frac{a}{r} \begin{bmatrix} a_{50} + \frac{\beta_1}{\lambda_0} a_{00} \\ a_{30} \\ a_{40} \\ a_{60} \lambda_0^2 \\ a_{70} \lambda_0^2 \end{bmatrix} (1-\eta_0),$$

$$\begin{bmatrix} \sigma_{zz}^{(0)}(r,z) \\ E_z^{(0)}(r,z) \\ H_z^{(0)}(r,z) \\ D_z^{(0)}(r,z) \\ B_z^{(0)}(r,z) \end{bmatrix} = \frac{2T_0}{\pi a_{00}} \begin{bmatrix} \left(a_{50} + \frac{\beta_1}{\lambda_0} a_{00} \right) \frac{1}{\lambda_0} \\ a_{30} \lambda_0 \\ a_{40} \lambda_0 \\ a_{60} \lambda_0 \\ a_{70} \lambda_0 \end{bmatrix} \left(\frac{\pi}{2} - \tan^{-1} \xi_0 \right),$$

$$\begin{bmatrix} \sigma_{rr}^{(0)}(r,z) \\ \sigma_{\theta\theta}^{(0)}(r,z) \end{bmatrix} = -\frac{2T_0}{\pi} \left(\frac{a_{50}}{a_{00}} \lambda_0 + \beta_1 \right) \left(\frac{\pi}{2} - \tan^{-1} \xi_0 \right) - \begin{bmatrix} (c_{11} - c_{12}) \frac{u_r^{(0)}(r,z)}{r} \\ (c_{11} - c_{12}) \frac{\partial u_r^{(0)}(r,z)}{\partial r} \end{bmatrix},$$

$$T(r,z) = T_0 \left(1 - \frac{2}{\pi} \tan^{-1} \xi_0 \right) \tag{3.9}$$

where

$$\xi_0 = \frac{1}{2}\sqrt{\left(\sqrt{\frac{\lambda_0^2 z^2}{a^2} + \left(\frac{r}{a}+1\right)^2} + \sqrt{\frac{\lambda_0^2 z^2}{a^2} + \left(\frac{r}{a}-1\right)^2}\right) - 4}, \quad \eta_0 = \frac{\lambda_0 z}{a\xi_0}. \tag{3.10}$$

The fundamental equations are single – direction coupling such that the thermal loading can change electro – magneto – elastic fields, the mechanical, electric and magnetic loadings cannot change the thermal field. This is a theory of uncoupled thermoelasticity of multifield material where the temperature field is independent of the electric displacement, and in addition, the inertial terms can be neglected.

To satisfy the zero – shear condition at $z = 0$, we find field defined by harmonic potential (Hankel integral)

$$\phi_i(r,z) = \int_0^\infty \xi^{-1} A_i(\xi) e^{-\lambda_i \xi z} J_0(\xi r)\, d\xi, \quad i = 1, 2, 3, 4, \tag{3.11}$$

in which

$$m_2^* A_i(\xi) = -\frac{2}{\pi} T_0 \tilde{\beta} \frac{\sin \xi a}{\xi^2} d_i^*, \qquad \tilde{\beta} = \frac{a_{50}}{a_{00}} + \frac{\beta_1}{\lambda_0} \tag{3.12}$$

where m_2^* and d_i^* are defined in the Appendix A by Eqs (A5).

Since

$$\sum a_{3i} d_i^* = \sum a_{4i} d_i^* = \sum \frac{a_{5i} d_i^*}{\lambda_i} = 0, \qquad m_2^* = \sum a_{5i} d_i^*, \qquad \sum \equiv \sum_{i=1}^{i=4}, \tag{3.13}$$

the solution satisfies the boundary conditions at $z = 0, r \geq 0$

$$\phi = \psi = \sigma_z = 0, \qquad \sigma_{zr} = -\frac{2}{\pi} \tilde{\beta} T_0 \left(\frac{a}{r} - \sqrt{\frac{a^2}{r^2} - 1} H\left(\frac{a}{r} - 1\right)\right) \tag{3.14}$$

where $H(\bullet)$ denotes Heaviside's unit function.

The full field in this case is given in analytical form, that is

$$u_r(r,z) = \frac{1}{\pi} T_0 \tilde{\beta} \frac{r}{m_2^*} \sum_{i=1}^{4} a_{1i} \lambda_i d_i^* \left(\frac{\pi}{2} - \tan^{-1}\xi_i + \frac{\xi_i}{1+\xi_i^2} \frac{1-\eta_i}{1+\eta_i}\right),$$

$$\begin{bmatrix} u_z(r,z) \\ -\varphi(r,z) \\ -\psi(r,z) \end{bmatrix} = -\frac{2}{\pi} T_0 \tilde{\beta} \frac{a}{m_2^*} \sum_{i=1}^{4} \begin{bmatrix} d_i^* \\ a_{3i} d_i^* \\ a_{4i} d_i^* \end{bmatrix} \left[\frac{1}{2} \ln \frac{1+\eta_i}{1-\eta_i} - \eta_i \left(1 - \xi_i \left(\frac{\pi}{2} - \tan^{-1}\frac{1}{\xi_i}\right)\right)\right],$$

$$\begin{bmatrix} \sigma_{zr}(r,z) \\ E_r(r,z) \\ H_r(r,z) \\ D_r(r,z) \\ B_r(r,z) \end{bmatrix} = -\frac{2}{\pi} T_0 \tilde{\beta} \frac{a}{r} \frac{1}{m_2^*} \sum_{i=1}^{4} \begin{bmatrix} a_{5i} d_i^* \\ a_{3i} d_i^* \\ a_{4i} d_i^* \\ a_{6i} \lambda_i^2 d_i^* \\ a_{7i} \lambda_i^2 d_i^* \end{bmatrix} (1-\eta_i),$$

$$\begin{bmatrix} \sigma_{zz}(r,z) \\ E_z(r,z) \\ H_z(r,z) \\ D_z(r,z) \\ B_z(r,z) \end{bmatrix} = -\frac{2}{\pi} T_0 \tilde{\beta} \frac{1}{m_2^*} \sum_{i=1}^{4} \begin{bmatrix} \frac{a_{5i}}{\lambda_i} d_i^* \\ a_{3i} d_i^* \lambda_i \\ a_{4i} d_i^* \lambda_i \\ a_{6i} d_i^* \lambda_i \\ a_{7i} d_i^* \lambda_i \end{bmatrix} \left(\frac{\pi}{2} - \tan^{-1}\xi_i\right), \tag{3.15}$$

$$\begin{bmatrix} \sigma_{rr}(r,z) \\ \sigma_{\theta\theta}(r,z) \end{bmatrix} = \frac{2}{\pi} T_0 \tilde{\beta} \frac{1}{m_2^*} \sum_{i=1}^{4} a_{5i} \lambda_i d_i^* \left(\frac{\pi}{2} - \tan^{-1}\xi_i\right) - \begin{bmatrix} (c_{11}-c_{12}) \frac{u_r(r,z)}{r} \\ (c_{11}-c_{12}) \frac{\partial u_r(r,z)}{\partial r} \end{bmatrix}.$$

The nonsingular thermoelastic coupled field can be obtained by superimposing the two parts as given by Eqs (3.9) and (3.15). For a multifield material and embedded interior thin heated inclusion subjected to temperature T_0 at the contact surface the normal and shear stresses, electric and magnetic potentials and electric displacement and magnetic induction in the axial direction are as follow

$$\sigma_{zz}(r,z) = T_0 \tilde{\beta} \left[\frac{1}{\lambda_0} \left(1 - \frac{2}{\pi} \tan^{-1}\xi_0\right) + \frac{2}{\pi} \frac{1}{m_2^*} \sum_{i=1}^{4} \frac{a_{5i}}{\lambda_i} d_i^* \tan^{-1}\xi_i \right],$$

$$\sigma_{zr}(r,z) = T_0 \tilde{\beta} \left(\eta_0 - \frac{1}{m_2^*} \sum_{i=1}^{4} a_{5i} d_i^* \eta_i \right),$$

$$u_z(r,z) = \frac{2T_0 a}{\pi} \left[\frac{1}{a_{00}} \ln \frac{1+\eta_0}{1-\eta_0} - \eta_0 \left(1 - \xi_0 \tan^{-1}\frac{1}{\xi_0}\right) + \right.$$
$$\left. + \frac{\tilde{\beta}}{m_2^*} \sum_{i=1}^{4} d_i^* \left(\frac{1}{2} \ln \frac{1+\eta_i}{1-\eta_i} - \eta_i \left(1 - \xi_i \tan^{-1}\frac{1}{\xi_i}\right)\right)\right],$$

$$\varphi(r,z) = -\frac{2T_0 a}{\pi} \left[\frac{a_{30}}{a_{00}} \left(\frac{1}{2} \ln \frac{1+\eta_0}{1-\eta_0} - \eta_0 \left(1 - \xi_0 \tan^{-1}\frac{1}{\xi_0}\right)\right) + \right.$$
$$\left. + \frac{\tilde{\beta}}{m_2^*} \sum_{i=1}^{4} a_{3i} d_i^* \left(\frac{1}{2} \ln \frac{1+\eta_i}{1-\eta_i} - \eta_i \left(1 - \xi_i \tan^{-1}\frac{1}{\xi_i}\right)\right)\right],$$

$$\psi(r,z) = -\frac{2T_0 a}{\pi}\left[\frac{a_{40}}{a_{00}}\left(\frac{1}{2}\ln\frac{1+\eta_0}{1-\eta_0} - \eta_0\left(1-\xi_0\tan^{-1}\frac{1}{\xi_0}\right)\right) + \right.$$

$$\left. + \frac{\tilde{\beta}}{m_2^*}\sum_{i=1}^{4} a_{4i} d_i^* \left(\frac{1}{2}\ln\frac{1+\eta_i}{1-\eta_i} - \eta_i\left(1-\xi_i\tan^{-1}\frac{1}{\xi_i}\right)\right)\right],$$

$$D_z(r,z) = T_0\left[\left(p_3 + p_3^*\right)\left(1-\frac{2}{\pi}\tan^{-1}\xi_0\right) - \frac{\tilde{\beta}}{m_2^*}\sum_{i=1}^{4} a_{6i}\lambda_i d_i^*\left(1-\frac{2}{\pi}\tan^{-1}\xi_i\right)\right],$$

$$B_z(r,z) = T_0\left[\left(\gamma_3 + \gamma_3^*\right)\left(1-\frac{2}{\pi}\tan^{-1}\xi_0\right) - \frac{\tilde{\beta}}{m_2^*}\sum_{i=1}^{4} a_{7i}\lambda_i d_i^*\left(1-\frac{2}{\pi}\tan^{-1}\xi_i\right)\right] \tag{3.16}$$

where

$$p_3^* = \frac{a_{60}^*\lambda_0}{a_{00}}, \quad \gamma_3^* = \frac{a_{70}^*\lambda_0}{a_{00}}, \quad \tilde{\beta} = \frac{\beta_1}{\lambda_0} + \beta_1^*, \quad \beta_1^* = \frac{a_{50}}{a_{00}},$$

$$a_{60}^* = e_{33} - e_{31}a_{10} + \varepsilon_{33}a_{30} + d_{33}a_{40},$$

$$a_{70}^* = q_{33} - q_{31}a_{10} + d_{33}a_{30} + \mu_{33}a_{40}. \tag{3.17}$$

The units of thermal moduli β_1 and β_1^*, pyroelectric and pyromagnetic constants p_3 and p_3^*, and γ_3 and γ_3^* are

$$\left[\beta_1,\beta_1^*\right] = 10^5\, N/\left(m^2K\right), \quad \left[p_3, p_3^*\right] = 10^{-6}\, C/\left(m^2K\right), \quad \left[\gamma_3,\gamma_3^*\right] = 10^{-4}\, N/\left(AmK\right) \tag{3.18}$$

On the plane $z=0$ we have

$$\begin{bmatrix}\sigma_{zz}\\ D_z\\ B_z\end{bmatrix} = \begin{bmatrix}\sigma_T\\ D_T\\ B_T\end{bmatrix} = T_0\begin{bmatrix}\tilde{\beta}/\lambda_0\\ p\\ \gamma\end{bmatrix}\left[1-\left(1-\frac{2}{\pi}\arcsin\left(\frac{a}{r}\right)H\left(1-\frac{a}{r}\right)\right)\right],$$

$$\sigma_{zr} \equiv 0$$

$$\begin{bmatrix}u_z\\ \varphi\\ \psi\end{bmatrix} = \begin{bmatrix}u_T\\ \varphi_T\\ \psi_T\end{bmatrix} = \frac{2}{\pi}T_0 a\begin{bmatrix}\tilde{\alpha}\\ -\tilde{e}\\ -\tilde{q}\end{bmatrix}\begin{cases}\ln\left(\frac{a}{r}+\sqrt{\frac{a^2}{r^2}-1}\right)-\sqrt{1-\frac{r^2}{a^2}}, & r \le a\\ 0, & r \ge a\end{cases} \tag{3.19}$$

where

$$p = p_3 + p_3^* - \frac{\tilde{\beta}}{m_2^*}\sum_{i=1}^{4} a_{6i}\lambda_i d_i^* = p_3 + p_3^* - p_3^{**}, \quad \left[10^{-6}\, C/m^2 K\right],$$

$$\gamma = \gamma_3 + \gamma_3^* - \frac{\beta}{m_2^*}\sum_{i=1}^{4} a_{7i}\lambda_i d_i^* = \gamma_3 + \gamma_3^* - \gamma_3^{**}, \quad \left[10^{-4}\, N/m^2 K\right],$$

$$\tilde{\alpha} = \frac{1}{a_{00}} + \frac{\tilde{\beta}}{m_2^*}\sum_{i=1}^{4} d_i^*, \quad \left[10^{-6}\, 1/K\right],$$

$$\tilde{e} = \frac{a_{30}}{a_{00}} + \frac{\tilde{\beta}}{m_2^*}\sum_{i=1}^{4} a_{3i} d_i^*, \quad \left[10^{-6}\, C/mK\right],$$

$$\tilde{q} = \frac{a_{40}}{a_{00}} + \frac{\tilde{\beta}}{m_2^*}\sum_{i=1}^{4} a_{4i} d_i^*, \quad \left[10^{-4}\, A/mK\right] \tag{3.20}$$

The generalized stresses and displacements (3.18) are presented graphically in Fig.2.

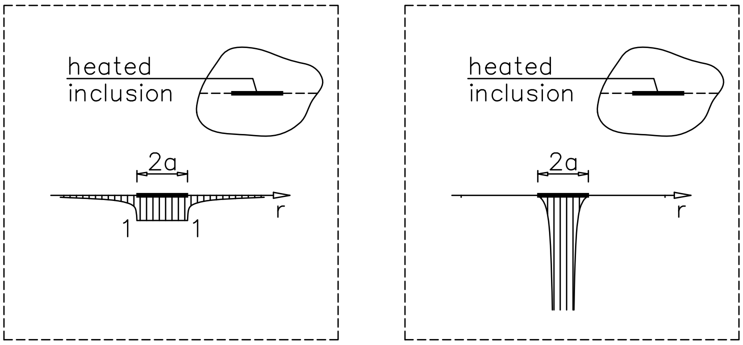

Fig.2. The generalized stresses and displacements on the plane $z = 0$; are given by Eq.(3.19).

4. The half – space problem

We assume that the multifield material changes the temperature on the boundary which is given in the circular region $r \le a$ as the constant cylindrical, revolution conical or revolution parabolic(see Fig.3).

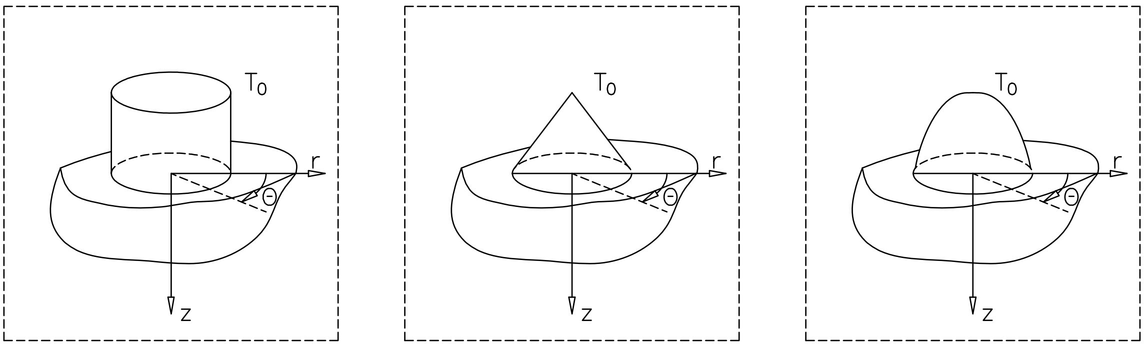

Fig.3. The half – space under temperature loading: a) cylindrical constant, b) revolution conical, c) revolution parabolic.

The electric permittivity and magnetic permeability of the external medium (usually air) is much less that the multifield half – space region. For example $\varepsilon_{11}/\varepsilon_{air} = 60/0{,}0885 = 680$ times lower for the PZT – 4 commercial piezoelectric and $\mu_{11}/\mu_{air} = 590/0{,}4\pi = 475$ times lower for a typical piezomagnetic. In consequence the electric displacement D_z and magnetic induction B_z must vanish on the boundary surface, which is also stress free.

Thus, we consider the following boundary conditions

$$\sigma_{zr} = 0, \quad \sigma_z + \sigma_T = 0, \quad D_z + D_T = 0, \quad B_z + B_T = 0, \quad z = 0, \quad r \geq 0 \tag{4.1}$$

where

$$\begin{bmatrix} \sigma_T \\ D_T \\ B_T \end{bmatrix} = T_0 \begin{bmatrix} \tilde{\beta}/\lambda_0 \\ p \\ \gamma \end{bmatrix} \left[1 - \left(1 - \frac{2}{\pi} \arcsin\left(\frac{a}{r}\right) H\left(1 - \frac{a}{r}\right)\right)\right]. \tag{4.2}$$

The quasi – harmonic functions needed for satisfying the boundary conditions are

$$\phi_i(r, \lambda_i z) = \int_0^\infty B_i(\xi) e^{-\lambda_i \xi z} J_0(\xi r) d\xi \tag{4.3}$$

where $J_0(\xi r)$ is the Bessel function and ξ is a parameter of the Hankel transform.

The mixed boundary conditions on the plane $z = 0$ give

$$\sigma_{rz}(r,0) = \sum_{i=1}^{4} \int_0^\infty \xi^2 B_i(\xi) J_1(\xi r) d\xi = 0, \qquad r \geq 0,$$

$$\begin{aligned} \{\sigma_{zz}, D_z, B_z(r,0)\} = -\left[1 - \left(1 - \frac{2}{\pi}\arcsin\left(\frac{a}{r}\right)H\left(1-\frac{a}{r}\right)\right)\right]\left\{\tilde{\beta}\frac{1}{\lambda_0}, p, \gamma\right\}T_0 = \\ = \sum_{i=1}^{4}\left\{\frac{a_{5i}}{\lambda_i}, a_{6i}\lambda_i, a_{7i}\lambda_i\right\}\int_0^\infty \xi^2 B_i(\xi) J_0(\xi r) d\xi. \end{aligned} \tag{4.4}$$

To satisfy the zero condition at $z=0$ for σ_{zz}, D_z and B_z we obtain the additional displacements u_T, φ_T and ψ_T. This thermal problem is obtained as given below.

Boundary conditions for generalized stresses yield

$$\begin{bmatrix} \xi B_1(\xi) \\ \xi B_2(\xi) \\ \xi B_3(\xi) \\ \xi B_4(\xi) \end{bmatrix} = -\frac{2}{\pi} T_0 \frac{\sin \xi a}{\xi^2} \begin{bmatrix} a_{51} & a_{52} & a_{53} & a_{54} \\ \dfrac{a_{51}}{\lambda_1} & \dfrac{a_{52}}{\lambda_2} & \dfrac{a_{53}}{\lambda_3} & \dfrac{a_{54}}{\lambda_4} \\ a_{61}\lambda_1 & a_{62}\lambda_2 & a_{63}\lambda_3 & a_{64}\lambda_4 \\ a_{71}\lambda_1 & a_{72}\lambda_2 & a_{73}\lambda_3 & a_{74}\lambda_4 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \tilde{\beta}\dfrac{1}{\lambda_0} \\ p \\ \gamma \end{bmatrix}, \tag{4.5}$$

or

$$\tilde{m}_2\xi B_i(\xi) = -\frac{2}{\pi}T_0\frac{\sin\xi a}{\xi^2}\Delta_i^T, \quad \Delta_i^T = \frac{1}{\lambda_0}\beta\tilde{d}_i + p\tilde{l}_i + \gamma\tilde{k}_i. \tag{4.6}$$

We have

$$\sum a_{5i}\tilde{d}_i = \sum a_{5i}\tilde{l}_i = \sum a_{5i}\tilde{k}_i = \sum a_{6i}\lambda_i\tilde{d}_i = \sum a_{6i}\lambda_i\tilde{k}_i = \sum a_{7i}\lambda_i\tilde{d}_i = \sum a_{7i}\lambda_i\tilde{l}_i = 0,$$
$$\tilde{m}_2 = \sum\frac{a_{5i}}{\lambda_i}\tilde{d}_i = \sum a_{6i}\lambda_i\tilde{l}_i = \sum a_{7i}\lambda_i\tilde{k}_i, \quad \sum = \sum_{i=1}^{4}, \quad \tilde{m}_2 = \det(M) \tag{4.7}$$

M is the 4×4 matrix with elements M_{ji} as in Eq.(4.5) (in this equation "-1" denotes the inverse matrix). The $\tilde{d}_i$, $\tilde{l}_i$ and $\tilde{k}_i$ are the corresponding algebraic cofactors of matrix M with elements M_{ji} for $j = 2, 3,$ and 4, respectively (see equations A7).

The physical thermal fields in the multifield half – space are obtained as follows

$$u_r^T(r,z) = -\frac{T_0 r a_{10}\lambda_0}{\pi a_{00}}\left(\frac{\pi}{2} - \tan^{-1}\xi_0 + \frac{\xi_0}{1+\xi_0}\frac{1-\eta_0}{1+\eta_0}\right) + $$
$$+\frac{T_0 r}{\pi}\sum_{i=1}^{4}a_{1i}\lambda_i\tilde{\Delta}_i^T\left(\frac{\pi}{2} - \tan^{-1}\xi_i + \frac{\xi_i}{1+\xi_i^2}\frac{1-\eta_i}{1+\eta_i}\right),$$

$$\begin{bmatrix} u_z^T(r,z) \\ \varphi(r,z) \\ \psi(r,z)\end{bmatrix} = \frac{2T_0 a}{\pi a_{00}}\begin{bmatrix} 1 \\ -a_{30} \\ -a_{40}\end{bmatrix}\left[\frac{1}{2}\ln\frac{1+\eta_0}{1-\eta_0} - \eta_0\left(1-\xi_0\tan^{-1}\frac{1}{\xi_0}\right)\right] + $$
$$-\frac{2T_0 a}{\pi}\sum_{i=1}^{4}\begin{bmatrix} 1 \\ -a_{3i} \\ -a_{4i}\end{bmatrix}\tilde{\Delta}_i^T\left[\frac{1}{2}\ln\frac{1+\eta_i}{1-\eta_i} - \eta_i\left(1-\xi_i\tan^{-1}\frac{1}{\xi_i}\right)\right],$$

$$\begin{bmatrix} \sigma_{zr}^T(r,z) \\ E_r^T(r,z) \\ H_r^T(r,z) \\ D_r^T(r,z) \\ B_r^T(r,z)\end{bmatrix} = \frac{2T_0}{\pi a_{00}}\frac{a}{r}\begin{bmatrix} a_{50} + \frac{\beta_1}{\lambda_0}a_{00} \\ a_{30} \\ a_{40} \\ a_{60}\lambda_0^2 \\ a_{70}\lambda_0^2\end{bmatrix}(1-\eta_0) - \frac{2T_0}{\pi}\frac{a}{r}\begin{bmatrix} a_{5i} \\ a_{3i} \\ a_{4i} \\ a_{6i}\lambda_i \\ a_{70}\lambda_i\end{bmatrix}\tilde{\Delta}_i^T(1-\eta_i),$$

$$\begin{bmatrix} \sigma_{zz}^T(r,z) \\ E_z^T(r,z) \\ H_z^T(r,z) \\ D_z^T(r,z) \\ B_z^T(r,z)\end{bmatrix} = \frac{2T_0}{\pi a_{00}}\begin{bmatrix} \left(a_{50} + \frac{\beta_1}{\lambda_0}a_{00}\right)\frac{1}{\lambda_0} \\ a_{30}\lambda_0 \\ a_{40}\lambda_0 \\ a_{60}\lambda_0 \\ a_{70}\lambda_0\end{bmatrix}\left(\frac{\pi}{2} - \tan^{-1}\xi_0\right) - \frac{2T_0}{\pi}\sum_{i=1}^{4}\begin{bmatrix} \frac{a_{5i}}{\lambda_i} \\ a_{3i}\lambda_i \\ a_{4i}\lambda_i \\ a_{6i}\lambda_i \\ a_{70}\lambda_i\end{bmatrix}\tilde{\Delta}_i^T\left(\frac{\pi}{2} - \tan^{-1}\xi_i\right),$$

$$\begin{bmatrix}\sigma_{rr}^T(r,z)\\ \sigma_{\theta\theta}^T(r,z)\end{bmatrix}=-\frac{2T_0}{\pi a_{00}}\left(a_{50}+\frac{\beta_1}{\lambda_0}a_{00}\right)\left(\frac{\pi}{2}-\tan^{-1}\xi_0\right)+$$

$$+\frac{2T_0}{\pi}\sum_{i=1}^{4}a_{5i}\lambda_i\tilde{\Delta}_i^T\left(\frac{\pi}{2}-\tan^{-1}\xi_i\right)-\left(c_{11}-c_{12}\right)\begin{bmatrix}\dfrac{u_r^T(r,z)}{r}\\ \dfrac{\partial u_r^T(r,z)}{\partial r}\end{bmatrix} \tag{4.8}$$

where

$$\tilde{\Delta}_i^T=\frac{\Delta_i^T}{\tilde{m}_2}+\frac{\beta\lambda_0 d_i^*}{m_2^*}=\beta\left(\frac{\tilde{d}}{\tilde{m}_2}+\lambda_0\frac{d_i^*}{m_2^*}\right)+\frac{p\tilde{l}_i+\gamma\tilde{k}_i}{\tilde{m}_2}. \tag{4.9}$$

Note that on the boundary $z=0$ the coordinates ξ_i and η_i are independent of eigenvalues λ_i and are

$$\begin{gathered}\xi=0,\quad r\le a,\quad a\xi=\sqrt{r^2-a^2},\quad r\ge a,\\ a\eta=\sqrt{a^2-r^2},\quad r\le a,\quad \eta=0,\quad r\ge a\end{gathered} \tag{4.10}$$

On the plane $z=0$ we obtain

$$\begin{bmatrix}u_z^T(r,0)\\ \varphi^T(r,0)\\ \psi^T(r,0)\end{bmatrix}=-\frac{2T_0 a}{\pi}\begin{bmatrix}\alpha\\ \tilde{p}\\ \tilde{\gamma}\end{bmatrix}\begin{cases}\ln\left(\dfrac{a}{r}+\sqrt{\dfrac{a^2}{r^2}-1}\right)-\sqrt{1-\dfrac{r^2}{a^2}}, & r\le a\\ 0 & ,\ r\ge a\end{cases}$$

$$\begin{bmatrix}\alpha\\ \tilde{p}\\ \tilde{\gamma}\end{bmatrix}=\frac{1}{a_{00}}\begin{bmatrix}1\\ -a_{30}\\ -a_{40}\end{bmatrix}-\sum_{i=1}^{4}\begin{bmatrix}1\\ -a_{3i}\\ -a_{4i}\end{bmatrix},$$

$$[\alpha]=10^{-6}\ 1/K,\quad [\tilde{p}]=10^{-6}\ V/mK,\quad [\tilde{\gamma}]=10^{-4}\ A/mK,$$

$$\begin{gathered}\sigma_{zz}^T(r,0)=T_0\left[1-\left(1-\frac{2}{\pi}\arcsin\frac{a}{r}\right)H\left(1-\frac{a}{r}\right)\right](\beta-\beta)=0,\\ \sigma_{zr}^T(r,0)=\frac{2}{\pi}T_0\frac{a-\sqrt{a^2-r^2}}{r}\left(\lambda_0\beta-\lambda_0\beta\right)=0,\\ D_z^T(r,0)=T_0\left[1-\left(1-\frac{2}{\pi}\arcsin\frac{a}{r}\right)H\left(1-\frac{a}{r}\right)\right]\left(p_3+p_3^*-p_3^{**}-p\right)=0,\\ B_z^T(r,0)=T_0\left[1-\left(1-\frac{2}{\pi}\arcsin\frac{a}{r}\right)H\left(1-\frac{a}{r}\right)\right]\left(\gamma_3+\gamma_3^*-\gamma_3^{**}-\gamma\right)=0.\end{gathered} \tag{4.11}$$

The coefficients α, $\tilde{p}$, and $\tilde{\gamma}$ may be called "the thermal coefficients of generalized compliances". The generalized displacements u, φ and ψ vanish on the boundary $z=0, r \geq a$ and as Fig.2 shows are regular except of the point $r=0$, where the solution has logarithmic singularity. All physical quantities satisfy the regularity conditions at infinity.

5. Punch problem

We assume that the cylindrical punch is flat ended, maintained at a constant electric and magnetic potential and temperature T_0 and loaded centrally by a concentrated force P and by a concentrated electric charge Q and total magnetic flux B.

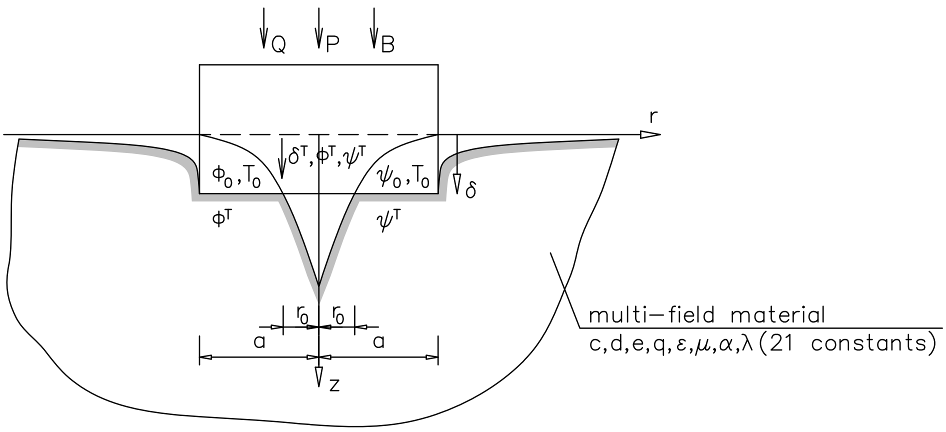

Fig.4. Punch on multifield half – space.

The contact region is annular

$$r_0 \leq r \leq a \tag{5.1}$$

where r_0 is determined by the condition (see Fig.4)

$$\delta = \delta^T(r_0) = \frac{2T_0 a}{\pi}\alpha\left[\ln\left(\frac{a}{r_0}+\sqrt{\frac{a^2}{r_0^2}-1}\right)-\sqrt{1-\frac{r_0^2}{a^2}}\right]. \tag{5.2}$$

The inner circumference of the annulus will shrink with increasing load.

The boundary conditions are

$$\begin{bmatrix} u_z(r,0) \\ \phi(r,0) \\ \psi(r,0) \end{bmatrix} = \begin{bmatrix} \delta \\ \phi_0 + \phi^T(r) \\ \psi_0 + \psi^T(r) \end{bmatrix}, \quad r_0 \leq r \leq a, \tag{5.3}$$

$$\sigma_{zz}(r,0) = 0, \quad r \geq 0, \tag{5.4}$$

$$\begin{bmatrix} \sigma_{zz}(r,0) \\ D_z(r,0) \\ B_z(r,0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad r < r_0, \quad r > a. \tag{5.5}$$

Using the Hankel transform method the integral equations become

$$\begin{bmatrix} u_z(r,0) \\ \varphi(r,0) \\ \psi(r,0) \end{bmatrix} = \int_0^\infty \xi \begin{bmatrix} -U(\xi) \\ \Phi(\xi) \\ \Psi(\xi) \end{bmatrix} J_0(r\xi)\,d\xi = \begin{bmatrix} \delta \\ \varphi_0 + \varphi^T(r) \\ \psi_0 + \psi^T(r) \end{bmatrix}, \quad r_0 \le r \le a, \tag{5.6}$$

$$\begin{bmatrix} \sigma_{zz}(r,0) \\ D_z(r,0) \\ B_z(r,0) \end{bmatrix} = [C] \int_0^\infty \xi \begin{bmatrix} U(\xi) \\ \Phi(\xi) \\ \Psi(\xi) \end{bmatrix} J_0(r\xi)\,d\xi = 0, \quad r_0 \le r \le a \tag{5.7}$$

where C is the indentation stiffness matrix defined as follows

$$[C] = \frac{1}{m_2} \sum_{i=1}^{4} \begin{bmatrix} a_{5i} d_i / \lambda_i & a_{5i} l_i / \lambda_i & a_{5i} k_i / \lambda_i \\ a_{6i} \lambda_i d_i & a_{6i} \lambda_i l_i & a_{6i} \lambda_i k_i \\ a_{7i} \lambda_i d_i & a_{7i} \lambda_i l_i & a_{7i} \lambda_i k_i \end{bmatrix} = \frac{1}{m_2} \begin{bmatrix} m & m_6 & \widetilde{m}_6 \\ m_5 & m_7 & m_8 \\ m_9 & m_{10} & m_{11} \end{bmatrix}, \tag{5.8}$$

and where d_i, l_i and k_i are the corresponding algebraic cofactors of the multifield compliance matrix $M(4 \times 4)$ with elements $M_{ji}, (i = 1, 2, 3, 4, j = 2, 3, 4, 5)$, for $j = 2$, $j = 3$ and $j = 4$, respectively

$$[M] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \\ a_{51} & a_{52} & a_{53} & a_{54} \end{bmatrix}. \tag{5.9}$$

Then, if $m_2 = \det(M)$, the material parameters d_i, l_i and k_i are

$$\begin{aligned} &M(d_1, d_2, d_3, d_4)^T = (m_2, 0, 0, 0)^T, \\ &M(l_1, l_2, l_3, l_4)^T = (0, m_2, 0, 0)^T, \\ &M(k_1, k_2, k_3, k_4)^T = (0, 0, m_2, 0)^T \end{aligned} \tag{5.10}$$

here T denotes the transpose of a matrix. The condition (5.4) is satisfied identically. We have

$$\begin{aligned} &m_2 = \sum d_i = \sum a_{3i} l_i = \sum a_{4i} k_i, \\ &\sum l_i = \sum k_i = \sum a_{3i} d_i = \sum a_{3i} k_i = \sum a_{4i} d_i = \sum a_{4i} l_i = \sum a_{5i} d_i = \sum a_{5i} l_i = \sum a_{5i} k_i = 0 \end{aligned} \tag{5.11}$$

where the sum is from 1 to 4. The coefficients d_i, l_i and k_i are given in the Appendix A by Eqs (A6).

The integral equations become

$$\int_0^\infty \xi \begin{bmatrix} -U(\xi) \\ \Phi(\xi) \\ \Psi(\xi) \end{bmatrix} J_0(r\xi)\,d\xi = \begin{bmatrix} \delta \\ \varphi_0 + \varphi^T(r) \\ \psi_0 + \psi^T(r) \end{bmatrix}, \quad r_0 \le r \le a \tag{5.12}$$

$$[C]\int_0^\infty \xi^2 \begin{bmatrix} U(\xi) \\ \Phi(\xi) \\ \Psi(\xi) \end{bmatrix} J_0(r\xi)\,d\xi = 0, \quad r < r_0, \quad r > a \tag{5.13}$$

Changing the variable $r \in \langle r_0, a\rangle$ on $\alpha \in \langle 0, \pi\rangle$ by equation

$$2r_c b\cos\alpha = r_c^2 + b^2 - r^2, \quad a = r_c + b, \quad r_0 = r_c - b, \tag{5.14}$$

and assuming that $\xi U(\xi)$, $\xi\varphi(\xi)$, $\xi\psi(\xi)$ can be presented by integral as follows

$$\xi \begin{bmatrix} U(\xi) \\ \varphi(\xi) \\ \psi(\xi) \end{bmatrix} = \int_0^\infty \begin{bmatrix} F_1(\beta) \\ F_2(\beta) \\ F_3(\beta) \end{bmatrix} J_0(\xi R)\,d\beta,$$

$$R^2 = r_c^2 + b^2 - 2r_c b\cos\beta, \tag{5.15}$$

we obtain

$$\begin{bmatrix} \sigma_{zz}(r,0) \\ D_z(r,0) \\ B_z(r,0) \end{bmatrix} = -[C]\int_0^\infty \begin{bmatrix} F_1(\beta) \\ F_2(\beta) \\ F_3(\beta) \end{bmatrix} \frac{\delta(R-r)}{\sqrt{Rr}}\,d\beta. \tag{5.16}$$

Here $F_1(\beta)$, $F_2(\beta)$ and $F_3(\beta)$ are arbitrary continuous functions, $\delta(R-r)$ is Dirac's delta function and the following formula is used

$$\int_0^\infty \xi J_0(\xi R) J_0(\xi r)\,d\xi = \frac{\delta(R-r)}{\sqrt{Rr}} \tag{5.17}$$

Then Eqs (5.13) are identically satisfied. Introducing the series representations

$$\begin{bmatrix} F_1(\beta) \\ F_2(\beta) \\ F_3(\beta) \end{bmatrix} = \sum_{n=0}^\infty \begin{bmatrix} a_n \\ b_n \\ c_n \end{bmatrix} \cos(n\beta), \tag{5.18}$$

we obtain from Eqs (5.16)

$$\begin{bmatrix} \sigma_{zz}(r,0) \\ D_z(r,0) \\ B_z(r,0) \end{bmatrix} = -[C]\frac{1}{r_c b \sin\alpha}\sum_{n=0}^{\infty}\begin{bmatrix} a_n \\ b_n \\ c_n \end{bmatrix}\cos(n\beta). \tag{5.19}$$

Substituting Eqs (5.18) and (5.15) into the boundary conditions (5.12), we obtain

$$\sum_{n=0}^{\infty}\begin{bmatrix} a_n \\ b_n \\ c_n \end{bmatrix}\int_0^{\infty} J_0(\xi r)\,d\xi\int_0^{\pi} J_0(\xi R)\cos(n\beta)\,d\beta = \begin{bmatrix} \delta \\ -\varphi_0 - \varphi^T(r) \\ -\psi_0 - \psi^T(r) \end{bmatrix}. \tag{5.20}$$

Using the formula (Gradsztejn and Ryzhik, 1965)

$$\int_0^{\pi} J_0(\xi R)\cos(n\beta)\,d\beta = \pi J_n(\xi r_c)J_n(\xi b) = \pi Z_n(\xi r_c, \xi b). \tag{5.21}$$

Then Eq.(5.20) becomes

$$\sum_{n=0}^{\infty}\begin{bmatrix} a_n \\ b_n \\ c_n \end{bmatrix}\int_0^{\infty} J_0(\xi r) Z_n(\xi r_c, \xi b)\,d\xi = \frac{1}{\pi}\begin{bmatrix} \delta \\ -\varphi_0 - \varphi^T(r) \\ -\psi_0 - \psi^T(r) \end{bmatrix}. \tag{5.22}$$

Introducing the new coefficients d_n and e_n instead of b_n and f_n and g_n instead of c_n as follows

$$\begin{aligned} b_n &= -\frac{a}{\pi}\varphi_0 d_n - \frac{2T_0 a^2}{\pi^2}\tilde{p} e_n, \\ c_n &= -\frac{a}{\pi}\psi_0 f_n - \frac{2T_0 a^2}{\pi^2}\tilde{\gamma} g_n, \end{aligned} \tag{5.23}$$

Eq.(5.22) are converted to five algebraic equations with respect to a_n, d_n, e_n, f_n and g_n

$$\sum_{n=0}^{\infty}\begin{bmatrix} a_n \\ d_n \\ e_n \\ f_n \\ g_n \end{bmatrix}\int_0^{\infty} J_0(\xi r) Z_n(\xi r_c, \xi b)\,d\xi = \begin{bmatrix} 1 \\ 1 \\ h(r) \\ 1 \\ h(r) \end{bmatrix}, \quad r_0 \le r \le a \tag{5.24}$$

where

$$h(r)=ln\left(\frac{a}{r}+\sqrt{\frac{a^2}{r^2}-1}\right)-\sqrt{1-\frac{r^2}{a^2}}, \quad r\le a. \tag{5.25}$$

We conclude that $a_n = d_n = f_n$ and $e_n = g_n$. By using Neumann's formula (Gradsztejn and Ryzhik, 1965)

$$J_0(\xi r)=Z_0(\xi r_c,\xi b)+2\sum_{m=1}^{\infty}Z_m(\xi r_c,\xi b)\cos(m\alpha), \quad 0\le\alpha\le\pi, \tag{5.26}$$

and the Fourier expansion for the function $h(r)$

$$h(r)=h_0+2\sum h_m\cos(m\alpha),$$
$$h_m=\frac{1}{\pi}\int_0^{\pi}h(r)\cos(m\alpha)\,d\alpha, \tag{5.27}$$

Eqs.(5.24) are converted to two simultaneous algebraic equations

$$\sum_{n=0}^{\infty}a_nA_{mn}=\delta_{0m},$$
$$\sum_{n=0}^{\infty}e_nA_{mn}=h_m, \qquad m=0,1,2,... \tag{5.28}$$

with the matrix

$$A_{mn}=\int_0^{\infty}J_m(\xi s_1)J_m(\xi s_2)J_n(\xi s_1)J_n(\xi s_2)\,d\xi,$$
$$s_{1,2}=\frac{1}{2}(1\pm s), \qquad s=\frac{r_0}{a}, \tag{5.29}$$

and where δ_{0m} is Kronecker's delta.

From the condition that $\sigma_{zz}(r,0)$ must be finite at $r=r_0$ $(\alpha=0)$ we conclude that the infinite series in Eqs (5.19) must be zero for $\alpha\to 0\,(r\to r_0+0)$. This gives the condition

$$\frac{\sum_{n=0}^{\infty}a_n}{\sum_{n=0}^{\infty}e_n}=\frac{\dfrac{2T_0a}{\pi\delta}\left(\tilde{p}\dfrac{m_6}{m}+\tilde{\gamma}\dfrac{\tilde{m}_6}{m}\right)}{1-\dfrac{m_6}{m}\dfrac{\phi_0}{\delta}-\dfrac{\tilde{m}_6}{m}\dfrac{\psi_0}{\delta}}. \tag{5.30}$$

The equation gives the relation between the depth of penetration of the punch δ, the boundary electric and magnetic potentials φ_0 and ψ_0 and the radius of the contact region r_0, since the sets a_n and e_n depend on r_0. Of course, temperature T_0 appears in this equation.

Finally, we obtain

a) contact stress

$$\sigma_{zz}(r,0) = -\frac{a\delta m}{\pi r_c b m_2}\frac{1}{\sin\alpha}\sum_{n=0}^{\infty}\left[a_n\left(1-\frac{\varphi_0}{\delta}\frac{m_6}{m}-\frac{\psi_0}{\delta}\frac{\tilde{m}_6}{m}\right)-e_n\frac{2T_0 a}{\pi\delta}\left(\tilde{p}\frac{m_6}{m}+\tilde{\gamma}\frac{\tilde{m}_6}{m}\right)\right], \tag{5.31}$$

b) electric displacement

$$D_z(r,0) = -\frac{a\delta m_5}{\pi r_c b m_2}\frac{1}{\sin\alpha}\sum_{n=0}^{\infty}\left[a_n\left(1-\frac{\varphi_0}{\delta}\frac{m_7}{m_5}-\frac{\psi_0}{\delta}\frac{m_8}{m_5}\right)-e_n\frac{2T_0 a}{\pi\delta}\left(\tilde{p}\frac{m_7}{m_5}+\tilde{\gamma}\frac{m_8}{m_5}\right)\right], \tag{5.32}$$

c) magnetic induction

$$B_z(r,0) = -\frac{a\delta m_9}{\pi r_c b m_2}\frac{1}{\sin\alpha}\sum_{n=0}^{\infty}\left[a_n\left(1-\frac{\phi_0}{\delta}\frac{m_{10}}{m_9}-\frac{\psi_0}{\delta}\frac{m_{11}}{m_9}\right)-e_n\frac{2T_0 a}{\pi\delta}\left(\tilde{p}\frac{m_{10}}{m_9}+\tilde{\gamma}\frac{m_{11}}{m_9}\right)\right], \tag{5.33}$$

d) displacement on the boundary $z=0$

$$u_z(r,0) = \delta\sum_{n=0}^{\infty}a_n I_0^n\left(\frac{r}{a}\right), \tag{5.34}$$

e) electric potential on the boundary $z=0$

$$\varphi(r,0) = -\varphi_0\sum_{n=0}^{\infty}a_n I_0^n\left(\frac{r}{a}\right), \tag{5.35}$$

f) magnetic potential on the boundary $z=0$

$$\psi(r,0) = -\psi_0\sum_{n=0}^{\infty}a_n I_0^n\left(\frac{r}{a}\right) \tag{5.36}$$

where the integrals

$$I_0^n\left(\frac{r}{a}\right) = \int_0^{\infty}J_0\left(\xi\frac{r}{a}\right)J_n(\xi s_1)J_n(\xi s_2)\,d\xi, \tag{5.37}$$

are presented analytically, as well as the matrix A_{mn} in Eq.(5.28), by Rogowski (2006).

It can be shown that the gradient of $u_z(r,0)$ is bounded for $r\to r_0+0$ and unbounded for $r\to r_0-0$ and for $r\to a+0$ tending, to plus or minus infinity, respectively. If in the contact region potentials φ_0 and ψ_0 appear the gradient of $u_z(r,0)$ tends to minus infinity at the left neighborhood of the contact circle $r = r_0$. This phenomenon occurs, among others, in adhesive contact problem.

The equilibrium of the punch yields

$$P = 2\pi a\delta \frac{m}{m_2}\left[a_0\left(1 - \frac{\varphi_0}{\delta}\frac{m_6}{m} - \frac{\psi_0}{\delta}\frac{\tilde{m}_6}{m}\right) - e_0\frac{2T_0 a}{\pi\delta}\left(\tilde{p}\frac{m_6}{m} + \tilde{\gamma}\frac{\tilde{m}_6}{m}\right)\right]. \tag{5.38}$$

The coefficients h_m are calculated from the formulae

$$h_m = \frac{r_c b}{a^2}\left(2\delta_{0m} + \delta_{1m}\right) \quad \text{for} \quad h(r) = 1 - \frac{r^2}{a^2},$$

$$h_m(0) = \frac{1}{4}\left(2\delta_{0m} + \delta_{1m}\right) \quad \text{for} \quad r_0 = 0, \tag{5.39}$$

$$h_m = \frac{1}{\pi a}\int_0^a\left(a - \sqrt{a^2 + b^2 - 2r_c b\cos\alpha}\right)\cos(m\alpha)\,d\alpha \quad \text{for} \quad h(r) = 1 - \frac{r}{a}.$$

The solution of the infinite systems of algebraic Eqs (5.28) for $r_0 = 0$ is

$$a_n = -\frac{4}{\pi\left(1 + \delta_{n0}\right)}\frac{1}{4n^2 - 1},$$

$$e_n = -\frac{4}{\pi\left(1 + \delta_{n0}\right)}\left(\frac{1}{4n^2 - 1} - \frac{6}{4n^2 - 9}\right) \tag{5.40}$$

where δ_{n0} is Kronecker's delta.

Equations (5.38) and (5.40) yield

$$P = 4\delta a\frac{m}{m_2}\left[1 - \frac{\varphi_0}{\delta}\frac{m_6}{m} - \frac{\psi_0}{\delta}\frac{\tilde{m}_6}{m} - \frac{2}{3}\frac{T_0 a}{\delta}\left(\tilde{p}\frac{m_6}{m} + \tilde{\gamma}\frac{\tilde{m}_6}{m}\right)\right], \tag{5.41}$$

which is the solution of the problem ("theoretically") of full contact.

Note that the solution (a_n, e_n) depends on the ratio of contact radii r_0/a and the inner radius r_0 is unknown. Notice that for an annular contact region the solution (a_n, e_n) of the simultaneous algebraic equations satisfies the inequalities

$$\left|a_n, e_n(s \to 1)\right| \le \left|a_n, e_n(s)\right| \le \left|a_n, e_n(0)\right|, \quad s = \frac{r_0}{a}. \tag{5.42}$$

The piezoelectric response amplitude m_6/m and piezomagnetic response amplitude $\tilde{m}_6/m$ for real materials are negative. This proves that increasing the applied electric and / or magnetic potentials will always reduce the indentation depth δ. Similarly, if $T_0\tilde{p}$ and / or $T_0\tilde{\gamma}$ are positive the indentation depth δ is smaller than that for isothermal problem. For negative $T_0\tilde{p}$ and / or $T_0\tilde{\gamma}$ the thermal generalized displacement φ^T and ψ^T change the sign and δ increases.

The piezoelectric and piezomagnetic response amplitudes are defined as follows

$$\mathrm{Re}\,R = \left(\frac{\partial\delta}{\partial\varphi_0}\right)_{\substack{P=const\\ \psi_0=const}} = \frac{m_6}{m} \quad \left[\frac{m}{V}\right],$$
$$PmR = \left(\frac{\partial\delta}{\partial\psi_0}\right)_{\substack{P=const\\ \varphi_0=const}} = \frac{\tilde{m}_6}{m} \quad \left[\frac{m}{A}\right]. \tag{5.43}$$

For $BaTiO_3 - CoFe_2O_4$ commercial composite we have

PeR = $-2.14\times10^{-9} m/V$,   PmR = $-7.7\times10^{-9} m/A$.

The indentation elastic stiffness coefficient

$$\frac{P}{4a\delta} = \frac{m}{m_2}, \tag{5.44}$$

assumes the values of 62.5GPa for the multifield composite $BaTiO_3 - CoFe_2O_4$ and 70.4GPa for a pure elastic transversely isotropic material with parameters c_{ij} only for this composite.

The indentation piezoelectric coefficient is

$$\frac{Q}{4a\delta} = \frac{m_5}{m_2} = 14{,}3\frac{C}{m^2}. \tag{5.45}$$

The indentation piezomagnetic coefficient is

$$\frac{B}{4a\delta} = \frac{m_9}{m_2} = 7.0\times10^3\frac{N}{Am}, \tag{5.46}$$

for this multifield material.

The total concentrated electric charge Q and total magnetic flux B in the contact region are obtained by integrating D_z and B_z over the circle of contact. We obtain

$$Q = 2\pi a\delta\frac{m_5}{m_2}\left[a_0\left(1-\frac{\varphi_0}{\delta}\frac{m_7}{m_5}-\frac{\psi_0}{\delta}\frac{m_8}{m_5}\right)-e_0\frac{2T_0a}{\pi\delta}\left(\tilde{p}\frac{m_7}{m_5}+\tilde{\gamma}\frac{m_8}{m_5}\right)\right],$$
$$B = 2\pi a\delta\frac{m_9}{m_2}\left[a_0\left(1-\frac{\varphi_0}{\delta}\frac{m_{10}}{m_9}-\frac{\psi_0}{\delta}\frac{m_{11}}{m_9}\right)-e_0\frac{2T_0a}{\pi\delta}\left(\tilde{p}\frac{m_{10}}{m_9}+\tilde{\gamma}\frac{m_{11}}{m_9}\right)\right]. \tag{5.47}$$

The coupling terms m_6, $\widetilde{m}_6$, m_8 (or m_5, m_9, m_{10}), which are the elements of the matrix on the right hand side of Eq.(5.8) are non – zero for multifield materials. This suggests that even in the absence of an applied electric and / or magnetic potentials, an electric and magnetic charge could be accumulated on the surface due to the applied mechanical load or displacement.

Similarly, an applied electric potential and / or magnetic potential could cause mechanical pressure at the contact region. Solving Eqs (5.38) and (5.47) with respect to δ, φ_0 and ψ_0, we obtain the corresponding generalized compliance relations.

Defining the stress, electric displacement and magnetic induction intensity factors as follows

$$\left[K_\sigma, K_D, K_B\right]^T = \lim_{r\to a^-}\sqrt{2(a-r)}\left[\sigma_z(r,0), D_z(r,0), B_z(r,0)\right]^T \tag{5.48}$$

we obtain

$$\left[K_\sigma, K_D, K_B\right]^T = \frac{1}{2\pi a\sqrt{a}}\left[P, Q, B\right]. \tag{5.49}$$

6. Single phase materials and multifield composite materials

Use the notation

$$e_1 = (e_{31}+e_{15})\lambda_i^2, \quad e_2 = e_{31}+e_{33}\lambda_i^2, \quad c = c_{11}+c_{13}\lambda_i^2, \quad \varepsilon_1 = \varepsilon_{11}-\varepsilon_{33}\lambda_i^2, \tag{6.1}$$

and define the matrix and its inverse

$$C_E = \begin{bmatrix} e_1 & \varepsilon_1 & 0 \\ 0 & 0 & -\infty \\ c & e_2 & 0 \end{bmatrix}, \qquad C_E^{-1} = \frac{1}{e_1 e_2 - c\varepsilon_1}\begin{bmatrix} e_2 & 0 & -\varepsilon_1 \\ -c & 0 & e_1 \\ 0 & 0 & 0 \end{bmatrix}. \tag{6.2}$$

Of course, $C_E C_E^{-1}$ is the square unit matrix.

Similarly,

$$q_1 = (q_{31}+q_{15})\lambda_i^2, \quad q_2 = q_{31}+q_{33}\lambda_i^2, \quad c = c_{11}+c_{13}\lambda_i^2, \quad \mu_1 = \mu_{11}-\mu_{33}\lambda_i^2, \tag{6.3}$$

$$C_H = \begin{bmatrix} 0 & -\infty & 0 \\ q_1 & 0 & \mu_1 \\ c & 0 & q_2 \end{bmatrix}, \qquad C_H^{-1} = \frac{1}{q_1 q_2 - c\mu_1}\begin{bmatrix} 0 & q_2 & -\mu_1 \\ 0 & 0 & 0 \\ 0 & -c & q_1 \end{bmatrix}. \tag{6.4}$$

Thus, we obtain

$$\begin{Bmatrix} a_{1i} \\ a_{3i} \end{Bmatrix}^E = \frac{1}{e_1 e_2 - c\varepsilon_1}\begin{bmatrix} e_2 & -\varepsilon_1 \\ -c & e_1 \end{bmatrix}\begin{Bmatrix} e_{33}\lambda_i^2 - e_{15} + p_3 a_{00}\lambda_i\delta_{i0} \\ c_{33}\lambda_i^2 + c_{13} - \left(\beta_3 + \beta_1\lambda_i^{-2}\right)a_{00}\lambda_i\delta_{i0} \end{Bmatrix}, \tag{6.5}$$

$$\begin{Bmatrix} a_{1i} \\ a_{4i} \end{Bmatrix}^H = \frac{1}{q_1 q_2 - c\mu_1} \begin{bmatrix} q_2 & -\mu_1 \\ -c & q_1 \end{bmatrix} \begin{Bmatrix} q_{33}\lambda_i^2 - q_{15} + \gamma_3 a_{00}\lambda_i \delta_{i0} \\ c_{33}\lambda_i^2 + c_{13} - \left(\beta_3 + \beta_1 \lambda_i^{-2}\right) a_{00}\lambda_i \delta_{i0} \end{Bmatrix}, \tag{6.6}$$

respectively, for piezoelectric and piezomagnetic thermoelastic materials.

Note that for a piezoelectric material $a_{4i} \equiv 0$, but a_{3i} define the coefficients a_{7i}, that is also the magnetic induction B_z, by the electromagnetic constant d_{11}. Similarly, for the piezomagnetic material where $a_{3i} \equiv 0$, a_{4i} define a_{6i}, that is also D_z, as a consequence of the electromagnetic effect (see Eqs (2.7)).

For a composite two – phase (E + H) multifield material the compliance matrix may be defined as

$$C_{EH}^{-1} = \frac{1}{2}\left(C_E^{-1} + C_H^{-1}\right). \tag{6.7}$$

The constitutive stiffness matrix for the composite will be the inversion of this compliance matrix. The solution presented here may be used for composite materials made of multifield materials.

7. Conclusions

The potential harmonic theory method has been generalized in this paper to analyze the thermal Green's functions for a multifield material. Green's functions are used to analyze the contact problem of a heated and conducting punch indenting a multifield half – space. The boundary value problem is converted to triple integral equations, which are reduced to simultaneous two infinite systems of algebraic equations. In the limiting case of full contact, which theoretically may occur, the closed form of solution is obtained. The expressions for displacements, stresses, electric and magnetic potentials and electric displacement, and magnetic fluxes are presented in terms of infinite series. Some important relationships between the applied or accompanied loads and indentation depth, constant electric potential and magnetic potential are established. It is worth mentioning here again that the general solution shall take another form for equal eigenvalues cases. However, one can also derive the corresponding results of equal eigenvalues directly from the ones of distinct eigenvalues, by utilizing the well known l'Hospital rule.

In the light of the analytical analysis the following conclusions can be formulated.

1. Increasing the applied electric and / or magnetic potentials will always reduce the indentation depth of the punch.
2. The indentation electric stiffness for the representative multifield material $BaTiO_2 – CoFe_2O_3$ is smaller than that for a pure elastic transversely isotropic material.
3. The hypothesis of the constant electric and magnetic potential in the contact region is equivalent to a centrally applied concentrated charge Q and magnetic induction B, which can be obtained by integrating electric displacement and / or magnetic flux over the annular contact region.
4. It can be seen from the obtained formulae that the complete solution can be separated into three parts: the first part corresponds to the normal displacement δ, the second to the electric potential ϕ_0 and the third to the magnetic potential ψ_0.
5. If the contact region electric and magnetic potentials occur (conducting punch), then the phenomenon similar to adhesive contact occurs in the left neighborhood of the inner radius of the contact region.

Nomenclature

a	– outer radius of the contact region or radius of the heated circular field
B	– magnetic flux applied to the punch
B_r, B_z	– magnetic inductions
C	– indentation stiffness matrix
D	– electric charge applied to the punch
D_r, D_z	– electric displacements
d	– electromagnetic constants
E	– electric field
e	– piezoelectric constants
q	– piezomagnetic constants
H	– magnetic field
J_m	– Bessel function of the first kind of order m
K_σ, K_B, K_D	– stress, electric displacements, magnetic induction intensity factors, respectively
P	– force applied to the punch
Q	– heat flux or electric charge applied to the punch
r_0	– inner radius of the contact region
T	– temperature
u_r, u_z	– components of displacement
z	– vertical coordinate
δ	– indentation depth
ε	– dielectric constants
ζ, η	– the oblate spheroidal coordinate system
λ_i	– eigenvalues of multifield material
μ	– magnetic constants
ξ	– Hankel parameter
σ	– stress tensor
φ	– electric potential
φ_0	– electric potential on the contact surface
ψ	– magnetic potential
ψ_0	– magnetic potential on the contact surface

Other symbols are defined in the text of the paper.

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From author

The Curie brothers discovered the piezoelectric effect in 1880 (Cady W. G., Piezoelectricity, New York, McGraw – Hill, 1964). Today, over a 100 piezoelectric materials or composites are known (Pohanka R. C., Smith P. L., Recent advances in piezoelectric ceramics, In: Levinson M. L., editor, Electronic ceramics, New York: Marcel Dekker, 1988).

The electro – mechanical, magneto – mechanical, electro – magnetic coupling of those materials, which I name "multifield materials", has an immense technological potential in designing "smart" materials and structures ranging from huge aerospace structures to miniaturized medical devices and miniatural medical apparatures. Multifield materials have been widely used for applications such as sensors, filters, ultrasonic generators and actuators, magnetic field probes, acoustic and ultrasonic devices, hydrophones. Also these materials have been employed as integrated active structural elements. These structures are capable of monitoring and adapting to their environment providing a "smart", response to the external conditions. Interested readers may refer to a state of the art survey by Rao S. S., Sunar M., Piezoelectricity and its use in disturbance sensing and control of flexible structures: a survey, Appl. Mech. Rev., 1994, 47, 113 – 123. One of the practical examples of a piezoelectric device is a piezoelectric accelerometer for triggering the onset of an airbag in tens of thousandths of a second during the accident. The advanced micro – electro – mechanical systems use piezoelectric materials in the latest technologies of smart / intelligent designs featuring miniaturization.

The physical law for piezoelectric materials has been explored by Nowacki W., in 1° Some general theorems in thermo – piezoelectricity, J. Therm., Stresses, 1978, 1, 171 – 182, 2° Foundations of linear piezoelectricity. In: Parkus H, editor, Electromagnetic interaction in elastic solids. Wien, Springer, 1979, 3° Mathematical models of phenomenological piezoelectricity. New problems in mechanics continua, Waterloo, Ontario: University of Waterloo Press, 1983, pp. 29 – 39.

Applications today: important roles in the design and health monitoring of ship and marine structures by establishing plentiful styles of devices such as sensors, actuators and power supplies with the responsibility of electro – magneto – mechanical energy conversion. Other fields of applications include: microwave electronics, optoelectronics, electronic instrumentation and scanning probe microscopy technique.

Due to exceptional functions of multifield materials such as flat frequency response and transformation of energy from one form to the other (mechanical, electric, magnetic or thermal energy) this type of composite exhibiting piezoelectric and piezomagnetic properties has found increasing applications in the following branches: aerospace, automotive, industries and submarines.

Nowadays, multifield composite materials have a wide range of applications in engineering science such as space planes, supersonic, air planes, rockets, missiles, fusion reactions and submarines.

This paper concerns some problems of mechanics of multifield materials. The results are new in the world science. I addresses it on the occasion of the 60th anniversary of my Department and 50 – years of my educational and research work.

Quo vadis mechanics?

BIMIMETING IN MATERIALS ENGINEERING

People observed and investigated products of nature to use its constructions in technology. "Bimimeting is about separating from nature good projects. It presents the way from biology to the nature", J. F. V. Vincent, Bimimeting modeling, Phil. Trans., R., Soc., London B, 358 (2003), 1597.

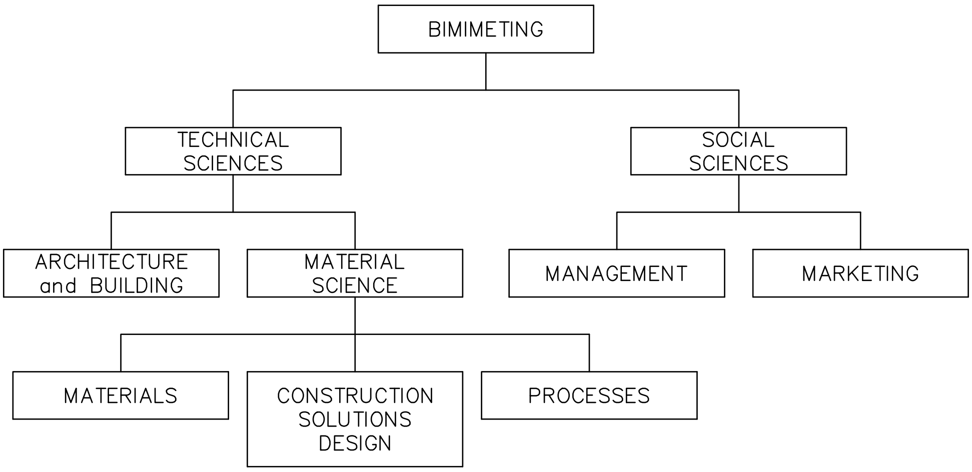
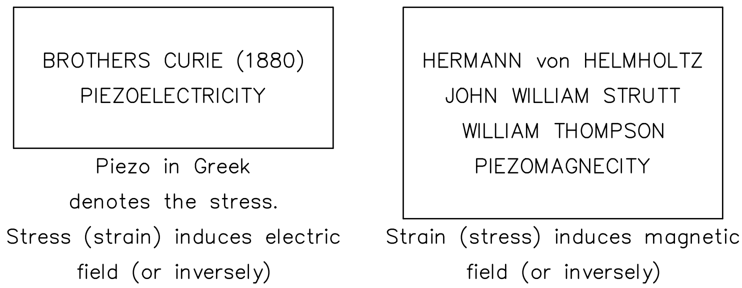
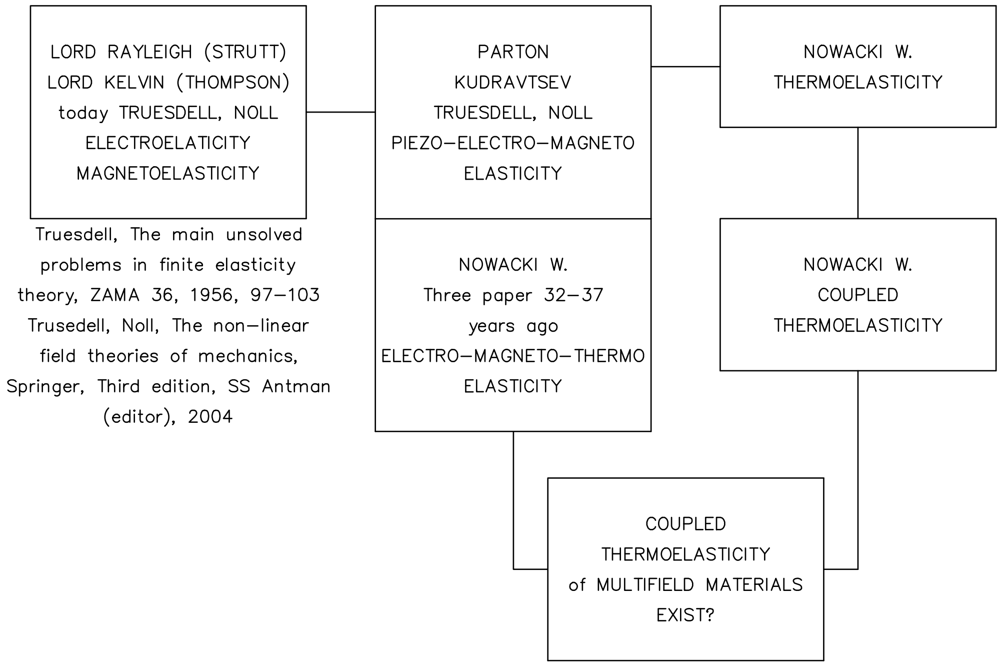


Fig.1. Bimimeting in dissimilar domains and disciplines of science (Konopka K., Wzorce z natury w technice i inżynierii materiałowej (Models from the nature in technology and material engineering), Oficyna Wydawnicza Politechniki Warszawskiej, Warszawa 2011).

MEANDERS OF MULTIFIELD MATERIALS



Def: Piezoelectricity is an electro – mechanical phenomenon, which couples elasticity and electricity through the existence of pressure induced electrical field or electric induced stress field.



Thermoelasticity – If temperature of the body increases (decreases) the displacements change and the stresses occur.

Does inverse phenomenon exist?

W. Nowacki published some papers on coupled thermoelasticity. Other scientists in experiments did not accept this theory. I think that such materials exist in nature, so the change of displacement, electric and magnetic fields changes the temperature of the body. Does inverse coupled thermomagnetoelectroelasticity exist or not? The answer is as follows: It does in theory, but materials in nature have not been discovered yet.

Appendix A. The material coefficients for mulifield materials

A1. The material parameters in the characteristic Eq.(2.6) are as follows

$$a = c_{44}\left[\mu_{33}e_{33}^2 + \varepsilon_{33}q_{33}^2 + c_{33}\mu_{33}\varepsilon_{33} - d_{33}\left(c_{33}d_{33} + 2e_{33}q_{33}\right)\right],$$

$$\begin{aligned}b = {} & \mu_{33}\left\{\left(e_{31}+e_{15}\right)\left[2c_{13}e_{33} - c_{33}\left(e_{31}+e_{15}\right)\right] + 2c_{44}e_{33}e_{31} - c_{11}e_{33}^2 - c_{33}c_{44}\varepsilon_{11}\right\} + \\ & +\varepsilon_{33}\left\{\left(q_{31}+q_{15}\right)\left[2c_{13}q_{33} - c_{33}\left(q_{31}+q_{15}\right)\right] + 2c_{44}q_{33}q_{31} - c_{11}q_{33}^2 - c_{33}c_{44}\mu_{11}\right\} + \\ & -\mu_{33}\varepsilon_{33}\tilde{c}^2 - \left(e_{31}+e_{15}\right)^2 q_{33}^2 - \left(q_{31}+q_{15}\right)^2 e_{33}^2 - c_{44}\mu_{11}e_{33}^2 - c_{44}\varepsilon_{11}q_{33}^2 + \\ & +2e_{33}q_{33}\left(q_{31}+q_{15}\right)\left(e_{31}+e_{15}\right) + d_{33}^2\tilde{c}^2 + 2c_{33}d_{33}\left(e_{31}+e_{15}\right)\left(q_{31}+q_{15}\right) + \\ & +2c_{44}c_{33}d_{11}d_{33} + 2e_{33}q_{33}\left(c_{44}d_{11} + c_{11}d_{33}\right) - 2d_{33}\left(c_{13}+c_{44}\right)\left[e_{33}\left(q_{31}+q_{15}\right) + q_{33}\left(e_{31}+e_{15}\right)\right],\end{aligned}$$

$$c=\mu_{33}\left\{2e_{15}\left[c_{11}e_{33}-c_{13}\left(e_{31}+e_{15}\right)\right]+c_{44}e_{31}^{2}+\varepsilon_{11}\tilde{c}^{2}\right\}+$$
$$+\varepsilon_{33}\left\{2q_{15}\left[c_{11}q_{33}-c_{13}\left(q_{31}+q_{15}\right)\right]+c_{44}q_{31}^{2}+\mu_{11}\tilde{c}^{2}\right\}+$$
$$+c_{33}c_{44}\mu_{11}\varepsilon_{11}+c_{11}c_{44}\mu_{33}\varepsilon_{33}+2\left(c_{13}+c_{44}\right)\left(q_{31}+q_{15}\right)\left(d_{11}e_{33}+d_{33}e_{15}-q_{33}\varepsilon_{11}\right)+$$
$$+2\left(c_{13}+c_{44}\right)\left(e_{31}+e_{15}\right)\left(d_{11}q_{33}+d_{33}q_{15}-e_{33}\mu_{11}\right)+$$
$$+\left(q_{31}+q_{15}\right)^{2}\left(c_{33}\varepsilon_{11}+2e_{33}e_{15}\right)+\left(e_{31}+e_{15}\right)^{2}\left(c_{33}\mu_{11}+2q_{33}q_{15}\right)+$$
$$-2\left(q_{31}+q_{15}\right)\left(e_{31}+e_{15}\right)\left(e_{33}q_{15}+q_{33}e_{15}+c_{33}d_{11}+c_{44}d_{33}\right)+$$
$$-2c_{11}d_{33}\left(e_{33}q_{15}+q_{33}e_{15}\right)-2c_{44}d_{11}\left(q_{33}e_{15}+e_{33}q_{15}\right)+$$
$$-2c_{11}d_{11}q_{33}e_{33}-2c_{44}d_{33}q_{15}e_{15}+2c_{44}q_{15}q_{33}\varepsilon_{11}+2c_{44}e_{15}e_{33}\mu_{11}+$$
$$+c_{11}q_{33}^{2}\varepsilon_{11}+c_{11}e_{33}^{2}\mu_{11}-2\tilde{c}^{2}d_{33}d_{11}-c_{11}c_{44}d_{33}^{2}-c_{44}c_{33}d_{11}^{2},$$

$$d=-c_{11}\mu_{33}\left(c_{44}\varepsilon_{11}+e_{15}^{2}\right)-c_{11}\varepsilon_{33}\left(c_{44}\mu_{11}+q_{15}^{2}\right)-c_{44}\left(e_{31}^{2}\mu_{11}+q_{31}^{2}\varepsilon_{11}\right)+$$
$$-e_{31}^{2}q_{15}^{2}-q_{31}^{2}e_{15}^{2}-\mu_{11}\varepsilon_{11}\tilde{c}^{2}+d_{11}\tilde{c}^{2}+2c_{11}c_{44}d_{11}d_{33}+$$
$$+2c_{13}q_{15}q_{31}\varepsilon_{11}+2c_{13}e_{15}e_{31}\mu_{11}-2c_{11}q_{15}q_{33}\varepsilon_{11}-2c_{11}e_{15}e_{33}\mu_{11}+$$
$$+2c_{13}q_{15}^{2}\varepsilon_{11}+2c_{13}e_{15}^{2}\mu_{11}+2e_{31}e_{15}q_{31}q_{15}+2c_{11}e_{15}q_{15}d_{33}+$$
$$+d_{11}\left[-2c_{13}e_{15}\left(q_{15}+q_{31}\right)-2c_{13}q_{15}\left(e_{15}+e_{31}\right)\right]+d_{11}\left[2c_{11}\left(e_{15}q_{33}+q_{15}e_{33}\right)+2c_{44}e_{31}q_{31}\right],$$

$$e=c_{11}\left[\mu_{11}e_{15}^{2}+\varepsilon_{11}q_{15}^{2}+c_{44}\varepsilon_{11}\mu_{11}-d_{11}\left(c_{44}d_{11}+2e_{15}q_{15}\right)\right],$$

$$\tilde{c}^{2}=c_{11}c_{33}-c_{13}\left(c_{13}+2c_{44}\right).$$

A2. The parameters a_1, b_1, c_1, d_1 (define a_{1i}), and a_2, b_2, c_2, d_2 in Eq.(2.7) are

$$a_1=\beta_1\left[c_{33}\left(\varepsilon_{33}\mu_{33}-d_{33}^{2}\right)+\mu_{33}e_{33}^{2}+\varepsilon_{33}q_{33}^{2}-2e_{33}d_{33}q_{33}\right]+$$
$$+\beta_3\left[-\left(c_{13}+c_{44}\right)\left(\varepsilon_{33}\mu_{33}-d_{33}^{2}\right)-\left(e_{31}+e_{15}\right)\left(\mu_{33}e_{33}-d_{33}q_{33}\right)-\left(q_{31}+q_{15}\right)\left(q_{33}\varepsilon_{33}-d_{33}e_{33}\right)\right]+$$
$$+\gamma_3\left[-\left(c_{13}+c_{44}\right)\left(d_{33}e_{33}-q_{33}\varepsilon_{33}\right)+\left(e_{31}+e_{15}\right)\left(d_{33}c_{33}+q_{33}e_{33}\right)-\left(q_{31}+q_{15}\right)\left(c_{33}\varepsilon_{33}+e_{33}^{2}\right)\right]+$$
$$+p_3\left[-\left(c_{13}+c_{44}\right)\left(d_{33}q_{33}-e_{33}\mu_{33}\right)+\left(q_{31}+q_{15}\right)\left(d_{33}c_{33}+q_{33}e_{33}\right)-\left(e_{31}+e_{15}\right)\left(c_{33}\mu_{33}+q_{33}^{2}\right)\right],$$

$$b_1 = \beta_1 \Big[c_{33} \left(2d_{11}d_{33} - \varepsilon_{33}\mu_{11} - \mu_{33}\varepsilon_{11} \right) + c_{44} \left(d_{33}^2 - \varepsilon_{33}\mu_{33} \right) - \varepsilon_{11}q_{33}^2 - \mu_{11}e_{33}^2 +$$
$$+ 2d_{33} \left(e_{33}q_{15} + q_{33}e_{15} \right) + 2d_{11}e_{33}q_{33} - 2q_{15}q_{33}\varepsilon_{33} - 2e_{15}e_{33}\mu_{33} \Big] +$$
$$+ \beta_3 \left[-\left(c_{13} + c_{44} \right)\left(2d_{11}d_{33} - \varepsilon_{33}\mu_{11} - \mu_{33}\varepsilon_{11} \right) + \left(q_{13} + q_{15} \right)\left(q_{15}\varepsilon_{33} + q_{33}\varepsilon_{15} - d_{11}e_{33} - d_{33}e_{15} \right) + \right.$$
$$\left. + \left(e_{31} + e_{15} \right)\left(e_{15}\mu_{33} + e_{33}\mu_{11} - d_{11}q_{33} - d_{11}q_{15} \right) \right] +$$
$$+ \gamma_3 \left[\left(c_{13} + c_{44} \right)\left(d_{11}e_{33} + d_{33}e_{15} - q_{15}\varepsilon_{33} - q_{33}\varepsilon_{11} \right) - \left(e_{31} + e_{15} \right)\left(c_{44}d_{33} + c_{33}d_{11} + q_{15}e_{33} + e_{15}q_{33} \right) + \right.$$
$$\left. + \left(q_{31} + q_{15} \right)\left(c_{44}\varepsilon_{33} + c_{33}\varepsilon_{11} + 2e_{15}e_{33} \right) \right] +$$
$$+ p_3 \left[\left(c_{13} + c_{44} \right)\left(d_{11}q_{33} + d_{33}q_{15} - e_{15}\mu_{33} - e_{33}\mu_{11} \right) - \left(q_{31} + q_{15} \right)\left(c_{44}d_{33} + c_{33}d_{11} + q_{15}e_{33} + e_{15}q_{33} \right) + \right.$$
$$\left. + \left(e_{31} + e_{15} \right)\left(c_{44}\mu_{33} + c_{33}\mu_{11} + 2q_{15}q_{33} \right) \right],$$

$$c_1 = \beta_1 \Big[c_{44} \left(\varepsilon_{11}\mu_{33} + \varepsilon_{33}\mu_{11} - 2d_{11}d_{33} \right) + c_{33} \left(\varepsilon_{11}\mu_{11} - d_{11}^2 \right) + \varepsilon_{33}q_{15}^2 + \mu_{33}e_{15}^2 +$$
$$- 2d_{11} \left(e_{15}q_{33} + q_{15}e_{33} \right) + 2q_{15}q_{33}\varepsilon_{11} + 2\mu_{11}e_{15}e_{33} \Big] +$$
$$+ \beta_3 \Big[\left(c_{13} + c_{44} \right)\left(d_{11}^2 - \varepsilon_{11}\mu_{11} \right) - \left(e_{31} + e_{15} \right)\left(\mu_{11}e_{15} - d_{11}q_{15} \right) - \left(q_{31} + q_{15} \right)\left(\varepsilon_{11}q_{15} - d_{11}e_{15} \right) \Big] +$$
$$+ \gamma_3 \Big[\left(c_{13} + c_{44} \right)\left(q_{15}\varepsilon_{11} - e_{15}d_{11} \right) + \left(e_{31} + e_{15} \right)\left(d_{11}c_{44} + q_{15}e_{15} \right) - \left(q_{31} + q_{15} \right)\left(c_{44}\varepsilon_{11} + e_{15}^2 \right) \Big] +$$
$$+ p_3 \Big[\left(c_{13} + c_{44} \right)\left(e_{15}\mu_{11} - q_{15}d_{11} \right) + \left(q_{31} + q_{15} \right)\left(d_{11}c_{44} + q_{15}e_{15} \right) - \left(e_{31} + e_{15} \right)\left(c_{44}\mu_{11} + q_{15}^2 \right) \Big],$$

$$d_1 = -\beta_1 \Big[c_{44} \left(\varepsilon_{11}\mu_{11} - d_{11}^2 \right) + \mu_{11}e_{15}^2 + \varepsilon_{11}q_{15}^2 - 2e_{15}q_{15}d_{11} \Big],$$

$$a_2 = c_{44} \Big[\beta_3 \left(\varepsilon_{33}\mu_{33} - d_{33}^2 \right) + \gamma_3 \left(d_{33}e_{33} + q_{33}\varepsilon_{33} \right) + p_3 \left(d_{33}q_{33} - e_{33}\mu_{33} \right) \Big],$$

$$b_2 = \beta_1 \Big[\left(c_{13} + c_{44} \right)\left(\varepsilon_{33}\mu_{33} - d_{33}^2 \right) - \left(e_{31} + e_{15} \right)\left(d_{33}q_{33} - \mu_{33}e_{33} \right) - \left(q_{31} + q_{15} \right)\left(d_{33}e_{33} - \varepsilon_{33}q_{33} \right) \Big] +$$
$$- \beta_3 \Big[c_{11} \left(\varepsilon_{33}\mu_{33} + d_{33}^2 \right) + c_{44} \left(\mu_{11}\varepsilon_{33} + \mu_{33}\varepsilon_{11} \right) - 2\left(q_{31} + q_{15} \right)\left(e_{31} + e_{15} \right) d_{33} +$$
$$+ \left(q_{31} + q_{15} \right)^2 \varepsilon_{33} + \left(e_{31} + e_{15} \right)^2 \mu_{33} \Big] +$$
$$- \gamma_3 \Big[c_{11} \left(d_{33}e_{33} - q_{33}\varepsilon_{33} \right) + c_{44} \left(d_{11}e_{33} + d_{33}e_{15} - q_{15}\varepsilon_{33} - q_{33}\varepsilon_{11} \right) - \left(c_{13} + c_{44} \right) d_{33} \left(e_{31} + e_{15} \right) +$$
$$- q_{33} \left(e_{31} + e_{15} \right)^2 + \varepsilon_{33} \left(q_{31} + q_{15} \right)\left(c_{13} + c_{44} \right) + e_{33} \left(q_{31} + q_{15} \right)\left(e_{31} + e_{15} \right) \Big] +$$
$$- p_3 \Big[c_{11} \left(d_{33}q_{33} - e_{33}\mu_{33} \right) + c_{44} \left(d_{11}q_{33} + d_{33}q_{15} - e_{15}\mu_{33} - e_{33}\mu_{11} \right) - \left(c_{13} + c_{44} \right) d_{33} \left(q_{31} + q_{15} \right) +$$
$$- e_{33} \left(q_{31} + q_{15} \right)^2 + \mu_{33} \left(e_{31} + e_{15} \right)\left(c_{13} + c_{44} \right) + q_{33} \left(q_{31} + q_{15} \right)\left(e_{31} + e_{15} \right) \Big],$$

$$c_2 = \beta_1\Big[-(c_{13}+c_{44})(\varepsilon_{11}\mu_{33}+\varepsilon_{33}\mu_{11})+(e_{31}+e_{15})(d_{11}q_{33}+d_{33}q_{15})+(q_{31}+q_{15})(d_{11}e_{33}+d_{33}e_{15})+$$
$$-(q_{31}+q_{15})(q_{15}\varepsilon_{33}+q_{33}\varepsilon_{11})-(e_{31}+e_{15})(e_{15}\mu_{33}+e_{33}\mu_{11})+2(c_{13}+c_{44})d_{11}d_{33}\Big]+$$
$$+\beta_3\Big[c_{44}\left(\varepsilon_{11}\mu_{11}+d_{11}^2\right)+c_{11}(\mu_{11}\varepsilon_{33}+\mu_{33}\varepsilon_{11})+\mu_{11}(e_{31}+e_{15})^2+\varepsilon_{11}(q_{31}+q_{15})^2+$$
$$-2(e_{31}+e_{15})(q_{31}+q_{15})d_{11}\Big]+$$
$$-\gamma_3\Big[c_{44}(q_{15}\varepsilon_{11}-e_{15}d_{11})+(e_{31}+e_{15})\big((c_{13}+c_{44})d_{11}+(e_{31}+e_{15})q_{15}\big)+$$
$$-(q_{31}+q_{15})\big((c_{13}+c_{44})\varepsilon_{11}+(e_{31}+e_{15})e_{15}\big)-c_{11}(d_{11}e_{33}+e_{15}d_{33}-\varepsilon_{11}q_{33}-\varepsilon_{33}q_{15})\Big]+$$
$$-p_3\Big[c_{44}(e_{15}\mu_{11}-q_{15}d_{11})+(q_{31}+q_{15})\big((c_{13}+c_{44})d_{11}+(q_{31}+q_{15})e_{15}\big)+$$
$$-(e_{31}+e_{15})\big((c_{13}+c_{44})\mu_{11}+(q_{31}+q_{15})q_{15}\big)-c_{11}(d_{11}q_{33}+q_{15}d_{33}-\mu_{11}e_{33}-\mu_{33}e_{15})\Big],$$

$$d_2 = \beta_1\Big[(c_{13}+c_{44})\varepsilon_{11}\mu_{11}-(e_{31}+e_{15})d_{11}q_{15}-(q_{31}+q_{15})d_{11}e_{15}+(q_{31}+q_{15})q_{15}\varepsilon_{11}+$$
$$+(e_{31}+e_{15})\mu_{11}e_{15}-(c_{13}+c_{44})d_{11}^2\Big]-\beta_3\Big[c_{11}\left(\varepsilon_{11}\mu_{11}-d_{11}^2\right)\Big]-\gamma_3\Big[c_{11}(e_{15}d_{11}-q_{15}\varepsilon_{11})\Big]+$$
$$-p_3\Big[c_{11}(q_{15}d_{11}-e_{15}\mu_{11})\Big].$$

A3. The parameters a_{3i} and a_{4i} in Eqs (2.1) are defined alternatively to those defined by Eqs (2.7) by parameter a_{1i} as follows

$$a_{3i} = \Bigg[\Big\{\beta_1\Big[(e_{31}+e_{15})\left(q_{33}\lambda_i^2-q_{15}\right)-(c_{13}+c_{44})\left(d_{11}-d_{33}\lambda_i^2\right)\Big]+$$
$$+\beta_3\Big[(e_{31}+e_{15})(q_{31}+q_{15})\lambda_i+\left(c_{44}\lambda_i^2-c_{11}\right)\left(d_{11}-d_{33}\lambda_i^2\right)\Big]+$$
$$+p_3\Big[\lambda_i(c_{13}+c_{44})(q_{31}+q_{15})+\left(c_{44}\lambda_i^2-c_{11}\right)\left(q_{33}\lambda_i^2-q_{15}\right)\Big]\Big\}a_{1i}\lambda_i+$$
$$+\beta_1\Big[\left(c_{33}\lambda_i^2-c_{44}\right)\left(d_{11}-d_{33}\lambda_i^2\right)-\left(q_{33}\lambda_i^2-q_{15}\right)\left(e_{33}\lambda_i^2-e_{15}\right)\Big]+$$
$$+\beta_3\lambda_i\Big[(c_{13}+c_{44})\left(d_{11}-d_{33}\lambda_i^2\right)-(q_{31}+q_{15})\left(e_{33}\lambda_i^2-e_{15}\right)\Big]+$$
$$+p_3\lambda_i\Big[(c_{13}+c_{44})\left(q_{33}\lambda_i^2-q_{15}\right)-(q_{31}+q_{15})\left(c_{33}\lambda_i^2-c_{44}\right)\Big]\Bigg]\times$$
$$\times\Big\{p_3\lambda_i\Big[(e_{31}+e_{15})\left(q_{33}\lambda_i^2-q_{15}\right)-(q_{31}+q_{15})\left(e_{33}\lambda_i^2-e_{15}\right)\Big]+$$
$$+\beta_1\Big[\left(e_{33}\lambda_i^2-e_{15}\right)\left(d_{11}-d_{33}\lambda_i^2\right)-\left(q_{33}\lambda_i^2-q_{15}\right)\left(\varepsilon_{11}-\varepsilon_{33}\lambda_i^2\right)\Big]+$$
$$+\beta_3\lambda_i\Big[(e_{31}+e_{15})\left(d_{11}-d_{33}\lambda_i^2\right)-(q_{31}+q_{15})\left(\varepsilon_{11}-\varepsilon_{33}\lambda_i^2\right)\Big]\Big\}^{-1},$$

$$a_{4i} = -\Big[\Big\{\beta_1\Big[(e_{31}+e_{15})\big(e_{33}\lambda_i^2 - e_{15}\big) - (c_{13}+c_{44})\big(\varepsilon_{11} - \varepsilon_{33}\lambda_i^2\big)\Big] +$$
$$+\beta_3\Big[(e_{31}+e_{15})^2\lambda_i + \big(c_{44}\lambda_i^2 - c_{11}\big)\big(\varepsilon_{11} - \varepsilon_{33}\lambda_i^2\big)\Big] +$$
$$+p_3\Big[\lambda_i(c_{13}+c_{44})(e_{31}+e_{15}) + \big(c_{44}\lambda_i^2 - c_{11}\big)\big(e_{33}\lambda_i^2 - e_{15}\big)\Big]\Big\}a_{1i}\lambda_i +$$
$$+\beta_1\Big[\big(c_{33}\lambda_i^2 - c_{44}\big)\big(\varepsilon_{11} - \varepsilon_{33}\lambda_i^2\big) - \big(e_{33}\lambda_i^2 - e_{15}\big)^2\Big] +$$
$$+\beta_3\lambda_i\Big[(c_{13}+c_{44})\big(\varepsilon_{11} - \varepsilon_{33}\lambda_i^2\big) - (e_{31}+e_{15})\big(e_{33}\lambda_i^2 - e_{15}\big)\Big] +$$
$$+p_3\lambda_i\Big[(c_{13}+c_{44})\big(e_{33}\lambda_i^2 - e_{15}\big) - (e_{31}+e_{15})\big(c_{33}\lambda_i^2 - c_{44}\big)\Big]\Big] \times$$
$$\times\Big\{p_3\lambda_i\Big[(e_{31}+e_{15})\big(q_{33}\lambda_i^2 - q_{15}\big) - (q_{31}+q_{15})\big(e_{33}\lambda_i^2 - e_{15}\big)\Big] +$$
$$+\beta_1\Big[\big(e_{33}\lambda_i^2 - e_{15}\big)\big(d_{11} - d_{33}\lambda_i^2\big) - \big(q_{33}\lambda_i^2 - q_{15}\big)\big(\varepsilon_{11} - \varepsilon_{33}\lambda_i^2\big)\Big] +$$
$$+\beta_3\lambda_i\Big[(e_{31}+e_{15})\big(d_{11} - d_{33}\lambda_i^2\big) - (q_{31}+q_{15})\big(\varepsilon_{11} - \varepsilon_{33}\lambda_i^2\big)\Big]\Big\}^{-1}.$$

A4. The roots of characteristic Eq.(2.6) are presented by formulae

$$\lambda_1^2 = -\frac{b}{4a} - \frac{1}{2}\sqrt{R_5 + R_6} - \frac{1}{2}\sqrt{2R_5 - R_6 + \frac{1}{4}\frac{R_7}{\sqrt{R_5 + R_6}}},$$

$$\lambda_2^2 = -\frac{b}{4a} - \frac{1}{2}\sqrt{R_5 + R_6} + \frac{1}{2}\sqrt{2R_5 - R_6 + \frac{1}{4}\frac{R_7}{\sqrt{R_5 + R_6}}},$$

$$\lambda_3^2 = -\frac{b}{4a} + \frac{1}{2}\sqrt{R_5 + R_6} - \frac{1}{2}\sqrt{2R_5 - R_6 - \frac{1}{4}\frac{R_7}{\sqrt{R_5 + R_6}}},$$

$$\lambda_4^2 = -\frac{b}{4a} + \frac{1}{2}\sqrt{R_5 + R_6} + \frac{1}{2}\sqrt{2R_5 - R_6 - \frac{1}{4}\frac{R_7}{\sqrt{R_5 + R_6}}}$$

where

$$R_1 = 2c^3 - 9bcd + 27ad^2 + 27b^2e - 72ace; \qquad R_2 = c^2 - 3bd + 12ae,$$

$$R_3 = \sqrt{R_1^2 - 4R_2^3}; \quad R_4 = \sqrt[3]{\frac{1}{2}(R_1 + R_3)},$$

$$R_5 = \frac{b^2}{4a^2} - \frac{2c}{3a}; \qquad R_6 = \frac{R_2}{3aR_4} + \frac{R_4}{3a}; \qquad R_7 = \frac{b^3}{a^3} - \frac{4bc}{a^2} + \frac{8d}{a}.$$

A5. The material parameters in Eqs (3.12) are as follows

$$d_1^* = a_{32}\left(\frac{a_{44}a_{53}}{\lambda_3} - \frac{a_{43}a_{54}}{\lambda_4}\right) + a_{33}\left(\frac{a_{42}a_{54}}{\lambda_4} - \frac{a_{44}a_{52}}{\lambda_2}\right) + a_{34}\left(\frac{a_{43}a_{52}}{\lambda_2} - \frac{a_{42}a_{53}}{\lambda_3}\right),$$

$$d_2^* = a_{31}\left(\frac{a_{43}a_{54}}{\lambda_4} - \frac{a_{44}a_{53}}{\lambda_3}\right) + a_{33}\left(\frac{a_{44}a_{51}}{\lambda_1} - \frac{a_{41}a_{54}}{\lambda_4}\right) + a_{34}\left(\frac{a_{41}a_{53}}{\lambda_3} - \frac{a_{43}a_{51}}{\lambda_1}\right),$$

$$d_3^* = a_{31}\left(\frac{a_{44}a_{52}}{\lambda_2} - \frac{a_{42}a_{54}}{\lambda_4}\right) + a_{32}\left(\frac{a_{41}a_{54}}{\lambda_4} - \frac{a_{44}a_{51}}{\lambda_1}\right) + a_{34}\left(\frac{a_{42}a_{51}}{\lambda_1} - \frac{a_{41}a_{52}}{\lambda_2}\right),$$

$$d_4^* = a_{31}\left(\frac{a_{42}a_{53}}{\lambda_3} - \frac{a_{43}a_{52}}{\lambda_2}\right) + a_{32}\left(\frac{a_{43}a_{51}}{\lambda_1} - \frac{a_{41}a_{53}}{\lambda_3}\right) + a_{33}\left(\frac{a_{41}a_{52}}{\lambda_2} - \frac{a_{42}a_{51}}{\lambda_1}\right),$$

$$m_2^* = \sum_{i=1}^{4} a_{5i} d_i^* \; .$$

A6. The material parameters d_i, l_i, and k_i in Eqs (5.8) and (5.11); $i = 1, 2, 3, 4$, are as follows

$$d_1 = a_{52}\left(a_{33}a_{44} - a_{43}a_{34}\right) + a_{53}\left(a_{34}a_{42} - a_{32}a_{44}\right) + a_{54}\left(a_{32}a_{43} - a_{33}a_{42}\right),$$

$$d_2 = a_{51}\left(a_{34}a_{43} - a_{33}a_{44}\right) + a_{53}\left(a_{31}a_{44} - a_{34}a_{41}\right) + a_{54}\left(a_{33}a_{41} - a_{31}a_{43}\right),$$

$$d_3 = a_{51}\left(a_{32}a_{44} - a_{34}a_{42}\right) + a_{52}\left(a_{34}a_{41} - a_{31}a_{44}\right) + a_{54}\left(a_{31}a_{42} - a_{32}a_{41}\right),$$

$$d_4 = a_{51}\left(a_{33}a_{42} - a_{32}a_{43}\right) + a_{52}\left(a_{31}a_{43} - a_{33}a_{41}\right) + a_{53}\left(a_{32}a_{41} - a_{31}a_{42}\right),$$

$$l_1 = a_{52}\left(a_{43} - a_{44}\right) + a_{53}\left(a_{44} - a_{42}\right) + a_{54}\left(a_{42} - a_{43}\right),$$

$$l_2 = a_{51}\left(a_{44} - a_{43}\right) + a_{53}\left(a_{41} - a_{44}\right) + a_{54}\left(a_{43} - a_{41}\right),$$

$$l_3 = a_{51}\left(a_{42} - a_{44}\right) + a_{52}\left(a_{44} - a_{41}\right) + a_{54}\left(a_{41} - a_{42}\right),$$

$$l_4 = a_{51}\left(a_{43} - a_{42}\right) + a_{52}\left(a_{41} - a_{43}\right) + a_{53}\left(a_{42} - a_{41}\right),$$

$$k_1 = a_{52}\left(a_{34} - a_{33}\right) + a_{53}\left(a_{32} - a_{34}\right) + a_{54}\left(a_{33} - a_{32}\right),$$

$$k_2 = a_{51}\left(a_{33} - a_{34}\right) + a_{53}\left(a_{34} - a_{31}\right) + a_{54}\left(a_{31} - a_{33}\right),$$

$$k_3 = a_{51}\left(a_{34} - a_{32}\right) + a_{52}\left(a_{31} - a_{34}\right) + a_{54}\left(a_{32} - a_{31}\right),$$

$$k_4 = a_{51}\left(a_{32} - a_{33}\right) + a_{52}\left(a_{33} - a_{31}\right) + a_{53}\left(a_{31} - a_{32}\right).$$

A7. The parameters in Eqs (4.6) have the following matrix forms

$$\tilde{d}_1 = -\begin{bmatrix} a_{52} & a_{53} & a_{54} \\ a_{62}\lambda_2 & a_{63}\lambda_3 & a_{64}\lambda_4 \\ a_{72}\lambda_2 & a_{73}\lambda_3 & a_{74}\lambda_4 \end{bmatrix}, \quad \tilde{d}_2 = \begin{bmatrix} a_{51} & a_{53} & a_{54} \\ a_{61}\lambda_1 & a_{63}\lambda_3 & a_{64}\lambda_4 \\ a_{71}\lambda_1 & a_{73}\lambda_3 & a_{74}\lambda_4 \end{bmatrix},$$

$$\tilde{d}_3 = -\begin{bmatrix} a_{51} & a_{52} & a_{54} \\ a_{61}\lambda_1 & a_{62}\lambda_2 & a_{64}\lambda_4 \\ a_{71}\lambda_1 & a_{72}\lambda_2 & a_{74}\lambda_4 \end{bmatrix}, \quad \tilde{d}_4 = \begin{bmatrix} a_{51} & a_{52} & a_{53} \\ a_{61}\lambda_1 & a_{62}\lambda_2 & a_{63}\lambda_3 \\ a_{71}\lambda_1 & a_{72}\lambda_2 & a_{73}\lambda_3 \end{bmatrix},$$

$$\tilde{l}_1 = \begin{bmatrix} a_{52} & a_{53} & a_{54} \\ \dfrac{a_{52}}{\lambda_2} & \dfrac{a_{53}}{\lambda_3} & \dfrac{a_{54}}{\lambda_4} \\ a_{72}\lambda_2 & a_{73}\lambda_3 & a_{74}\lambda_4 \end{bmatrix}, \quad \tilde{l}_2 = -\begin{bmatrix} a_{51} & a_{53} & a_{54} \\ \dfrac{a_{51}}{\lambda_1} & \dfrac{a_{53}}{\lambda_3} & \dfrac{a_{54}}{\lambda_4} \\ a_{71}\lambda_1 & a_{73}\lambda_3 & a_{74}\lambda_4 \end{bmatrix},$$

$$\tilde{l}_3 = \begin{bmatrix} a_{51} & a_{52} & a_{54} \\ \dfrac{a_{51}}{\lambda_1} & \dfrac{a_{52}}{\lambda_2} & \dfrac{a_{54}}{\lambda_4} \\ a_{71}\lambda_1 & a_{72}\lambda_2 & a_{74}\lambda_4 \end{bmatrix}, \quad \tilde{l}_4 = -\begin{bmatrix} a_{51} & a_{52} & a_{53} \\ \dfrac{a_{51}}{\lambda_1} & \dfrac{a_{52}}{\lambda_2} & \dfrac{a_{53}}{\lambda_3} \\ a_{71}\lambda_1 & a_{72}\lambda_2 & a_{73}\lambda_3 \end{bmatrix},$$

$$\tilde{k}_1 = -\begin{bmatrix} a_{52} & a_{53} & a_{54} \\ \dfrac{a_{52}}{\lambda_2} & \dfrac{a_{53}}{\lambda_3} & \dfrac{a_{54}}{\lambda_4} \\ a_{62}\lambda_2 & a_{63}\lambda_3 & a_{64}\lambda_4 \end{bmatrix}, \quad \tilde{k}_2 = \begin{bmatrix} a_{51} & a_{53} & a_{54} \\ \dfrac{a_{51}}{\lambda_1} & \dfrac{a_{53}}{\lambda_3} & \dfrac{a_{54}}{\lambda_4} \\ a_{61}\lambda_1 & a_{63}\lambda_3 & a_{64}\lambda_4 \end{bmatrix},$$

$$\tilde{k}_3 = -\begin{bmatrix} a_{51} & a_{52} & a_{54} \\ \dfrac{a_{51}}{\lambda_1} & \dfrac{a_{52}}{\lambda_2} & \dfrac{a_{54}}{\lambda_4} \\ a_{61}\lambda_1 & a_{62}\lambda_2 & a_{64}\lambda_4 \end{bmatrix}, \quad \tilde{k}_4 = \begin{bmatrix} a_{51} & a_{52} & a_{53} \\ \dfrac{a_{51}}{\lambda_1} & \dfrac{a_{52}}{\lambda_2} & \dfrac{a_{53}}{\lambda_3} \\ a_{61}\lambda_1 & a_{62}\lambda_2 & a_{63}\lambda_3 \end{bmatrix}$$

A8. The following integrals are used

$$\int_0^\infty \frac{d}{d\xi}\left(\frac{\sin \xi a}{\xi}\right) e^{-\xi\lambda_i z} J_0(r\xi)\, d\xi = -a n_i \left[1 - \zeta_i\left(\frac{\pi}{2} - \tan^{-1}\zeta_i\right)\right],$$

$$\int_0^\infty \frac{d}{d\xi}\left(\frac{\sin \xi a}{\xi}\right) e^{-\xi\lambda_i z} J_1(r\xi)\, d\xi = -\frac{r}{2}\left(\frac{\pi}{2} - \tan^{-1}\zeta_i - \frac{\zeta_i}{1+\zeta_i^2}\right),$$

$$\int_0^\infty \xi \frac{d}{d\xi}\left(\frac{\sin \xi a}{\xi}\right) e^{-\xi\lambda_i z} J_0(r\xi)\, d\xi = -\frac{\pi}{2} + \tan^{-1}\zeta_i + \frac{\zeta_i}{\zeta_i^2 + \eta_i^2},$$

$$\int_0^\infty \xi \frac{d}{d\xi}\left(\frac{\sin \xi a}{\xi}\right) e^{-\xi\lambda_i z} J_1(r\xi)\, d\xi = -\frac{r}{a}\frac{\eta_i}{\left(1+\zeta_i^2\right)\left(\zeta_i^2+\eta_i^2\right)},$$

$$\int_0^\infty \frac{\sin \xi a}{\xi^2} e^{-\xi\lambda_i z} J_0(r\xi)\, d\xi = -\frac{a}{2}\ln\frac{1+\eta_i}{1-\eta_i} + a\eta_i\left[1-\zeta_i\left(\frac{\pi}{2}-\tan^{-1}\zeta_i\right)\right],$$

$$\int_0^\infty \frac{\sin \xi a}{\xi^2} e^{-\xi\lambda_i z} J_1(r\xi)\, d\xi = \frac{a^2}{r}\zeta_i(1-\eta_i) + \frac{r}{2}\left(\frac{\pi}{2}-\tan^{-1}\zeta_i - \frac{\zeta_i}{1+\zeta_i^2}\right),$$

$$\int_0^\infty \frac{\sin \xi a}{\xi} e^{-\xi\lambda_i z} J_0(r\xi)\, d\xi = \frac{\pi}{2}-\tan^{-1}\zeta_i,$$

$$\int_0^\infty \frac{\sin \xi a}{\xi} e^{-\xi\lambda_i z} J_1(r\xi)\, d\xi = \frac{a}{r}(1-\eta_i)$$

where

$$\zeta_i(r,z,a,\lambda_i) = \frac{1}{\sqrt{2}a}\sqrt{\sqrt{\left(r^2+\lambda_i^2 z^2-a^2\right)^2+4\lambda_i^2 z^2 a^2}+\left(r^2+\lambda_i^2 z^2-a^2\right)},$$

$$\eta_i(r,z,a,\lambda_i) = \frac{1}{\sqrt{2}a}\sqrt{\sqrt{\left(r^2+\lambda_i^2 z^2-a^2\right)^2+4\lambda_i^2 z^2 a^2}-\left(r^2+\lambda_i^2 z^2-a^2\right)}$$

and λ_i are the roots of Eq.(2.6) with positive real parts.

Note that the improper integrals (A.8) are new analytical results obtained by the author in his textbooks. In addition, the improper integrals $(A.8)_5$ and $(A.8)_6$ are not presented earlier. The integrals (A.8) are useful in other problems of mechanics, which may be solved with the use of the integral Hankel transformation and integral equation technique method.

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