SEVERAL ALGORITHMS FOR MULTIPLICATION OF INCIDENCE CONDITIONS

Kirill YAKOVENKO[1/] **Vladimir VOLKOV**[2/]

[1/] Omsk F. M. Dostoevsky State University
prosp. Mira 55a, Omsk, Russia 644077
email: kirill.yakovenko@gmail.com
[2/] Siberian State Automobile and Road Construction Academy
prosp. Mira 5, SibADI, Omsk, Russia 644080
email: volkov_vy39@mail.ru

Abstract. We present several algorithms which allow calculating multiplications of some incidence conditions that define a variety of objects.

Keywords: enumerative geometry, incidence condition, multiplication of incidence conditions, types of incidence conditions, algorithms

1 Preamble

In general case mathematical manipulations and computations precede a direct geometrical construction during solving of a geometry problem or a geometry task, they allow reasoned approach a determination of correctness of a task and a choice of the optimal algorithm for solving. The result proceeding from analyze of some given conditions and manifolds determined in the task is designing of grafo-mathematical algorithms for solving a given problem or task.

The using of enumerative geometry as a basis of executable calculations allows applying the basic method of the descriptive geometry which is symbolical representation of geometrical conditions [1].

It is offered to use ***e*** character for a designation of incidence conditions and for a designation of appropriate manifolds thereat the common incidence condition is symbolically represented in the form [1]:

$$\boldsymbol{e}^{m,\ m-1,\dots,1,\ 0}_{a_m,\ a_{m-1},\dots,a_1,a_0}\ . \tag{1}$$

where number of upper and lower indexes coincides and its values are positive natural numbers. *m, m*-1, …, 0 indexes determine dimension of the linear manifold and all its sub manifolds but a_i is a dimensions of a manifolds which are linear sub manifolds of a desired manifold.

The alternative superposition of one or other of the two dependent conditions is represented by sum of two selected characters of any order and superposition of two conditions at one time is represented as result of a product of two characters in any order. This multiplication submits to the distributive law while the associative law is both valid for the sum and for the product of such characters. So characters in two operations submit to common algebra. Subtraction of characters can appear in the equations between characters of conditions but division of characters doesn't arise.

In given article, algorithms of multiplication of incidence conditions and examples of them applying are considered for four common types.

2 The first type of multiplication of incidence conditions

The first type of multiplication is a multiplication of two general conditions of an intersection of *m*-planes with *k*- and *s*-planes in a point:

$$e^{m,m-1,\ldots,\;\;1,\;\;0}_{n,\;n-1,\ldots,n-m+1,k} \cdot e^{m,m-1,\ldots,\;\;1,\;\;0}_{n,\;n-1,\ldots,n-m+1,s} = \sum_{k_1,k_0} e^{m,m-1,\ldots,\;\;2,\;\;1,\;0}_{n,\;n-1,\ldots,n-m-2,k_1,k_0} \qquad (2)$$

where $k_0+k_1=k+s+1$ and k_0,k_1 accept values $0 \le k_0 < k < k_1 \le n-m+1$. In [1] you can see that expression (1) can be reduced to

$$e^{m,m-1,\ldots,\;\;1,\;\;0}_{n,\;n-1,\ldots,n-m+1,k} \cdot e^{m,m-1,\ldots,\;\;1,\;\;0}_{n,\;n-1,\ldots,n-m+1,s} = \sum_{i=0}^{k} e^{m,m-1,\ldots,\;\;1,\;\;0}_{n,\;n-1,\ldots,s+i+1,k-i}$$

if s+k ≤ n–m, s ≥ k and it can be deduced to

$$e^{m,m-1,\ldots,\;\;1,\;\;0}_{n,\;n-1,\ldots,n-m+1,k} \cdot e^{m,m-1,\ldots,\;\;1,\;\;0}_{n,\;n-1,\ldots,n-m+1,s} = \sum_{i=0}^{n-m-s} e^{m,m-1,\ldots,\;\;1,\;\;0}_{n,\;n-1,\ldots,s+i+1,k-i}$$

if s+k >n–m, s≥ k.

Based on the above stated it is possible to make the following algorithm to calculate a multiplication of virtual incidence conditions of the form 1:

Require: e_1, e_2
Ensure: $e=e_1 \cdot e_2$

if e_1 and e_2 satisfy the type 1 **then**
    if $k > s$ **then**
        replace e_1 with e_2 and vice versa
    e is initialized by empty list
    if $s+k+1 \le n-m+1$ **then**
        for $i=0$ **to** k **do**
            append $e^{m,m-1,\ldots,\;1,\;0}_{n,\;n-1,\ldots,s+i+1,k-i}$ to e
    else
        for $i=0$ **to** $n-m-s$ **do**
            append $e^{m,m-1,\ldots,\;1,\;0}_{n,\;n-1,\ldots,n-m-i+1,s+k+m+i-n}$ to e
else
    it’s error, you should choose another algorithm

Algorithm 1. C***alculation of*** the multiplication of two incidence conditions of an intersection of *m*-planes with *k*- and *s*-planes in a point.

Let's consider application of the given algorithm on examples of products of incidence conditions of straight lines and planes in three- and four-dimensional spaces.

For three-dimensional space we assume that it is necessary to determine the number of lines that intersect a plane and other line. This condition in formalized form looks as $e^{1,0}_{3,2} \cdot e^{1,0}_{3,1}$, where $e^{1,0}_{3,2}$ means condition of intercross of a plane and a line and $e^{1,0}_{3,1}$ is condition of intersection of two straight lines. Applying the algorithm we find that both incidence conditions correspond to the type 1 where $k=2$ and $s=1$, $k>s$ and we can write down our expression as $e^{1,0}_{3,1} \cdot e^{1,0}_{3,2}$. Then based on the fact that $s+k=3$, n-$m=2$ and 3>2 summation limits are from 0 to $n-m-s$, i.e. from 0 to 0 and this in turn means that there is the unique solution $e^{1,0}_{3,1}$.

Let's ***presume*** for four-dimensional space that it is necessary to determine the number of planes passing through the given straight line and a point. This condition in formalized form is as $e^{2,1,0}_{4,3,0} \cdot e^{2,1,0}_{4,3,2}$ where $e^{2,1,0}_{4,3,0}$ means condition of the plane passing through the point and $e^{2,1,0}_{4,3,2}$ defines the common type of plane which intersects other plane in four-dimensional space. Applying the algorithm we find that both incidence conditions correspond to the type 1 where k=0 and s=2, recognizing that $s+k=2$, $n–m=2$ and $2 \geq 2$ summation limits are from 0 to k, i.e. from 0 to 0, hence there is only one solution $e^{2,1,0}_{4,2,0}$ which means that the required plane will intersect other plane in a straight line.

3 The second type of multiplication of incidence conditions

The second type of multiplication is a multiplication of two incidence conditions of m-plane with p-, r-planes that have the following form:

$$e^{m,\quad m-1,\quad ...,\quad 1,\quad 0}_{n-p,\, n-p-1,\, ...,\, n-p-m+1,\, n-p-m} \cdot e^{m,\quad m-1,\quad ...,\quad 1,\quad 0}_{n-r,\, n-r-1,\, ...,\, n-r-m+1,\, n-r-m}, \qquad (3)$$

where $n–p \geq m$, $n–r \geq m$.

This reduction is the most simple reduction of the presented. Based on the above stated conditions of existence a (2) reduction we receives that manifold of m-planes at $n–p–r \geq m$ satisfies them, in a different way conditions are defined over it means that their dimension is more then dimension of Grassmann's manifold of m-planes in n-dimensional space. Hence the multiplication equals following:

$$e^{m,\quad m-1,\quad ...,\quad 1,\quad 0}_{n-p-r,\, n-p-r-1,\, ...,\, n-p-r-m+1,\, n-p-r-m},$$

at $n–p–r>m$. If $n–p–r=m$ then we receives fundamental incidence condition $e^{m,m-1,...,1,0}_{m,m-1,...,1,0}$.

Require: e_1, e_2
Ensure: $e=e_1 \cdot e_2$

1. **if** e_1 and e_2 satisfy the type 2 **then**
2.   **if** $n – p – r > m$ **then**
3. $e = e^{m,\quad m-1,\quad ...,\quad 1,\quad 0}_{n-p-r,\, n-p-r-1,\, ...,\, n-p-r-m+1,\, n-p-r-m}$
4.   **else if** $n–p–r=m$ **then**
5. $e = e^{m,m-1,...,1,0}_{m,m-1,...,1,0}$
6. **else**
7.   it's error, you should choose another algorithm

Algorithm 2. C***alculation of*** the multiplication of two incidence conditions of m-plane with p-, r-planes.

Let's consider application of the given algorithm on examples of products of incidence conditions of straight lines and planes in three- and four-dimensional spaces.

For three-dimensional space we assume that it is necessary to determine the number of lines which intersect given plane at a point and belong to other plane. This condition in formalized form looks as $e^{1,0}_{2,1} \cdot e^{1,0}_{3,2}$ where $e^{1,0}_{3,2}$ is condition of intersection of a straight line and a plane in general position in three-dimensional space and $e^{1,0}_{2,1}$ defines condition of belonging a straight line to a plane. Applying the algorithm we find that both incidence conditions correspond to the type 2 where *p*=0, *r*=1 and there is only one solution and start out with *n*–*p*–*r*=2 > *m*=1 the solution is $e_{2,1}^{1,0}$.

Let's ***presume*** for four-dimensional space that it is necessary to determine the number of lines which intersect a plane at point and belong to a plane. This condition in formalized form is $e_{2,1}^{1,0} \cdot e_{3,2}^{1,0}$ where $e_{3,2}^{1,0}$ means condition of incidence of a straight line lying in three-dimensional space to a plane and $e_{2,1}^{1,0}$ means intersection of a straight line and a plane in four-dimensional space. Applying the algorithm we find that both incidence conditions correspond to the type 2 where *p*=1, *r*=2 hence there is the unique solution and as *n*–*p*–*r*=1 = *m*=1, the solution is fundamental incidence condition $e_{1,0}^{1,0}$ that means a general provisions straight line in four-dimensional space.

4 The third type of multiplication of incidence conditions

The third type of multiplication is a multiplication of two incidence conditions one of which is condition of intersections of *m*-plane with *k*-plane at point and the second is common incidence condition.

$$e_{n,\ n-1,\ldots,n-m+1,k}^{m,\ m-1,\ldots,\ 1,\ 0} \cdot e_{a_m,\ a_{m-1},\ldots,a_1,a_0}^{m,\ m-1,\ldots,\ 1,\ 0} = \sum e_{c_m,\ c_{m-1},\ldots,c_1,c_0}^{m,\ m-1,\ldots,\ 1,\ 0}, \qquad (4)$$

where $c_m, \ldots, c_0$ indexes should satisfy following conditions:

$$a_{m-1} < c_m \le a_m,\ a_{m-2} < c_{m-1} \le a_{m-1},\ \ldots,\ 0 \le c_0 \le a_0,$$

$$\sum_{i=0}^{m} c_i = \sum_{i=0}^{m} a_i - n + m + k.$$

Require: e_1, e_2

Ensure: $e=e_1 \cdot e_2$

1. **if** e_1 satisfies the type 1 **then**
2. e is initialized by empty list
3. $sum = \sum_{i=0}^{m} a_i - n + m + k.$
4. **repeat**
5. **if** $\sum_{i=0}^{m} c_i$ is equal *sum* **then**
6. append $e_{c_m,\ c_{m-1},\ldots,c_1,c_0}^{m,\ m-1,\ldots,\ 1,\ 0}$ to e
7. **until** there are all possible sets of c_i which satisfy conditions $a_{m-1} < c_m \le a_m, \ldots, 0 \le c_0 \le a_0$
8. **else if** e_2 satisfies the type 1 **then**
9. replace e_1 with e_2 and vice versa then start over
10. **else**
11. it's error, you should choose another algorithm

Algorithm 3. C***alculation of*** the multiplication of two incidence conditions one of which is condition of intersections of *m*-plane with *k*-plane at point.

So if you can't get any numbers that satisfy the above conditions, these conditions aren't compatible. Hence the criterion of consistency of the given conditions is given conditions set.

Let's consider application of the given algorithm on examples of products of incidence conditions of straight lines and planes in three- and four-dimensional spaces.

For three-dimensional space we assume that it is necessary to determine the number of planes which intersect a plane at line passing through a given point. This condition in formalized form looks as $e^{2,1,0}_{3,2,0} \cdot e^{2,1,0}_{3,1,0}$ where $e^{2,1,0}_{3,2,0}$ means condition of a plane passing through a point and $e^{2,1,0}_{3,1,0}$ means the incidence of a plane and a straight line. Applying the algorithm we find that the first incidence condition corresponds to the type 1 and the second incidence condition is common incidence condition then let's calculate a sum of indexes c_i: $\sum_{i=0}^{m} a_i - n + m + k = (3+1+0) - 3 + 2 + 0 = 3$ and plot the conditions for c_i: $1 < c_2 \leq 3$, $0 < c_1 \leq 1$, $0 \leq c_0 \leq 0$ which show that there are two sets of indexes satisfying the given conditions, namely, they are 3,1,0 and 2,1,0. Compare the sum of the obtained indexes with the estimated: 3+1+0=4≠3 and 2+1+0=3, hence there is only one solution and it's $e^{2,1,0}_{2,1,0}$ that is fundamental incidence condition and means a common plane in tree-dimensional space.

Let's ***presume*** for four-dimensional space that it is necessary to determine the number of planes that intersect pair wise in a straight line and intersect other straight line at a point. This condition in formalized form is $e^{2,1,0}_{4,3,1} \cdot e^{2,1,0}_{4,2,1}$ where $e^{2,1,0}_{4,3,1}$ means condition of intersection of a plane and a straight line at a point and $e^{2,1,0}_{4,2,1}$ means that planes intersect pair wise in a straight line. Applying the algorithm we find that the first incidence condition corresponds to the type 1 and the second incidence condition is common incidence condition then calculate a sum of indexes c_i: $\sum_{i=0}^{m} a_i - n + m + k = (4+2+1) - 4 + 2 + 1 = 6$ and plot the conditions for c_i: $2 < c_2 \leq 4$, $1 < c_1 \leq 2$, $0 \leq c_0 \leq 1$. They show that there are four sets of indexes satisfying the given conditions, namely, they are 4,2,1; 4,2,0; 3,2,1; 3,2,0. Compare the sum of the obtained indexes with the estimated: 4+2+1=7≠6, 4+2+0=6, 3+2+1=6, 3+2+0=5≠6. Hence there are two solutions $e^{2,1,0}_{4,2,0}$ and $e^{2,1,0}_{3,2,1}$, where $e^{2,1,0}_{4,2,0}$ is planes intersect pair wise in a straight line and this line passes through a given point and $e^{2,1,0}_{3,2,1}$ means condition of a plane lying in hyperspace of four-dimensional space.

5 The multiplication of two common incidence conditions

It's a multiplication of two common incidence conditions for m-planes in n-dimensional space which is expressed in following type:

$$e^{m,\quad m-1,\ldots,\,1,\;0}_{a_m\,,\;a_{m-1},\ldots,a_1,a_0} \cdot e^{m,\quad m-1,\ldots,\,1,\;0}_{b_m\,,\;b_{m-1},\ldots,b_1,b_0}. \tag{5}$$

This multiplication reduces to a multiplication of the third kind if we expand one of these conditions into a determinant consisting of the third kind conditions:

$$e^{m,\quad m-1,\ldots,\,1,\;0}_{a_m\,,\;a_{m-1},\ldots,a_1,a_0} = \begin{vmatrix} e_{(\ldots),k_{ij}} & \cdots \\ \vdots & \ddots \end{vmatrix}, \tag{6}$$

where $(\ldots)=n,n-1,\ldots,n-m+1$, $k_{ij}=a_i-i+j$ $(i,j)=(0,1,\ldots,m)$.

If $k_{ij} < 0$ or $k_{i,j} > n-m$, incidence conditions forming the determination have no sense and are equal zero. So common condition is represented as a determinant in which multiplications are interpreted as intersections.

Require: e_1, e_2

Ensure: $e=e_1 \cdot e_2$

1. e is initialized by empty list
2. decompose the e_1 condition to the determinant of the (6) form
3. represent matrix determinant in a type
$$\sum_{a_1,a_2,\ldots,a_n}(-1)^{N(a_1,a_2,\ldots,a_n)} \cdot a_{a_1 1} \cdot \ldots \cdot a_{a_n n}$$
4. **for all** e_i summands of got determinant **do**
5. e_s is initialized by empty list
6. **for all** $e_{i,j}$ factors of e_i summand **do**
7. multiply $e_{i,j} \cdot e_{i,j+1}$ using the 3 algorithm
8. append a result of multiplication to e_s
9. multiply $e_2 \cdot e_s$ using the 3 algorithm
10. append a result of multiplication to e

Algorithm 4. C***alculation of*** the multiplication of two common incidence conditions.

Let's consider application of the given algorithm on example of products of incidence conditions of planes in four-dimensional space. Let's assume that it's necessarily to determine a number of planes which intersect a plane in common position in straight line and which are incidence of other plane at a point. This condition in formalized form is $e_{4,2,0}^{2,1,0} \cdot e_{4,2,1}^{2,1,0}$ where $e_{4,2,0}^{2,1,0}$ means conditions of incidence of two planes at a point and $e_{4,2,1}^{2,1,0}$ is condition of intersection of planes in straight line. Expand the condition $e_{4,2,1}^{2,1,0}$ into determinant consisting from incidence conditions of the third type:

$$e_{4,2,1}^{2,1,0} = \begin{vmatrix} e_{4,3,1}^{2,1,0} & e_{4,3,0}^{2,1,0} & 0 \\ e_{4,3,2}^{2,1,0} & e_{4,3,1}^{2,1,0} & e_{4,3,0}^{2,1,0} \\ 0 & 0 & e_{4,3,2}^{2,1,0} \end{vmatrix},$$

this determinant is reduced to $e_{4,3,1}^{2,1,0} \cdot e_{4,3,1}^{2,1,0} \cdot e_{4,3,2}^{2,1,0} - e_{4,3,0}^{2,1,0} \cdot e_{4,3,2}^{2,1,0} \cdot e_{4,3,2}^{2,1,0}$ by using Sarrusa's rule (triangles rule). That in turn means that initial expression is reduced to:

$$e_{4,2,1}^{2,1,0} \cdot \left(e_{4,3,1}^{2,1,0} \cdot e_{4,3,1}^{2,1,0} \cdot e_{4,3,2}^{2,1,0} - e_{4,3,0}^{2,1,0} \cdot e_{4,3,2}^{2,1,0} \cdot e_{4,3,2}^{2,1,0}\right) =$$
$$e_{4,2,1}^{2,1,0} \cdot e_{4,3,1}^{2,1,0} \cdot e_{4,3,1}^{2,1,0} \cdot e_{4,3,2}^{2,1,0} - e_{4,2,1}^{2,1,0} \cdot e_{4,3,0}^{2,1,0} \cdot e_{4,3,2}^{2,1,0} \cdot e_{4,3,2}^{2,1,0}.$$

Then we sequentially multiply members of computed expression and as one member always is incidence condition of third type, it's possible to receive that result of this expression is

$$2 \cdot e_{3,1,0}^{2,1,0} - e_{3,1,0}^{2,1,0} = e_{3,1,0}^{2,1,0}$$

Hence there is only one solution $e_{3,1,0}^{2,1,0}$ which means a plane in three-dimensional space which passes through a straight line.

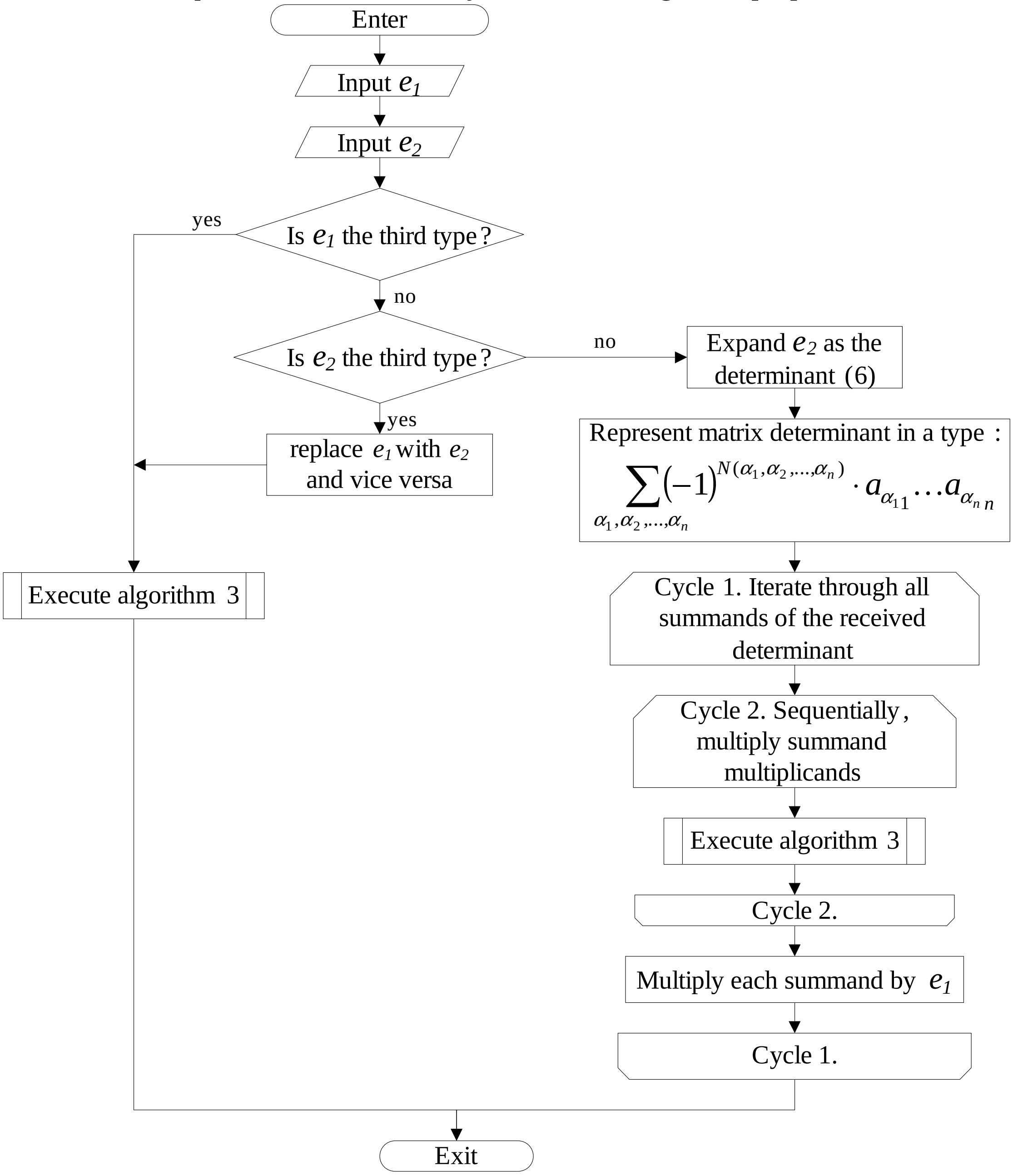
The generalized algorithm of multiplication of incidence conditions

As we can notice, algorithm 3 allows to calculate result for multiplication of incidence conditions of first and second types and algorithm 4 gives possibility to expand incidence condition as determinant consisting of the third kind incidence conditions. Algorithm 3 and 4 in total make it possible to calculate a multiplication of any virtual incidence conditions.

Based on all above stated, it's possible to produce generalized algorithm of multiplication of two incidence conditions which can be represented by the (1) block diagram.

6 Issue

Typically the direct geometric construction is preceded by mathematical calculations and computations which allow to approach to the correct definition of a task and to the choice of a optimal solution algorithm with well-reasoned. The presented algorithms make it possible to automate the process of research and synthesis of different varieties of geometric objects and also calculate the required dimensions of objects and their algebraic properties.



Picture 1. Block diagram of multiplication algorithm of two incidence conditions

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KILKA ALGORYTMÓW MNOŻENIA WARUNKÓW INCYDENCJI

W pracy przedstawiono kilka algorytmów, które pozwalają wyznaczać iloczyny pewnych warunków incydencji określających różne obiekty geometryczne.