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BASIC SUMS AS PARAMETERS CHARACTERIZING

Abstract. We discuss the method of distributions' characterization of geometric objects on the plane. The method is based on the basic sums. These basic sums perform the same role in description of image as the n -point correlation functions. In this paper we present some applications of this theory.

1. Introduction

Description of images' geometry¹ it's fundamental meaning in different branch of science i.e. engineering materials, medicine, biology, astronomy etc. In theory pattern recognition dominant attitude whose objective is to extract visible features from analyzed image. The next step in image analysis is the process of extracting the hidden features – invisible to the human, but significant (for example isotropy of material). This issue we can reduce to the construction of set G containing some parameters which describe geometry. We can treat this set as a representation of the analyzing image, because n -point correlation functions entirely describe geometry of considered images. Computational difficulties in the n -point correlation functions for $n > 3$ confine practical applications of this approach. In this paper apply alternative definition approach based on the G set

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¹Without loss of generality, we can consider only two-phase images [16, p. 24].

consisting of the elements e_p called *basic sums* (or *e-sums*). This term was introduced in [10] and systematically investigated in [6, 7, 11–13]. The *e-sums* are functionals on the correlation functions which determine the effective properties of media. We use this representation for images containing non-overlapping circles, but in the further part we will extend the method to other shapes.

2. Basic sums

Consider a lattice \mathcal{Q} which is determined on the complex plane by two vectors ω_1 and ω_2 (for definiteness, it is assumed that $\text{Im}[\omega_2/\omega_1] > 0$). We introduce the cell $(0, 0)$ as the parallelogram:

$$Q_{(0,0)} := \left\{ z = s_1\omega_1 + s_2\omega_2 : -\frac{1}{2} < s_k < \frac{1}{2} \ (k = 1, 2) \right\}.$$

The lattice \mathcal{Q} consists of the cells

$$Q_{(m_1, m_2)} := \{ z \in \mathbb{C} : z - m_1\omega_1 - m_2\omega_2 \in Q_{(0,0)} \}$$

where m_1, m_2 run over integer numbers. We will consider unit cells which include N non-overlapping geometric objects. This distribution will be realized in the torus topology (see Figure 1 case *a*). Therefore, we will treat the analysed two-phase image as the set of geometric objects on the complex plane \mathbb{C} and study its restriction to the cell $Q_{(0,0)}$.

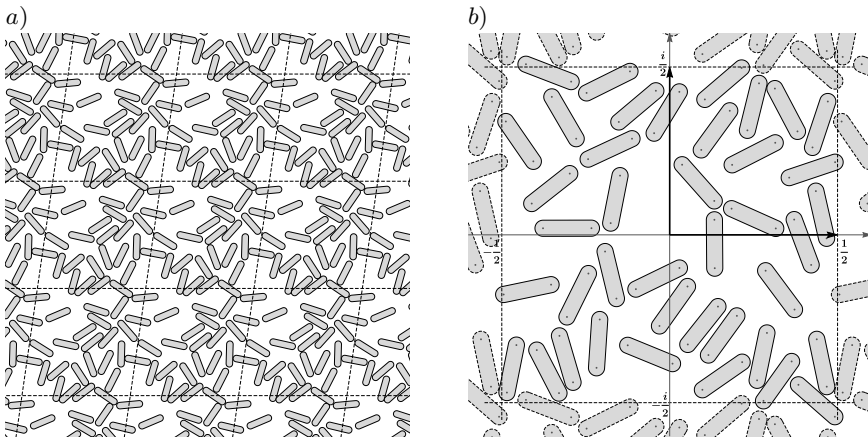


Fig. 1. *a*) – double periodic lattice \mathcal{Q} ; *b*) – example of cell $Q_{(0,0)}$ (square cell)

Before we introduce the basic sum we will provide some additional definitions. Let \mathcal{Q} will be the lattice described in the previous paragraph. Consider the *lattice sums* which for even values of n can be calculated using the following formulas [10]:

$$\begin{aligned} S_2 &= \left(\frac{\pi}{\omega_1}\right)^2 \left(\frac{1}{3} - 8 \sum_{m=1}^{\infty} \frac{mq^{2m}}{1-q^{2m}}\right), \\ S_4 &= \frac{1}{60} \left(\frac{\pi}{\omega_1}\right)^4 \left(\frac{4}{3} - 320 \sum_{m=1}^{\infty} \frac{m^3 q^{2m}}{1-q^{2m}}\right), \\ S_6 &= \frac{1}{140} \left(\frac{\pi}{\omega_1}\right)^6 \left(\frac{8}{27} - \frac{448}{3} \sum_{m=1}^{\infty} \frac{m^5 q^{2m}}{1-q^{2m}}\right), \end{aligned}$$

where $q = e^{\frac{\pi i \omega_2}{\omega_1}}$. Sums S_{2n} ($n \geq 4$) are calculated by the recurrence formula:

$$S_{2n} = \frac{3}{(2n+1)(2n-1)(n-3)} \sum_{m=2}^{n-2} (2m-1)(2n-2m-1) S_{2m} S_{2(n-m)}.$$

For odd values n it is known that $S_n = 0$.

The following definition of *Weierstrass function* \wp is based on the lattice sums [8]:

$$\wp(z) = \frac{1}{z^2} + \sum_{m=2}^{\infty} (2m-1) S_{2m} z^{2m-2}.$$

Function \wp is double periodic.

The Eisenstein functions E_m ($m = 2, 3, \dots$) and the Weierstrass functions are related by the following dependence:

$$E_2(z) = \wp(z) + S_2, \quad E_m(z) = \frac{(-1)^m}{(m-1)!} \frac{d^{m-2} \wp(z)}{dz^{m-2}} \quad (m = 3, 4, \dots). \quad (1)$$

Each of functions (1) is double periodic and has a pole of order n at $z = 0$. The value at zero of function E_n is defined as $E_n(0) := S_n$.

Consider points $a_k = x_k + iy_k$ ($k = 1, 2, \dots, N$) in the cell $Q_{(0,0)}$. Let q be a natural number, k_0, k_1, \dots, k_q run over integer numbers from 1 to N , and $p_j > 1$ ($j = 1, 2, \dots, q$) are integer numbers. The following sum was introduced by Mityushev [3, 10]:

$$\begin{aligned} e_{p_1, p_2, \dots, p_q}^{v_0, v_1, \dots, v_q} &= \sum_{k_0, k_1, \dots, k_q} v_0^{s_0} v_1^{s_1} v_2^{s_2} \dots v_q^{s_q} E_{p_1}(a_{k_0} - a_{k_1}) \overline{E_{p_2}(a_{k_1} - a_{k_2})} \times \\ &\times E_{p_3}(a_{k_2} - a_{k_3}) \dots C^{q+1} E_{p_q}(a_{k_{q-1}} - a_{k_q}), \end{aligned} \quad (2)$$

where functions E_m ($m = 2, 3, \dots$) are the *Eisenstein functions* corresponding to double periodic cell $Q_{(0,0)}$. Indexes s_n ($n = 0, 1, 2, \dots, q$) are provided by recurrence formulas

$$\begin{aligned} s_0 &= 1, \\ s_n &= p_n - s_{n-1}, \quad n = 1, 2, \dots, q. \end{aligned}$$

Symbol \mathbf{C} denotes operator of complex conjugation. The sum (2) is called *basic sum* (or *e-sum*) of the *multi-index* $\mathbf{p} = (p_1, p_2, \dots, p_q)$ and number $|\mathbf{p}| = \frac{p_1 + p_2 + \dots + p_q}{2}$ is called *the order of the basic sum*. When the points a_k ($k = 1, 2, \dots, N$) are interpreted as centers of the circles with radii r_k ($k = 1, 2, \dots, N$), the values $0 < v_k \leq 1$ ($k = 1, 2, \dots, N$), describe polydispersity of circles [3] and are defined as $\frac{r_k}{R}$, where R denotes radius of the biggest circle.

Example 1. If we consider N circles with the same radii, the sums e_2 and $e_{3,3,2}$ have the form:

$$\begin{aligned} e_2 &= \sum_{k_0=1}^N \sum_{k_1=1}^N E_2(a_{k_0} - a_{k_1}), \\ e_{3,3,2} &= \sum_{k_0=1}^N \sum_{k_1=1}^N \sum_{k_2=1}^N \sum_{k_3=1}^N E_3(a_{k_0} - a_{k_1}) \overline{E_3(a_{k_1} - a_{k_2})} E_2(a_{k_2} - a_{k_3}). \end{aligned}$$

Now we explain why the *e-sums* can be treated as parameters describing the distribution of geometric objects. In the papers [2, 9] the explicit functional form for the effective conductivity of a macroscopically isotropic composite was obtained. Considered a two-dimensional composite contains non-overlapping circular inclusions and it is a cross-section of the three-dimensional fiber composite. The above functional has the form of a series which is dependent on concentrations of inclusions ν

$$\widehat{\lambda} = 1 + 2\rho\nu + 2\rho\nu \sum_{q=1}^{\infty} A_q \nu^q, \quad (3)$$

where $\rho = \frac{\lambda_i - \lambda_m}{\lambda_i + \lambda_m}$ stands for the *Bergman parameter* [1], λ_i and λ_m denote conductivity (permeability) of inclusions and host, respectively. The constants A_q are linear combinations of basic sums $e_{\mathbf{p}}$, which coefficients depend on ρ . First five

A_q have the following form

$$\begin{aligned} A_1 &= \frac{\rho}{\pi} e_2, & A_2 &= \frac{\rho^2}{\pi^2} e_{2,2}, & A_3 &= \frac{1}{\pi^3} [-2\rho^2 e_{3,3} + \rho^3 e_{2,2,2}], \\ A_4 &= \frac{1}{\pi^4} [3\rho^2 e_{4,4} - 2\rho^3 (e_{3,3,2} + e_{2,3,3}) + \rho^4 e_{2,2,2,2}], \\ A_5 &= \frac{1}{\pi^5} [-4\rho^2 e_{5,5} + \rho^3 (3e_{4,4,2} + 6e_{3,4,3} + 3e_{2,4,4}) - \\ &\quad - 2\rho^4 (e_{3,3,2,2} + e_{2,3,3,2} + e_{2,2,3,3}) + \rho^5 e_{2,2,2,2,2}]. \end{aligned}$$

The next coefficients A_k can be written in closed form by application of the algorithm presented in [15]. Optimized and fast algorithm of calculation was developed in [12]. We consider the set \mathcal{M}_e multi-indexes $\mathbf{p} = (p_1, p_2, \dots, p_q)$ and $G := \{e_{\mathbf{p}}, \mathbf{p} \in \mathcal{M}_e\}$. We say that a multi-index belongs to \mathcal{M}_e if it is a factor of a coefficient in the expansion (3).

The basic sums and distributions of any shape geometric objects

We confine ourselves to study two-phase images where we can separate phase consisting a set of non-overlapping geometric objects of fixed shapes and sizes and the remaining part of the image. More precisely, we consider a set of objects $\{\mathcal{D}_k, k = 1, 2, 3, \dots\}$ where each object \mathcal{D}_k has a fixed geometry. We assume that the distribution of objects in space is random. Thus, the deterministic elements \mathcal{D}_k are introduced independently but the set $\{\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_2, \dots\}$ is introduced randomly. The diversity of random locations is expressed by joint probabilistic distributions of the non-overlapping objects \mathcal{D}_k . Using basis sums to describe the geometry of this two-phase image is justified when the geometric objects are circles. However, any geometric object, can be approximated by the circles with different radii (with some precision). Introduce the concept of error of approximation. Let $\mathcal{D} \subset \mathbb{C}$ will be a geometric object with nonzero area. The following number

$$\varepsilon := \frac{|P_{\mathcal{D}} - P_{\mathcal{K}_n}|}{P_{\mathcal{D}}},$$

is called the error of approximation of a geometric object \mathcal{D} by set of n circles \mathcal{K}_n , where $P_{\mathcal{D}}$ is the area of the geometric object \mathcal{D} and $P_{\mathcal{K}_n}$ is the sum of the areas of the circles \mathcal{K}_n . We can construct a general algorithms of approximation for object \mathcal{D} , for example by circles of the same radii or of two different radii, etc. Of course, approximation should have a small error, but should also adequately represent the considered geometric object. We calculate some of the basic sums (2)

for distributions of geometric objects in a cell, $Q_{(0,0)}$. In a given cell, we have N objects and each of them is approximated by n circles hence we have to calculate the value of e -sums for $N \cdot n$ circles. We are interested in such an approximation for which the value of error is acceptable and the number of approximating the circles is as small as possible. For example, we consider *stadium* – see Figure 2. In the Section 3, we will describe some distributions of such geometric objects by e -sums and introduce formal definition of stadium. We will approximate this stadium by two circles (see case b) on the Figure 2).

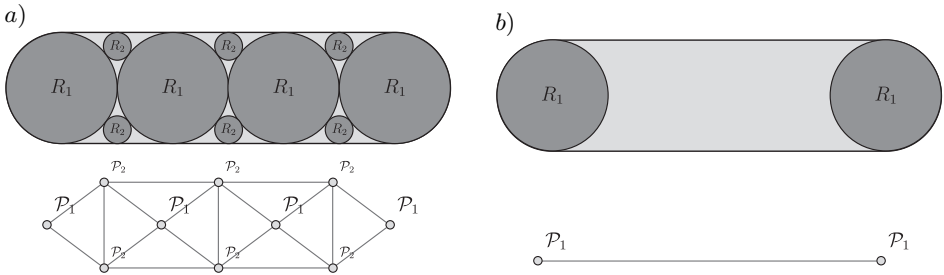


Fig. 2. a) – approximation of unit stadium with radius $R_1 = \frac{\sqrt{\pi+12}}{\pi+12}$ and the distances between the centers $\mu = 6R_1$ by circles with two different radii R_1 and $R_2 = \frac{R_1}{4}$. Error of approximation $\varepsilon = 0.0663355$. Below, Delaunay graph with weights of vertex (defined as the area of a circle) represents this approximation; b) – approximation of the same stadium as in case a), by two circles with a radius of R_1 . Below, Delaunay graph corresponding to this approximation. Error of approximation $\varepsilon = 0.585038$

In order to represent the approximation of the object \mathcal{D} we use the so-called *Delaunay graph* for circles' centers of the approximation. In this graph a weight equal to areas of circles from approximations is assigned to each vertex (see Figure 2).

3. Example of an algorithm generating random distribution of stadiums

In this section we present and characterize distributions of geometric objects called *stadium*. The study of such distributions is related to the collective behavior of bacteria [4].

First, we introduce a formal definition of stadium. Let $R > 0$ be a real number and $a, b \in \mathbb{C}$ be such that $a \neq b$. The *stadium* $\mathcal{S}((a, b), R) \subset \mathbb{C}$ is defined as follows

$$\begin{aligned} \mathcal{S}((a, b), R) := & \left\{ z \in \mathbb{C} : |z - a| \leq R \vee \right. \\ & \vee |z - b| \leq R \vee \left(\operatorname{Im}[z - a] \cdot \operatorname{Im}[a - b] + \right. \\ & + \operatorname{Re}[z - a] \cdot \operatorname{Re}[a - b] \leq 0 \wedge \operatorname{Im}[z - b] \cdot \operatorname{Im}[a - b] + \\ & + \operatorname{Re}[z - b] \cdot \operatorname{Re}[a - b] \geq 0 \wedge \left. \operatorname{Im}[(b - a) \cdot \operatorname{Re}[z] + \right. \\ & \left. \left. + (z - b) \cdot \operatorname{Re}[a] + (a - z) \cdot \operatorname{Re}[b]] \right| \leq R \cdot |a - b| \right) \left. \right\}, \end{aligned}$$

where R is called its *radius* and the (a, b) its *centers*. Now, we will describe the algorithm for generating distributions of stadiums, where their some directions are in some way determined. Consider a lattice \mathcal{Q} which is defined by two fundamental translation vectors $\omega_1 = 1$ and $\omega_2 = i$. Now, we have a square cell $Q_{(0,0)}$ (see Figure 1 case *b*). Consider N non-overlapping stadiums $\mathcal{S}_k = \mathcal{S}_k((a_k, b_k), r)$ of radius r with centers $a_k, b_k \in Q_{(0,0)}$ and such that $|a_k - b_k| = \mu = 6r$. More precisely, we obtain the distribution of the variable $\mathbf{v} = (a_1, a_2, \dots, a_N, a_{N+1}, a_{N+2}, \dots, a_{2N})$ where $a_k = b_{k-N}$ for $k = N+1, N+2, \dots, 2N$, with the restrictions $|a_i - a_{N+i}| = \mu$ ($i = 1, 2, \dots, N$) and provided that stadiums \mathcal{S}_k are non-overlapping. It should be noted that stadiums are considered in the double periodic torus topology when the opposite sides of $Q_{(0,0)}$ are glued by pairs. The random variable \mathbf{v} can be statistically realized for large N by Monte Carlo method to get numerical results.

We characterize distributions generated by this algorithm by the basic sums. We select a point a_1 as the realization of a random variable uniformly distributed in the cell $Q_{(0,0)}$ and the angle α_1 as the realization of a random variable uniformly distributed in $(-\pi, \pi]$. We define the first stadium $\mathcal{S}_1((a_1, b_1)r)$, where $b_1 = a_1 + \mu e^{i\alpha_1}$. k -th stadium ($k = 2, 3, \dots, N$) is selected according to the following rules:

1. We randomly take a point a_k uniformly distributed in $Q_{(0,0)}$.
2. We create a list of L stadiums centers c_j , such that $(\mu + 2r) \leq |a_k - c_j| \leq 3(\mu + 2r)$ and organize them as follows. As the first item in the list we put center c_j , which is located to the nearest a_k and at the next position of list we put those centers of stadiums beginning from this which was chosen at the latest and ending in this, which was chosen at the earliest. If the list L is empty, go to step 4, otherwise we realise the procedure described in step 3.

3. We iterate through the list L defining $\alpha_j = \text{Arg}(a_k - c_j)$ if for some j , the stadium $\mathcal{S}((a_k, b = a_k \pm \mu e^{i\alpha_j}))$ (where we choose plus or minus so that b lies closer to c_j) does not cross stadiums \mathcal{S}_i , ($i = 1, 2, \dots, k-1$) is a k -th stadium of generated distribution we choose $\mathcal{S}_k((a_k, b_k), r)$ where $b_k := b$ and finish process of selection k -th stadium. Otherwise, we move on next step.
4. If steps from 1 to 3 were done only once, we repeat them, and if twice, we go to step 5.
5. We select a point a_k as the realization of a random variable uniformly distributed in the cell $Q_{(0,0)}$ and the angle α_k as the realization of a random variable uniformly distributed in $(-\pi, \pi]$. We define stadium $\mathcal{S}_k((a_k, b_k), r)$. If $\bigcap_{i=1}^k \mathcal{S}_i \neq \emptyset$ as k -th stadium of generated distribution choose \mathcal{S}_k and finish process of selection k -th stadium.
6. Repeat this procedure until it will be finished by choice of k -th stadium in step No. 3 or No. 5.

Let ν stands for the concentration of stadiums in the unit cell, i.e.

$$\nu = N(r^2\pi + 2r\nu). \quad (4)$$

Table 1
The averaged basic sums for various concentrations

ν	$\text{Re} \langle \mathbf{e}_2 \rangle$	$\langle \mathbf{e}_{2,2} \rangle$	$\langle \mathbf{e}_{3,3} \rangle$	$\langle \mathbf{e}_{4,4} \rangle$
0.05	3.17182	207.594	-3487.48	98177.0
0.1	3.13031	79.5483	-691.009	9900.70
0.15	3.12769	47.3060	-271.648	2634.18
0.2	3.13401	33.5164	-139.067	1023.83
0.25	3.14044	26.2781	-82.6753	488.258
0.3	3.13264	22.0212	-54.9372	272.772
0.35	3.14658	19.4481	-39.1987	166.575
0.4	3.14245	17.6875	-29.6321	110.858

The algorithm described above generates a probability distribution \mathcal{U}_ν depending on the concentration (4). In order to describe this distribution we need

Table 2
The averaged basic sums for various concentrations

ν	Re $\langle e_{2,2,2} \rangle$	Re $\langle e_{3,3,2} \rangle$	$\langle e_{2,2,2,2} \rangle$
0.05	1306.38	-11127.3	166078
0.1	465.662	-2168.06	19594.6
0.15	263.374	-835.543	6002.47
0.2	179.511	-433.533	2712.60
0.25	133.876	-259.218	1514.52
0.3	107.359	-170.819	985.459
0.35	91.2993	-123.670	711.790
0.4	80.0331	-93.1401	553.584

theoretically all the probabilistic moments, whose number is infinite. In practice, a finite number of moments can be analysed. We will use the following e -sums corresponding to the precision at $O(\nu^3)$ [6, 7]².

In order to characterize the distribution of stadiums \mathcal{U}_ν , we calculate the basic sums using Monte Carlo method with a fixed parameters (the number of object in a cell (N) and number of experiments (M)). Following [4] we assume $N = 500$ and $M = 400$.

The averaged e -sums e_2 , $e_{2,2}$, $e_{3,3}$, $e_{4,4}$, $e_{2,2,2}$, $e_{3,3,2}$ and $e_{2,2,2,2}$ are computed for the distributions \mathcal{U}_ν for the concentrations $\nu \in \{0.05, 0.1, 0.15, 0.2, 0.25, 0.3, 0.35, 0.4\}$.

The results are shown in Tables 1 and 2. These are the fundamental parameters of the uniform non-overlapping distribution \mathcal{U}_ν of stadiums on the plane.

4. Bacteria's distribution and conclusion

In papers [4, 5] distributions of some swimming bacteria were studied. In first of them the locations of bacteria were modeled by segments randomly embedded in medium without overlapping and in second of them by non-overlapping stadiums. The sizes of geometric objects were determined by the observed sizes of bacteria. First, the values of e -sums for the observed experimental locations of bacteria in very thin liquid film (31 film frames of bacteria [14]) were computed. Second, the

²We omit the sum $e_{2,3,3}$, because according to the Lemma 2 (see [12, p. 10]), we have $e_{2,3,3} = e_{3,3,2}$.

values of e -sums for the disordered distributions³ were computed. The obtained sets of the basic sums were different. Based on this, there was concluded that behavior of bacteria is not disordered, but it is collective.

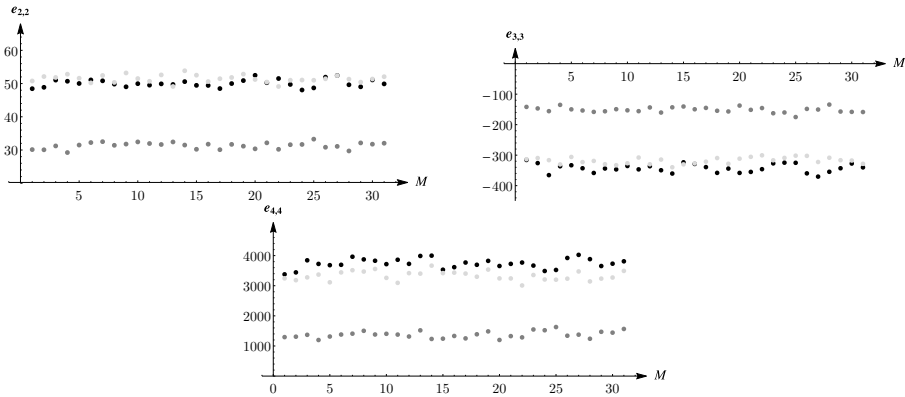


Fig. 3. The comparison of the values $e_{2,2}$, $e_{3,3}$ and $e_{4,4}$ for 3 sets of 31 samples (M) distributions of stadiums (black points – the real distributions of bacteria, grey points – distributions generated by the algorithm based on the RSA method, lightgrey points – distributions generated by the algorithm presented in Section 3)

The natural question is whether we can generate such distributions stadiums that approximate real locations of bacteria. The Figure 3 shows comparison the values of some e -sums for 3 sets of 31 samples distributions of stadiums: first set are real distributions of bacteria, second are distributions generated by the algorithm based on the RSA method and third are generated by algorithm presented in Section 3.⁴ We can see that the algorithm described in Section 3 generate the distributions of stadiums which seem to be very similar to the real distributions of bacteria. This algorithm is based on the observation that bacteria are reproduced by division which determines the orientation of their locations. In order to improve the model we need more information about behavior of these organisms.

³The distributions were generated by the algorithms based on RSA method (*random sequential adsorption*). Therefore we can assume that the locations of geometric objects are totally random-disordered.

⁴There were assumed the following values of parameters for the stadiums $r = 0.00214096$ and $\mu = 6r$ (it is based on real sizes of bacteria). Because the average number of bacteria in the frame is 1965 so in order to generate this distributions by the two algorithm there was assumed $N = 1965$.

References

1. Bergman D.J, Dunn K.J.: *Bulk effective dielectric constant of a composite with periodic micro-geometry*. Phys. Rev. Ser. B **45** (1992), 13262–13271.
2. Berlyand L., Mityushev V.V.: *Generalized Clausius-Mossotti formula for random composite with circular fibers*. J. Stat. Phys. **102** (2001), 115–145.
3. Berlyand L., Mityushev V.V.: *Increase and decrease of the effective conductivity of two phase composites due to polydispersity*. J. Stat. Phys. **118** (2005), 481–509.
4. Czapla R.: *Random sets of stadiums in square and collective behavior of bacteria*. IEEE/ACM Transactions on Computational Biology and Bioinformatics (in print).
5. Czapla R., Mityushev V.V.: *A criterion of collective behavior of bacteria*. Math. Biosci. Eng. **14** (2017), 277–287.
6. Czapla R., Mityushev V.V., Nawalaniec W.: *Effective conductivity of random twodimensional composites with circular nonoverlapping inclusions*. Comput. Mater. Sci. **63** (2012), 118–126.
7. Czapla R., Mityushev V.V., Nawalaniec W.: *Simulation of representative volume elements for random 2D composites with circular nonoverlapping inclusions*. Theor. Appl. Inform. **24** (2012), 227–242.
8. Hurwitz A., Courant R.: *Theory of Functions*. Moscow, Nauka 1968.
9. Mityushev V.V.: *Transport properties of doubly periodic arrays of circular cylinders and optimal design problems*. Appl. Math. Optim. **44** (2001), 17–31.
10. Mityushev V.V.: *Representative cell in mechanics of composites and generalized Eisenstein-Rayleigh sums*. Complex Var. Elliptic Equ. **51** (2006), 1033–1045.
11. Mityushev V.V., Nawalaniec W.: *Basic sums and their random dynamic changes in description of microstructure of 2D composites*. Comput. Mater. Sci. **97** (2015), 64–74.
12. Nawalaniec W.: *Algorithms for computing symbolic representations of basic esums and their application to composites*. J. Symbolic Comput. **74** (2016), 328–345.
13. Rylko N.: *Representative volume element in 2D for disks and in 3D for balls*. J. Mech. Mater. Struct. **9** (2014), 427–439.

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14. Sokolov A., Aranson I.S., Kessler J.O., Goldstein R.E.: *Concentration dependence of the collective dynamics of swimming bacteria*. Phys. Rev. Lett. **98**, 158102 (2007), 1–4.
 15. Szczepkowski J., Malevich A.E., Mityushev V.V.: *Macroscopic properties of similar arrays of cylinders*. Quart. J. Appl. Math. Mech. **56** (2003), 617–628.
 16. Torquato S.: *Random Hetrogeneous Materials: Microstructure and Macroscopic Properties*. Spinger-Verlag, New York 2002.