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STATISTICAL DISTRIBUTIONS AND RELIABILITY FUNCTIONS WITH TYPE-2 FUZZY PARAMETERS

ROZKŁADY STATYSTYCZNE I FUNKCJE NIEZAWODNOŚCI O PARAMETRACH ROZMYTYCH TYPU-2

Type-2 fuzzy sets were initially given by Zadeh as an extension of type-1 fuzzy sets. There is a growing interest in type-2 fuzzy set and its memberships (named secondary memberships) to handle the uncertainty in type-1 fuzzy set and its primary membership values. However, arithmetical operators on type-2 fuzzy sets have computational complexity due to third dimension of these sets. In this study, we present some mathematical operators which can be easily applied to type-2 fuzzy sets and numbers. Also, mathematical functions of type-2 fuzzy numbers are given according to their monotonicity. These functions are adapted to reliability and distribution functions of the random variables with the type-2 fuzzy parameters. These functions are applied to Exponential, Chi-square, Weibull distributions with respect to monotonicity of the parameters of these distributions.

Keywords: *type-2 fuzzy number, type-2 fuzzy parameters, (α,β)-cuts, fuzzy probability distribution, fuzzy reliability function.*

Zbiory rozmyte typu 2 po raz pierwszy zaproponował Zadeh jako rozszerzenie zbiorów rozmytych typu 1. Zbiory rozmyte typu 2 oraz ich funkcje przynależności (zwane wtórnymi funkcjami przynależności) cieszą się rosnącym zainteresowaniem, ponieważ pozwalają na modelowanie niepewności w zbiorze rozmytym typu 1 oraz wartości pierwotnych funkcji przynależności do takiego zbioru. Ich wadą jest złożoność obliczeniowa operatorów arytmetycznych wynikająca z trójwymiarowości tych zbiorów. W artykule przedstawiono operatory matematyczne, które można z powodzeniem stosować w odniesieniu do zbiorów i liczb rozmytych typu 2. Podano również funkcje matematyczne liczb rozmytych typu 2 zgodnie z ich monotonicznością. Funkcje te są dostosowane do funkcji niezawodności i rozkładu zmiennych losowych z parametrami rozmytymi typu 2. Można je stosować do opisu rozkładów wykładniczych, chi-kwadrat, oraz Weibulla w odniesieniu do monotoniczności parametrów tych rozkładów.

Słowa kluczowe: *liczba rozmyta typu-2, parametry rozmyte typu-2, (α, β) -cięcia, rozmyty rozkład prawdopodobieństwa, rozmyta funkcja niezawodności.*

1. Introduction

Fuzzy sets are useful and effective tools to model the uncertainty problem in real-life applications. The most common fuzzy sets used in these applications are known as type-1 fuzzy sets (T1FSs). Since the membership degrees of T1FSs are crisp numbers, recently, type-2 fuzzy sets (T2FSs) are also preferred by many researchers to express uncertainty in T1FSs. Zadeh [11] introduced T2FS as an extension version of the conventional T1FS. Some important studies about T2FSs can be given as Aisbett et al. [1], Hamrawi [3], Karnik and Mendel [5], Wu and Mendel [9]. Furthermore, some applications of T2FSs can be found in Tao et al. [6], Türkşen [7], Wagenkneckt and Hartmann [8], Wu and Mendel [10]. Also, basic operations on T2FSs were studied by Blewitt et.al. [2], Karnik and Mendel [5]. However, type-2 fuzzy numbers (T2FNs) are required to make theoretical inference about modelling uncertainty. In the literature, limited number of studies can be found related to the operators on T2FNs, e.g., Kardan et al. [4]. Mostly, it is difficult to use these operators on T2FNs due to the computational complexity of T2FSs.

This study introduces practical and innovative solutions for arithmetical operations on T2FN using the (α, β)-cut definition. Thus, type-2 fuzzy parameter-based distributions and reliability functions are proposed with regards to their monotonicity. Therefore, we present a novel perspective to perform various arithmetical operators on type-2 fuzzy numbers. The basis of this perspective is structured by an (α, β)-cut definition. This definition is derived from the type-1 operations on three type-1 membership functions (lower, upper and type-1 membership functions) of T2FN. Some operations (such as sum, subtraction, multiplication, division) for T2FNs are determined using the (α, β)-cuts. Then, the membership functions of T2FNs are structured by the (α, β)-cuts. Besides, we utilize this (α, β)-cut definition to form fuzzy function of T2FN under some assumptions. Finally, we give some applications of probability distributions when some parameters of the distributions are the T2FNs.

This paper is organized as follows: Section 2 provides mathematical background of type-2 fuzzy sets and numbers, (α, β)-cut definition of T2FN, fuzzy function of T2FN with its monotonicity. The applications based on the statistical distributions and reliability functions of T2FN are given in Section 3. Conclusion is drawn in Section 4.

2. Methodology

In this section, we give a brief overview of type-2 fuzzy sets and fuzzy numbers. Also, we propose some arithmetical operators on type-2 triangular fuzzy number (T2TFN) using (α, β)-cuts which are easily obtained by operators on type-1 fuzzy number, fuzzy function of T2TFN with its monotonicity.

2.1. Type-1 and Type-2 Fuzzy Sets

A fuzzy set, B, in a universe of discourse X is defined as $B = \{(x,\mu_B(x));x \in X\}$, where $\mu_B : X \rightarrow [0,1]$ and $\mu_B(x)$: the membership value of $x \in X$ in the fuzzy set B. The well-known issue is the type-1 fuzzy sets (T1FSs) are often represented by crisp numbers ranging from [0, 1]. Zadeh [11] defined the concept of a type-2 fuzzy set (T2FS) as an extension version of an ordinary fuzzy set, i.e., T1FS. A T2FS, $\tilde{B}$, is defined as in (1):

$$\tilde{B} = \int_{x\in X}\int_{u\in J_x} \mu_{\tilde{B}}(x,u)/(x,u), \qquad J_x \subseteq [0,1] \tag{1}$$

where J_x: primary membership function ($u \in J_x \subseteq [0,1]$), $\mu_{\tilde{B}}(x,u)$: secondary membership function and $0 \le \mu_{\tilde{B}}(x,u) \le 1$, $\iint$ denotes union overall admissible domain values x and u. Interval T2FS (IT2FS) is a special case of the T2FS where all $\mu_{\tilde{B}}(x,u) = 1$ in (1).

The footprint of uncertainty (*FOU*) is obtained from union of all primary memberships as shown in (2):

$$FOU(\tilde{B}) = \int_{x\in X} J_x \tag{2}$$

FOU is bounded by two type-1 membership functions (type-1 MFs): i) lower MF $\underline{\mu_{\tilde{B}}(x)}$ and ii) upper MF $\overline{\mu_{\tilde{B}}(x)}$ as shown in (3):

$$FOU(\tilde{B}) = \int_{x\in X}[\underline{\mu_{\tilde{B}}(x)},\overline{\mu_{\tilde{B}}(x)}] \tag{3}$$

2.2. Type-2 Triangular Fuzzy Numbers

In this study, particularly, we utilized T2TFNs to define some parameters of the probability distributions. A T2TFN, $\tilde{\tilde{B}}$, as an extension version of type-1 triangular fuzzy number $\tilde{B}$ T1TFN), and its type-1 MFs are illustrated in Fig. 1.

T2TFN has three membership functions as shown in Fig. 1: i) Lower membership function (LMF) of $\tilde{\tilde{B}}$ ii) Upper membership function (UMF) of $\tilde{\tilde{B}}$ and iii) Type-1 membership function (T1MF) of $\tilde{B}$. The notations of these functions are given in Table 1.

2.3. (α, β)-Cuts of Type-2 Triangular Fuzzy Numbers

In this section, we present (*α, β*)-cut definition of a T2TFN. We perform arithmetical operations on T2TFNs using (*α, β*)-cuts.

α-cut of a T1TFN $B=(a_{10},a,a_{20})$ is defined as $B_\alpha = \{\forall x \in U \mid \mu_B(x) \ge \alpha\}$. Lower and upper *α*-cut of a T1FN B can be written as follows:

Table 1. The type-1 membership functions of T2TFN

$LMF(\tilde{\tilde{B}}),\ \underline{\tilde{\tilde{B}}}$	$UMF(\tilde{\tilde{B}}),\ \overline{\tilde{\tilde{B}}}$	$T1MF(\breve{B}),\ \breve{B}$
$\underline{\mu}(x)=\begin{cases}0, & x\le \underline{a}_{12}\ and\ x\ge \underline{a}_{21}\\ \frac{x-\underline{a}_{12}}{a-\underline{a}_{12}}, & \underline{a}_{12}\le x<a\\ \frac{\underline{a}_{21}-x}{\underline{a}_{21}-a}, & a\le x\le \underline{a}_{21}\end{cases}$	$\overline{\mu}(x)=\begin{cases}0, & x\le \overline{a}_{11}\ and\ x\ge \overline{a}_{22}\\ \frac{x-\overline{a}_{11}}{a-\overline{a}_{11}}, & \overline{a}_{11}\le x<a\\ \frac{\overline{a}_{22}-x}{\overline{a}_{22}-a}, & a\le x\le \overline{a}_{22}\end{cases}$	$\mu(x)=\begin{cases}0, & x\le a_{10}\ and\ x\ge a_{20}\\ \frac{x-a_{10}}{a-a_{10}}, & a_{10}\le x<a\\ \frac{a_{20}-x}{a_{20}-a}, & a\le x\le a_{20}\end{cases}$

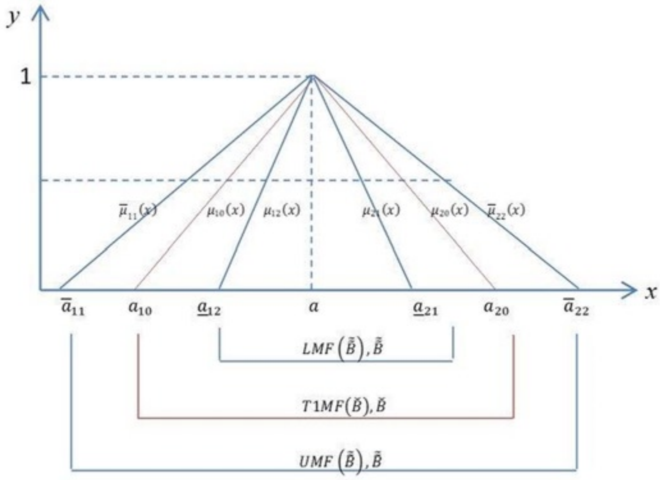

Fig. 1. T2TFN and its type-1 membership functions

$$a_{10}(\alpha) = a_{10} + (a - a_{10})\alpha \quad a_{20}(\alpha) = a_{20} - (a_{20} - a)\alpha$$

Therefore, B_α can be represented as follows:

$$B_\alpha = [a_{10}(\alpha), a_{20}(\alpha)], \alpha \in [0,1]$$

Definition 2.1. (*(α, β)-cut of T2TFN*) Based on *α*-cuts of T1FNs, (*α, β*)-cut of a T2TFN $\tilde{\tilde{B}}$ on R can be given as in (4), where $\alpha,\beta \in [0,1]$:

$$\left[\tilde{\tilde{B}}\right]_\alpha^\beta = \left[\left[\underline{\underline{a}}_{11}(\alpha,\beta),\overline{\overline{a}}_{12}(\alpha,\beta)\right];\left[\underline{\underline{a}}_{21}(\alpha,\beta),\overline{\overline{a}}_{22}(\alpha,\beta)\right]\right] \tag{4}$$

Here,

$$\underline{\underline{a}}_{11}(\alpha,\beta) = \overline{a}_{11}(\alpha) + \beta\left(a_{10}(\alpha) - \overline{a}_{11}(\alpha)\right) \tag{5}$$

$$\overline{\overline{a}}_{12}(\alpha,\beta) = \underline{a}_{12}(\alpha) + \beta\left(a_{10}(\alpha) - \underline{a}_{12}(\alpha)\right) \tag{6}$$

$$\underline{\underline{a}}_{21}(\alpha,\beta) = \underline{a}_{21}(\alpha) + \beta\left(a_{20}(\alpha) - \underline{a}_{21}(\alpha)\right) \tag{7}$$

$$\overline{\overline{a}}_{22}(\alpha,\beta) = \overline{a}_{22}(\alpha) + \beta\left(a_{20}(\alpha) - \overline{a}_{22}(\alpha)\right) \tag{8}$$

Moreover, $\underline{a}_{ij}(\alpha), \overline{a}_{ii}(\alpha), a_{i0}(\alpha)$ can be represented as follows:

$$\underline{a}_{ij}(\alpha) = \underline{a}_{ij} + \alpha\left(a - \underline{a}_{ij}\right), i = 1, 2;\ j = 1, 2,\ i \neq j; \tag{9}$$

$$\overline{a}_{ii}(\alpha) = \overline{a}_{ii} + \alpha\left(a - \overline{a}_{ii}\right),\ i = 1, 2; \tag{10}$$

$$a_{i0}(\alpha) = a_{i0} + \alpha\left(a - a_0\right),\ \textit{for } i = 1, 2 \tag{11}$$

Note that, the initial constants $\underline{a}_{ij}, \overline{a}_{ii}, a_{i0}$ are given in Table 1. The β-cuts for vertical slice of T2TFN are illustrated in Fig. 2.

a) b)

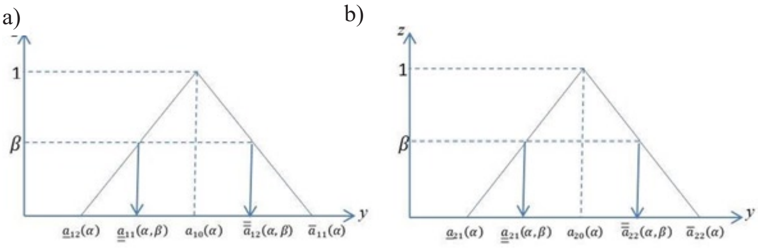

Fig. 2. The β-cuts of (a) LMF, (b) UMF of T2TFN

Finally, (α,β) cuts of T2TFN $\left(\left[\tilde{\tilde{B}}\right]_\alpha^\beta \text{ shown as in Eq.}(4)\right)$ are found as in (12)-(13) for LMF and in (14)-(15) for UMF:

$$\underline{\underline{a}}_{11}(\alpha,\beta) = \overline{a}_{11} + \alpha\left(a - \overline{a}_{11}\right) - \beta(1-\alpha)\left(\overline{a}_{11} - a_{10}\right), \tag{12}$$

$$\overline{\overline{a}}_{12}(\alpha,\beta) = \underline{a}_{12} + \alpha\left(a - \underline{a}_{12}\right) + \beta(1-\alpha)\left(\underline{a}_{12} - a_{10}\right), \tag{13}$$

$$\underline{\underline{a}}_{21}(\alpha,\beta) = \underline{a}_{21} + \alpha\left(a - \underline{a}_{21}\right) + \beta(1-\alpha)\left(\underline{a}_{21} - a_{20}\right), \tag{14}$$

$$\overline{\overline{a}}_{22}(\alpha,\beta) = \overline{a}_{22} + \alpha\left(a - \overline{a}_{22}\right) - \beta(1-\alpha)\left(\overline{a}_{22} - a_{20}\right). \tag{15}$$

2.4. Fuzzy Function of Type-2 Triangular Fuzzy Number

In this section, we present how to determine fuzzy function of a T2TFN under some assumptions.

Definition 2.2. Let *f(x)* be a function of a variable *X* and *X* be a type-2 fuzzy number. The following rules can be obtained for *f(.)* at the point of *X*=*a* under the monotonicity of the function *f*.

Rule-1: Let *f(x)* be monotone increasing function.

Consider $\left[\tilde{\tilde{a}}\right]_\alpha^\beta = \left[\left[\underline{\underline{a}}_{11}(\alpha,\beta), \overline{\overline{a}}_{12}(\alpha,\beta)\right];\left[\underline{\underline{a}}_{21}(\alpha,\beta), \overline{\overline{a}}_{22}(\alpha,\beta)\right]\right]$. The fuzzy function of $\left[\tilde{\tilde{a}}\right]_\alpha^\beta$ is described as follows:

$$\left[f\left(\tilde{\tilde{a}}\right)\right]_\alpha^\beta = \left[\left[f\left(\underline{\underline{a}}_{11}(\alpha,\beta)\right), f\left(\overline{\overline{a}}_{12}(\alpha,\beta)\right)\right];\left[f\left(\underline{\underline{a}}_{21}(\alpha,\beta)\right), f\left(\overline{\overline{a}}_{22}(\alpha,\beta)\right)\right]\right].$$

$f\left(\underline{\underline{a}}_{11}(\alpha,\beta)\right)\Big|_{\beta=0} = x$ then $\underline{\underline{a}}_{11}(\alpha,0) = f^{-1}(x), \Rightarrow \alpha = \mu_{11}\left(f^{-1}(x)\right) \equiv \mu_{f11}(x)$

Similarly, the following membership functions are found:

$$\mu_{f12}(x) = \mu_{12}\left(f^{-1}(x)\right),$$

$$\mu_{f21}(x) = \mu_{21}\left(f^{-1}(x)\right),$$

$$\mu_{f22}(x) = \mu_{22}\left(f^{-1}(x)\right),$$

Rule-2: Let *f(x)* be monotone decreasing function. Consider $\left[\tilde{\tilde{a}}\right]_\alpha^\beta = \left[\left[\underline{\underline{a}}_{11}(\alpha,\beta), \overline{\overline{a}}_{12}(\alpha,\beta)\right];\left[\underline{\underline{a}}_{21}(\alpha,\beta), \overline{\overline{a}}_{22}(\alpha,\beta)\right]\right]$. The fuzzy function of $\left[\tilde{\tilde{a}}\right]_\alpha^\beta$ is described as follows:

$$\left[f\left(\tilde{\tilde{a}}\right)\right]_\alpha^\beta = \left[\left[f\left(\overline{\overline{a}}_{22}(\alpha,\beta)\right), f\left(\underline{\underline{a}}_{21}(\alpha,\beta)\right)\right];\left[f\left(\overline{\overline{a}}_{12}(\alpha,\beta)\right), f\left(\underline{\underline{a}}_{11}(\alpha,\beta)\right)\right]\right]$$

If $f\left(\overline{\overline{a}}_{22}(\alpha,\beta)\right)\Big|_{\beta=0} = x$ then $\overline{\overline{a}}_{22}(\alpha,0) = f^{-1}(x)$, $\Rightarrow \alpha = \mu_{22}\left(f^{-1}(x)\right) = \mu_{f11}(x)$.

Similarly, the following membership functions are found:

$$\mu_{f12}(x) = \mu_{21}\left(f^{-1}(x)\right),$$

$$\mu_{f21}(x) = \mu_{12}\left(f^{-1}(x)\right),$$

$$\mu_{f22}(x) = \mu_{11}\left(f^{-1}(x)\right),$$

Rule-3: Let *f(x)*be non-monotone function. However, consider that *f(x)* is monotone function for each $[d_i, d_{i+1}]$, where $-\infty < d_1 < d_2 < \ldots < d_i < \ldots < \infty$. Accordingly, Rule-1 is applied to *f(x)* if *f(x)* is monotone increasing function at $[d_i, d_{i+1}]$, Rule-2 is applied to *f(x)* if *f(x)* is monotone decreasing function at $[d_i, d_{i+1}]$. An illustration of non-monotone function is given in Fig. 3.

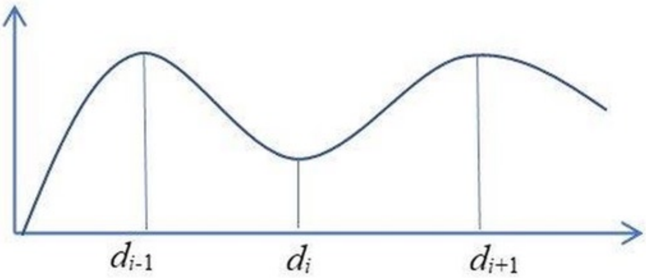

Fig. 3. An illustration of non-monotone function

Definition 2.3. Let $F(x;a)$ be distribution function with parameter-*a* defined by T2TFN. (α,β)-cut of the parameter-a ($\left[\tilde{\tilde{a}}\right]_\alpha^\beta$) is defined as in (4). Then, reliability function with T2TFN-defined parameter ($R(x;a)$) can be determined as in (16):

$$R(x;a) = 1 - F(x;a) \tag{16}$$

The following rules can be obtained for whether R is monotone by the parameter-a.

Rule-1: Let $R(x;a)$ be monotone increasing function by the parameter-a. If the parameter is defined as in (4) then:

$$\left[R(x;\tilde{\tilde{a}})\right]_{\alpha}^{\beta}=\left[\left[R_{11}(x;a;(\alpha,\beta)),R_{12}(x;a;(\alpha,\beta))\right];\left[R_{21}(x;a;(\alpha,\beta)),R_{22}(x;a;(\alpha,\beta))\right]\right].$$

Here:

$$R_{11}(x;a;(\alpha,\beta))=R\left(x;\underline{\underline{a}}_{11}(\alpha,\beta)\right)$$

$$R_{12}(x;a;(\alpha,\beta))=R\left(x;\overline{\overline{a}}_{12}(\alpha,\beta)\right)$$

$$R_{21}(x;a;(\alpha,\beta))=R\left(x;\underline{\underline{a}}_{21}(\alpha,\beta)\right)$$

$$R_{22}(x;a;(\alpha,\beta))=R\left(x;\overline{\overline{a}}_{22}(\alpha,\beta)\right)$$

Rule-2: Let $R(x;a)$ be monotone decreasing function by the parameter-a. According to T2TFN-definedparameter as in (4), reliability function will be obtained as follows:

$$\left[R(x;\tilde{\tilde{a}})\right]_{\alpha}^{\beta}=\left[\left[R_{11}(x;a;(\alpha,\beta)),R_{12}(x;a;(\alpha,\beta))\right];\left[R_{21}(x;a;(\alpha,\beta)),R_{22}(x;a;(\alpha,\beta))\right]\right]$$

Here

$$R_{11}(x;a;(\alpha,\beta))=R\left(x;\overline{\overline{a}}_{22}(\alpha,\beta)\right)$$

$$R_{12}(x;a;(\alpha,\beta))=R\left(x;\underline{\underline{a}}_{21}(\alpha,\beta)\right)$$

$$R_{21}(x;a;(\alpha,\beta))=R\left(x;\overline{\overline{a}}_{12}(\alpha,\beta)\right)$$

$$R_{22}(x;a;(\alpha,\beta))=R\left(x;\underline{\underline{a}}_{11}(\alpha,\beta)\right)$$

Rule-3: Let *R(x;a)* be non-monotone function. However, consider that *R(x;a)* is monotone function for each $[d_i,d_{i+1}]$, where $-\infty<d_1<d_2<\ldots<d_i<\ldots<\infty$. Accordingly, Rule-1 is applied to *R(x;a)* if *R(x;a)* is monotone increasing function at $[d_i,d_{i+1}]$, Rule-2 is applied to *R(x;a)* if *R(x;a)* is monotone decreasing function at $[d_i,d_{i+1}]$.

Distribution function with T2TFN-defined parameter (*F(x;a)*) can be determined by definition 2.3. In this study, Matlab R2018b programming language was used for the illustrations and calculations in section 3.

3. Experimental Results

In this section, we demonstrate how to apply to the well-known three probability distributions when some parameters of these distributions are chosen as T2TFN.

3.1. Experiment-1

Let the distribution function of exponential variate X be as $F(x;a)$ and the parameter a be type-2 fuzzy number. The distribution function $F(x;a)$ can be written as in (17):

$$F(x;a)=P(X\le x)=1-e^{-xa},x>0 \tag{17}$$

Thus, this distribution function is an increasing function according to a. Therefore, the (α,β)-cut of distribution function F can be written as in (18)-(21):

$$F_{11}(x;a;(\alpha,\beta))=F\left(x;\underline{\underline{a}}_{11}(\alpha,\beta)\right)=1-e^{-x\underline{\underline{a}}_{11}(\alpha,\beta)}, \tag{18}$$

$$F_{12}(x;a;(\alpha,\beta))=F\left(x;\overline{\overline{a}}_{12}(\alpha,\beta)\right)=1-e^{-x\overline{\overline{a}}_{12}(\alpha,\beta)}, \tag{19}$$

$$F_{21}(x;a;(\alpha,\beta))=F\left(x;\underline{\underline{a}}_{21}(\alpha,\beta)\right)=1-e^{-x\underline{\underline{a}}_{21}(\alpha,\beta)}, \tag{20}$$

$$F_{22}(x;a;(\alpha,\beta))=F\left(x;\overline{\overline{a}}_{22}(\alpha,\beta)\right)=1-e^{-x\overline{\overline{a}}_{22}(\alpha,\beta)}, \tag{21}$$

(α,β)-cuts for distribution function F of an Exponential variate is illustrated in Fig. 4.

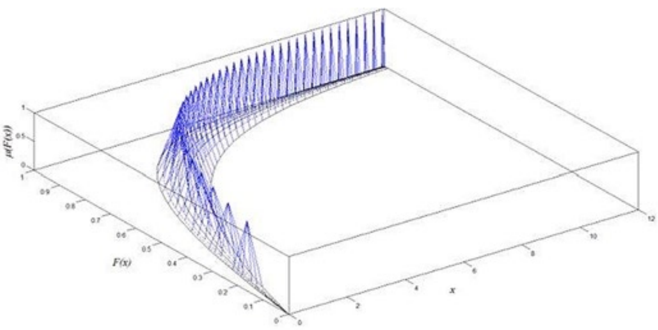

Fig. 4. (α,β)-cuts for distribution function F of an Exponential variate

The reliability function R for exponential distribution can be written as follows:

$$R(x;a)=P(X>x)=e^{-xa},\ x>0 \tag{22}$$

It is easy to see that, $R(x;a)$ is a decreasing function according to parameter a. Therefore, in this case, the (α,β)-cut of reliability function $R(x;a)$ can be written as in (23)-(26):

$$R_{11}(x;a;(\alpha,\beta))=R\left(x;\overline{\overline{a}}_{22}(\alpha,\beta)\right)=e^{-x\overline{\overline{a}}_{22}(\alpha,\beta)}, \tag{23}$$

$$R_{12}(x;a;(\alpha,\beta))=R\left(x;\underline{\underline{a}}_{21}(\alpha,\beta)\right)=e^{-x\underline{\underline{a}}_{21}(\alpha,\beta)}, \tag{24}$$

$$R_{21}(x;a;(\alpha,\beta))=R\left(x;\overline{\overline{a}}_{12}(\alpha,\beta)\right)=e^{-x\overline{\overline{a}}_{12}(\alpha,\beta)}, \tag{25}$$

$$R_{22}\left(x;a;(\alpha,\beta)\right)=R\left(x;\underline{\underline{a}}_{11}(\alpha,\beta)\right)=e^{-x\underline{\underline{a}}_{11}(\alpha,\beta)}. \quad (26)$$

(α,β) cuts for reliability function R of an Exponential variate is illustrated in Fig.5.

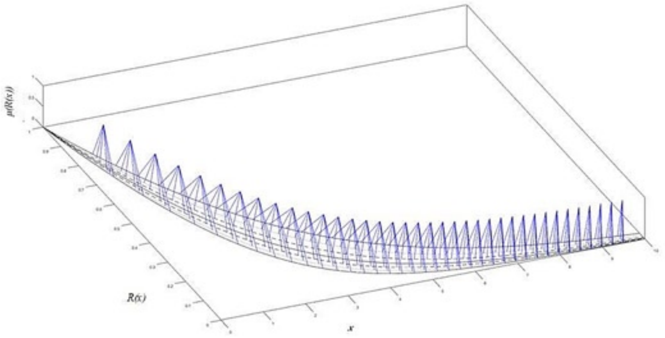

Fig. 5. (α,β)-cuts for Reliability function R of an Exponential variate.

3.2. Experiment-2

Let the distribution function of chi-square variate X be as $F(x;a)$ and the parameter a be type-2 fuzzy number (T2FN). The distribution function $F(x;a)$ can be written as in (27):

$$F(x;a)=P(X\le x)=\frac{1}{2^{\frac{a}{2}}\Gamma\left(\frac{a}{2}\right)}\int_0^x t^{\frac{a}{2}-1}e^{-t/2}dt \quad (27)$$

Thus, this distribution function is an decreasing function according to a. The (α,β)-cut of the parameter a is as in (4).

By using the (α,β)-cut of parameter a, the (α,β)-cut of distribution function F can be written as in (28)-(31):

$$F_{11}\left(x;a;(\alpha,\beta)\right)=F\left(x;\overline{\overline{a}}_{22}(\alpha,\beta)\right)=\frac{1}{2^{\frac{\overline{\overline{a}}_{22}(\alpha,\beta)}{2}}\Gamma\left(\frac{\overline{\overline{a}}_{22}(\alpha,\beta)}{2}\right)}\int_0^x t^{\frac{\overline{\overline{a}}_{22}(\alpha,\beta)}{2}-1}e^{-t/2}dt, \quad (28)$$

$$F_{12}\left(x;a;(\alpha,\beta)\right)=F\left(x;\underline{\underline{a}}_{21}(\alpha,\beta)\right)=\frac{1}{2^{\frac{\underline{\underline{a}}_{21}(\alpha,\beta)}{2}}\Gamma\left(\frac{\underline{\underline{a}}_{21}(\alpha,\beta)}{2}\right)}\int_0^x t^{\frac{\underline{\underline{a}}_{21}(\alpha,\beta)}{2}-1}e^{-t/2}dt, \quad (29)$$

$$F_{21}\left(x;a;(\alpha,\beta)\right)=F\left(x;\overline{\overline{a}}_{12}(\alpha,\beta)\right)=\frac{1}{2^{\frac{\overline{\overline{a}}_{12}(\alpha,\beta)}{2}}\Gamma\left(\frac{\overline{\overline{a}}_{12}(\alpha,\beta)}{2}\right)}\int_0^x t^{\frac{\overline{\overline{a}}_{12}(\alpha,\beta)}{2}-1}e^{-t/2}dt, \quad (30)$$

$$F_{22}\left(x;a;(\alpha,\beta)\right)=F\left(x;\underline{\underline{a}}_{11}(\alpha,\beta)\right)=\frac{1}{2^{\frac{\underline{\underline{a}}_{11}(\alpha,\beta)}{2}}\Gamma\left(\frac{\underline{\underline{a}}_{11}(\alpha,\beta)}{2}\right)}\int_0^x t^{\frac{\underline{\underline{a}}_{11}(\alpha,\beta)}{2}-1}e^{-t/2}dt, \quad (31)$$

(α,β)-cuts for distribution function F of a Chi-square variate is illustrated in Fig. 6.

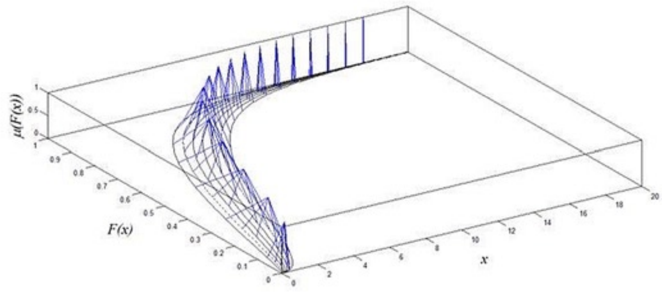

Fig. 6. (α,β)-cuts for distribution function F of a Chi-square variate

The reliability function R for chi-square distribution is an increasing function according to parameter a.

$$R(x;a)=P(X>x)=1-\frac{1}{2^{\frac{a}{2}}\Gamma\left(\frac{a}{2}\right)}\int_0^x t^{\frac{a}{2}-1}e^{-t/2}dt \quad (32)$$

Therefore, (α,β)-cut of R can be written as in (33)-(36):

$$R\left(x;a;(\alpha,\beta)\right)=R\left(x;\underline{\underline{a}}_{11}(\alpha,\beta)\right)=1-\frac{1}{2^{\frac{\underline{\underline{a}}_{11}(\alpha,\beta)}{2}}\Gamma\left(\frac{\underline{\underline{a}}_{11}(\alpha,\beta)}{2}\right)}\int_0^x t^{\frac{\underline{\underline{a}}_{11}(\alpha,\beta)}{2}-1}e^{-t/2}dt, \quad (33)$$

$$R_{12}\left(x;a;(\alpha,\beta)\right)=R\left(x;\overline{\overline{a}}_{12}(\alpha,\beta)\right)=1-\frac{1}{2^{\frac{\overline{\overline{a}}_{12}(\alpha,\beta)}{2}}\Gamma\left(\frac{\overline{\overline{a}}_{12}(\alpha,\beta)}{2}\right)}\int_0^x t^{\frac{\overline{\overline{a}}_{12}(\alpha,\beta)}{2}-1}e^{-t/2}dt, \quad (34)$$

$$R_{21}\left(x;a;(\alpha,\beta)\right)=R\left(x;\underline{\underline{a}}_{21}(\alpha,\beta)\right)=1-\frac{1}{2^{\frac{\underline{\underline{a}}_{21}(\alpha,\beta)}{2}}\Gamma\left(\frac{\underline{\underline{a}}_{21}(\alpha,\beta)}{2}\right)}\int_0^x t^{\frac{\underline{\underline{a}}_{21}(\alpha,\beta)}{2}-1}e^{-t/2}dt, \quad (35)$$

$$R_{22}\left(x;a;(\alpha,\beta)\right)=R\left(x;\overline{\overline{a}}_{22}(\alpha,\beta)\right)=1-\frac{1}{2^{\frac{\overline{\overline{a}}_{22}(\alpha,\beta)}{2}}\Gamma\left(\frac{\overline{\overline{a}}_{22}(\alpha,\beta)}{2}\right)}\int_0^x t^{\frac{\overline{\overline{a}}_{22}(\alpha,\beta)}{2}-1}e^{-t/2}dt. \quad (36)$$

(α,β)-cuts for reliability function R of a Chi-square variate is illustrated in Fig. 7.

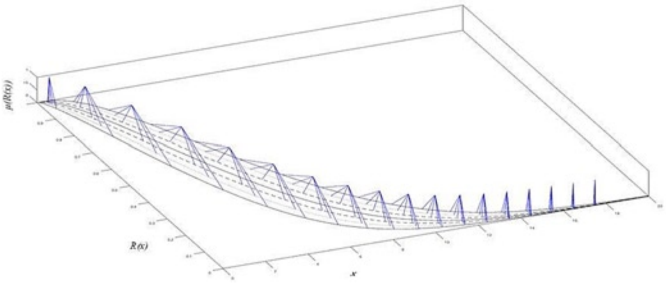

Fig. 7. (α,β)-cuts for Reliability function R of a Chi-Square variate

3.3. Experiment-3

Let the distribution function of Weibull variate X be as $F(x;(a,\sigma))$ and the parameter a be type-2 fuzzy number. The distribution function $F(x;(a,\sigma))$ can be written as follows:

$$F(x;(a,\sigma)) = P(X \le x) = 1 - e^{-\left(\frac{x}{a}\right)^{\sigma}} \tag{37}$$

It is seen that, this distribution function is a decreasing function according to parameter a. The (α,β)-cut of the parameter a is as in (4).

In this case, the (α,β)-cut of distribution function F can be written as in (38)-(41):

$$F_{11}(x;a;(\alpha,\beta)) = F\left(x;\overline{\overline{a}}_{22}(\alpha,\beta)\right) = 1 - e^{-\left(\frac{x}{\overline{\overline{a}}_{22}(\alpha,\beta)}\right)^{\sigma}}, \tag{38}$$

$$F_{12}(x;a;(\alpha,\beta)) = F\left(x;\underline{\underline{a}}_{21}(\alpha,\beta)\right) = 1 - e^{-\left(\frac{x}{\underline{\underline{a}}_{21}(\alpha,\beta)}\right)^{\sigma}}, \tag{39}$$

$$F_{21}(x;a;(\alpha,\beta)) = F\left(x;\overline{\overline{a}}_{12}(\alpha,\beta)\right) = 1 - e^{-\left(\frac{x}{\overline{\overline{a}}_{12}(\alpha,\beta)}\right)^{\sigma}}, \tag{40}$$

$$F_{22}(x;a;(\alpha,\beta)) = F\left(x;\underline{\underline{a}}_{11}(\alpha,\beta)\right) = 1 - e^{-\left(\frac{x}{\underline{\underline{a}}_{11}(\alpha,\beta)}\right)^{\sigma}}, \tag{41}$$

(α,β)-cuts for distribution function F of a Weibull variate is illustrated in Fig. 8.

The reliability function $R(x;(a,\sigma))$ of Weibull distribution has the following form:

$$R(x;(a,\sigma)) = P(X > x) = e^{-\left(\frac{x}{a}\right)^{\sigma}} \tag{42}$$

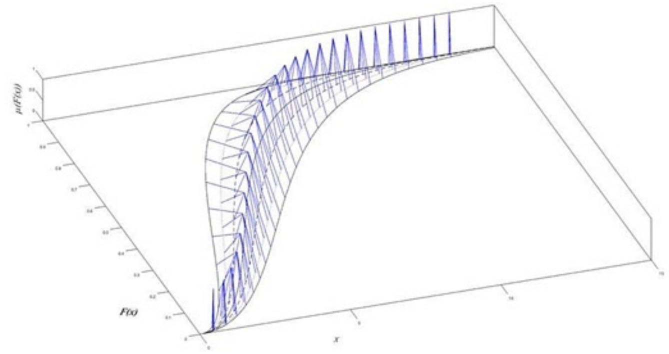

Fig. 8. (α,β)-cuts for distribution function F of a Weibull variate

It is seen from (42) that the reliability function $R(x;(a,\sigma))$ is an increasing function according to parameter a. Therefore, (α,β)-cut of reliability function R can be written as in (43)-(46):

$$R_{11}(x;a;(\alpha,\beta)) = R\left(x;\underline{\underline{a}}_{11}(\alpha,\beta)\right) = e^{-\left(\frac{x}{\underline{\underline{a}}_{11}(\alpha,\beta)}\right)^{\sigma}} \tag{43}$$

$$R_{12}(x;a;(\alpha,\beta)) = R\left(x;\overline{\overline{a}}_{12}(\alpha,\beta)\right) = e^{-\left(\frac{x}{\overline{\overline{a}}_{12}(\alpha,\beta)}\right)^{\sigma}} \tag{44}$$

$$R_{21}(x;a;(\alpha,\beta)) = R\left(x;\underline{\underline{a}}_{21}(\alpha,\beta)\right) = e^{-\left(\frac{x}{\underline{\underline{a}}_{21}(\alpha,\beta)}\right)^{\sigma}} \tag{45}$$

$$R_{22}(x;a;(\alpha,\beta)) = R\left(x;\overline{\overline{a}}_{22}(\alpha,\beta)\right) = e^{-\left(\frac{x}{\overline{\overline{a}}_{22}(\alpha,\beta)}\right)^{\sigma}} \tag{46}$$

(α,β)-cuts for reliability function R of a Weibull variate is illustrated in Fig. 9.

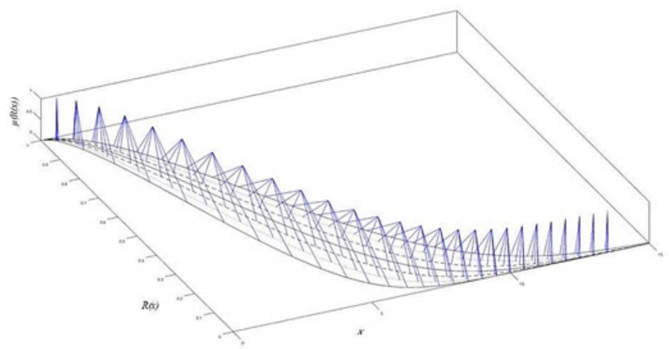

Fig. 9. (α,β)-cuts for Reliability function R of a Weibull variate

4. Conclusion

In this study, we presented (α, β)-cut definition of a T2TFN which is easily defined by type-1 membership functions of the T2TFN. Therefore, the computational complexity due to third dimension of T2FSs was reduced by using this (α, β)-cut definition. According to this definition, we proposed fuzzy function of T2TFN under its structural rules: i) monotone decreasing, ii) monotone increasing, iii) non-monotone functions. We provided how to apply the (α, β)-cut to some well-known statistical distributions and reliability functions, where the parameters are defined as T2TFN. The proposed T2TFN parameter-based distributions and reliability functions can be used for various application areas related to survival analysis, reliability-based system design, machine productivity. In future studies, the reliability function with type-2 fuzzy parameter, expressing by non-triangular membership function, will be investigated.

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