BULLETIN OF THE POLISH ACADEMY OF SCIENCES
TECHNICAL SCIENCES, Vol. 69(4), 2021, Article number: e137731
DOI: 10.24425/bpasts.2021.137731

CONTROL AND INFORMATICS

Positive electrical circuits with the chain structure and cyclic Metzler state matrices

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Abstract. The cyclicity of the state matrices of positive linear electrical circuits with the chain structure is considered. Two classes of positive linear electrical circuits with the chain structure and cyclic Metzler state matrices are analyzed. Some new properties of these classes of positive electrical circuits are established. The results are extended to fractional linear electrical circuits.

Key words: cyclic matrix; positive; electrical circuit; fractional; Metzler matrix.

1. INTRODUCTION

A dynamical system and an electrical circuit are called positive if their state variables take nonnegative values for all nonnegative inputs and nonnegative initial conditions. The positive linear systems have been investigated in [1–9]. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models having positive linear behavior can be found in engineering, management science, economics, social sciences, biology and medicine, etc.

Mathematical fundamentals of the fractional calculus are given in the monographs [6,7,10,11]. Fractional dynamical systems have been investigated in [4–7,10,12–16].

Positive linear systems with different fractional orders have been addressed in [4,6,7,15].

In this paper the positive electrical circuits of integer and fractional orders with the chain structure and cyclic Metzler state matrices are investigated. The paper is organized as follows. In Section 2 some definitions and theorems concerning positive and cyclic matrices are recalled. New results concerning positive electrical circuits with the chain structure and cyclic Metzler state matrices are presented in Section 3. An extension of these results to fractional positive electrical circuits is given in Section 4. Concluding remarks are given in Section 5.

The following notation will be used: $\mathbb{R}$ – the set of real numbers, $\mathbb{R}^{n\times m}$ – the set of $n\times m$ real matrices, $\mathbb{M}_n$ – the set of $n\times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries), $\mathbb{I}_n$ – the $n\times n$ identity matrix.

2. PRELIMINARIES

Consider the continuous-time linear system

$$\dot{x} = Ax + Bu, \tag{1a}$$

$$y = Cx, \tag{1b}$$

where $x = x(t) \in \mathbb{R}^n$, $u = u(t) \in \mathbb{R}^m$, $y = y(t) \in \mathbb{R}^p$ are the state, input and output vectors and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$.

Definition 1. [3,6,7] The continuous-time linear system (1) is called (internally) positive if $x(t) \in \mathbb{R}_+^n$, $y(t) \in \mathbb{R}_+^p$, $t \geq 0$ for any initial conditions $x(0) \in \mathbb{R}_+^n$ and all inputs $u(t) \in \mathbb{R}_+^m$, $t \geq 0$.

Theorem 1. [3,6,7] The continuous-time linear system (1) is positive if and only if

$$A \in \mathbb{M}_n, \quad B \in \mathbb{R}_+^{n\times m}, \quad C \in \mathbb{R}_+^{p\times n}. \tag{2}$$

Definition 2. [3,6,7] The positive continuous-time system (1) for $u(t) = 0$ is called asymptotically stable if

$$\lim_{t\to\infty} x(t) = 0 \quad \text{for any} \quad x(0) \in \mathbb{R}_+^n. \tag{3}$$

Theorem 2. [3,6,7] The positive continuous-time linear system (1) for $u(t) = 0$ is asymptotically stable if and only if one of the following equivalent conditions is satisfied:

1. All coefficients of the characteristic polynomial
$$p_n(s) = \det[\mathbb{I}_n s - A] = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0 \tag{4}$$
are positive, i.e. $a_i > 0$ for $i = 0,1,\ldots,n-1$.
2. There exists strictly positive vector $\lambda^T = [\lambda_1 \quad \ldots \quad \lambda_n]$, $\lambda_k > 0$, $k = 1,\ldots,n$ such that
$$A\lambda < 0 \quad \text{or} \quad \lambda^T A < 0. \tag{5}$$
If the matrix A is nonsingular, then we can choose $\lambda = A^{-1}c$, where $c \in \mathbb{R}^n$ is strictly positive.

Let

$$\varphi(s) = \det[\mathbb{I}_n s - A] = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0 \tag{6}$$

be the characteristic polynomial of the matrix A.

The minimal polynomial $\psi(s)$ of the matrix A is related to the characteristic polynomial (6) by

$$\psi(s) = \frac{\varphi(s)}{D_{n-1}(s)}, \tag{7}$$

where $D_{n-1}(s)$ is the greatest common divisor of all $n-1$ order minors of the matrix $[\mathbb{I}_n s - A]$ [3,5,17].

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Manuscript submitted 2021-03-09, revised 2021-06-01, initially accepted for publication 2021-06-09, published in August 2021

From (7) it follows that $\psi(s) = \varphi(s)$ if and only if $D_{n-1}(s) = 1$.

Definition 3. The matrix A is called cyclic if $\psi(s) = \varphi(s)$.

Definition 4. The matrix A has the Frobenius canonical form if it has one of the following forms [18]:

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix},$$

$$A_2 = A_1^T = \begin{bmatrix} 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & \dots & 0 & -a_1 \\ 0 & 1 & \dots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -a_{n-1} \end{bmatrix},$$

$$A_3 = \begin{bmatrix} -a_{n-1} & -a_{n-2} & \dots & -a_1 & -a_0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}, \tag{8}$$

$$A_4 = A_3^T = \begin{bmatrix} -a_{n-1} & 1 & 0 & \dots & 0 \\ -a_{n-2} & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_1 & 0 & 0 & \dots & 1 \\ -a_0 & 0 & 0 & \dots & 0 \end{bmatrix}.$$

The inverse matrices of the matrices (8) have also the Frobenius canonical forms [12, 18, 19]:

$$A_1^{-1} = \begin{bmatrix} -\frac{a_1}{a_0} & -\frac{a_2}{a_0} & \dots & -\frac{a_{n-1}}{a_0} & -\frac{1}{a_0} \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix},$$

$$A_2^{-1} = \begin{bmatrix} -\frac{a_1}{a_0} & 1 & 0 & \dots & 0 \\ -\frac{a_2}{a_0} & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{a_{n-1}}{a_0} & 0 & 0 & \dots & 1 \\ -\frac{1}{a_0} & 0 & 0 & \dots & 0 \end{bmatrix},$$

$$A_3^{-1} = \begin{bmatrix} 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \\ -\frac{1}{a_0} & -\frac{a_{n-1}}{a_0} & \dots & -\frac{a_2}{a_0} & -\frac{a_1}{a_0} \end{bmatrix},$$

$$A_4^{-1} = \begin{bmatrix} 0 & \dots & 0 & 0 & -\frac{1}{a_0} \\ 1 & \dots & 0 & 0 & -\frac{a_{n-1}}{a_0} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \dots & 1 & 0 & -\frac{a_2}{a_0} \\ 0 & \dots & 0 & 1 & -\frac{a_1}{a_0} \end{bmatrix}. \tag{9}$$

Note that the greatest common divisor of the matrices (8) and (9) is $D_{n-1}(s) = 1$.

Theorem 3. The real matrix

$$\overline{A}_1 = \begin{bmatrix} a_{11} & a_{12} & 0 & \dots & 0 & 0 \\ a_{21} & a_{22} & a_{23} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-2,1} & a_{n-2,2} & a_{n-2,3} & \dots & a_{n-2,n-1} & 0 \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \dots & a_{n-1,n-1} & a_{n-1,n} \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{n,n-1} & a_{nn} \end{bmatrix} \tag{10a}$$

and

$$\overline{A}_2 = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1,n-2} & a_{1,n-1} & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2,n-2} & a_{2,n-1} & a_{2n} \\ 0 & a_{32} & \dots & a_{3,n-2} & a_{3,n-1} & a_{3n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & a_{n,n-1} & a_{nn} \end{bmatrix} \tag{10b}$$

are cyclic matrices if the matrix (10a) satisfies the condition

$$a_{12}, a_{23}, \dots, a_{n-2,n-1}, a_{n-1,n} \neq 0 \tag{11a}$$

and the matrix (10b)

$$a_{21}, a_{32}, \dots, a_{n-1,n-2}, a_{n,n-1} \neq 0, \tag{11b}$$

respectively.

Proof. If the condition (11a) is satisfied then the greatest common divisor of all $n-1$ order minors of the matrix $[\mathbb{I}_n s - A_1]$ for A_1 defined by (10a) is $D_{n-1}(s) = 1$. In this case from (7) we have $\psi(s) = \varphi(s)$ and by Definition 3 the matrix is cyclic. Proof for the matrix (10b) is dual. □

Remark 1. Every square matrix with only one nonzero entry in each row and in each column and its inverse are cyclic matrices [7].

Examples of such matrices are the matrices (8).

Definition 5. The system is called normal if its matrix A is cyclic.

Normal systems have very useful properties and play an important role in technical sciences [2–4, 17].

3. POSITIVE ELECTRICAL CIRCUITS WITH THE CHAIN STRUCTURE AND CYCLIC STATE MATRICES

Consider the electrical circuit shown in Fig. 1 with given resistances $R_1, R_2, \dots, R_n$, inductances $L_1, L_2, \dots, L_n$ and source voltage $e = e(t)$.

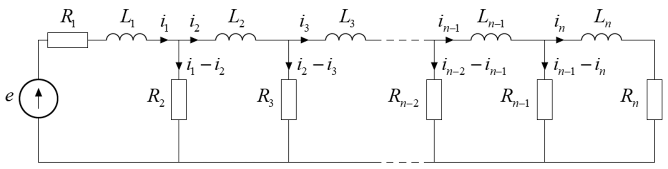

Fig. 1. Electrical circuit with inductances

Using Kirchhoff's laws we may write for the electrical circuit the equations

$$\begin{aligned}
&L_1 \frac{di_1}{dt} + (R_1+R_2)i_1 - R_2 i_2 = e,\\
&L_2 \frac{di_2}{dt} + (R_2+R_3)i_2 - R_2 i_1 - R_3 i_3 = 0,\\
&\vdots \\
&L_{n-1} \frac{di_{n-1}}{dt} + R_{n-1}(i_{n-1}-i_n) + R_{n-2}(i_{n-1}-i_{n-2}) = 0,\\
&L_n \frac{di_n}{dt} + R_{n-1}(i_n - i_{n-1}) + R_n i_n = 0,
\end{aligned} \tag{12}$$

which can be written in the form

$$\frac{dx_L}{dt} = A_1 x_L + B_1 e, \tag{13a}$$

where

$$A_1 = \begin{bmatrix} a_1 & \frac{R_2}{L_1} & 0 & \dots & 0 & 0 & 0 \\ \frac{R_2}{L_2} & a_2 & \frac{R_3}{L_2} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \frac{R_{n-2}}{L_{n-1}} & a_{n-1} & \frac{R_{n-1}}{L_{n-1}} \\ 0 & 0 & 0 & \dots & 0 & \frac{R_{n-1}}{L_n} & a_n \end{bmatrix},$$

$$a_1 = -\frac{R_1+R_2}{L_1}, \quad a_2 = -\frac{R_2+R_3}{L_2}, \tag{13b}$$

$$a_{n-1} = -\frac{R_{n-2}+R_{n-1}}{L_{n-1}}, \quad a_n = -\frac{R_{n-1}+R_n}{L_n},$$

$$B_1 = \begin{bmatrix} \frac{1}{L_1} & 0 & \dots & 0 & 0 \end{bmatrix}^T,$$

$$x_L = \begin{bmatrix} i_1 & i_2 & \dots & i_{n-1} & i_n \end{bmatrix}^T.$$

As the output $y = y(t)$ of the electrical circuit the voltage on the resistance R_n is chosen

$$y = R_n i_n = \bar{C}_1 x_L, \quad \bar{C}_1 = \begin{bmatrix} 0 & \dots & 0 & R_n \end{bmatrix}. \tag{13c}$$

Note that the matrix A_1 is the asymptotically stable Metzler matrix and the matrices $B_1 \in \mathbb{R}_+^{n\times 1}$, $\bar{C}_1 \in \mathbb{R}_+^{1\times n}$, and the electrical circuit is positive and asymptotically stable. The matrix A_1 satisfies the condition of Theorem 3 and it is a cyclic matrix. Therefore, the following theorem has been proved.

Theorem 4. The electrical circuit shown in Fig. 1 is positive with asymptotically stable cyclic Metzler state matrix A_1.

Consider the electrical circuit shown in Fig. 2 with given resistances R_1, R_2, $\dots$, R_{n-1} capacitances C_1, C_2, $\dots$, C_n, and source voltage $e = e(t)$.

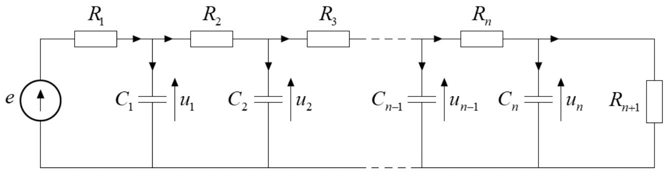

Fig. 2. Electrical circuit with capacitances

Using Kirchhoff's laws we may write for the electrical circuit the equations

$$\begin{aligned}
\frac{e-u_1}{R_1} &= \frac{u_1-u_2}{R_2} + C_1\frac{du_1}{dt},\\
\frac{u_1-u_2}{R_2} &= \frac{u_2-u_3}{R_3} + C_2\frac{du_2}{dt},\\
&\vdots \\
\frac{u_{n-1}-u_n}{R_n} &= \frac{u_n}{R_{n+1}} + C_n\frac{du_n}{dt},
\end{aligned} \tag{14a}$$

which can written in the form

$$\frac{dx_C}{dt} = A_2 x_C + B_2 e, \tag{14b}$$

where

$$A_2 = \begin{bmatrix} a_1 & \frac{1}{C_1R_2} & 0 & \dots & 0 & 0 & 0 \\ \frac{1}{C_2R_2} & a_2 & \frac{1}{C_2R_3} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \frac{1}{C_{n-1}R_{n-1}} & a_{n-1} & \frac{1}{C_{n-1}R_n} \\ 0 & 0 & 0 & \dots & 0 & \frac{1}{C_nR_n} & a_n \end{bmatrix}, \tag{14c}$$

$$a_n = -\frac{1}{C_n}\left(\frac{1}{R_n} + \frac{1}{R_{n+1}}\right),$$

$$B_2 = \begin{bmatrix} \frac{1}{R_1C_1} & 0 & \dots & 0 & 0 \end{bmatrix}^T,$$

$$x_C = \begin{bmatrix} u_1 & u_2 & \dots & u_{n-1} & u_n \end{bmatrix}^T.$$

As the output $y = y(t)$ of the electrical circuit we choose the voltage on the resistance R_{n+1}

$$y = u_n = \bar{C}_2 x_C, \qquad \bar{C}_2 = \begin{bmatrix} 0 & 0 & \dots & 1 \end{bmatrix}. \tag{14d}$$

The matrix A_2 is cyclic asymptotically stable Metzler matrix and $B_2 \in \mathbb{R}_+^{n\times 1}$, $\bar{C}_2 \in \mathbb{R}_+^{1\times n}$. The electrical circuit shown in Fig. 2 is positive and asymptotically stable. Therefore, the following theorem has been proved.

Theorem 5. The electrical circuits shown in Fig. 2 is positive with asymptotically stable cyclic Metzler state matrix A_2.

The following example shows that the presented above considerations can be extended to linear electrical circuits composed of resistances, inductances, capacitances and source voltages.

Example 1. Consider the electrical circuit shown in Fig. 3 with given resistances $R_1, R_2, \dots, R_6$, inductances L_1, L_2, capacitances C_1, C_2 and source voltage $e = e(t)$.

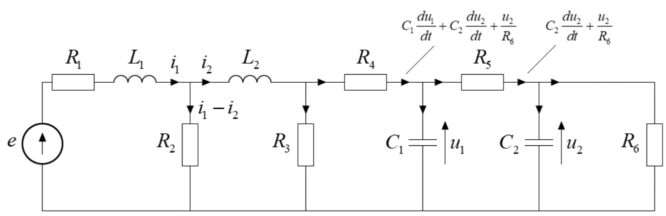

Fig. 3. Linear electrical circuit

Using Kirchhoff's laws we may write for the electrical circuit the following equations

$$\begin{aligned}
&R_1 i_1 + L_1 \frac{di_1}{dt} + R_2(i_1 - i_2) = e,\\
&L_2 \frac{di_2}{dt} + R_3\left(i_2 - C_1\frac{du_1}{dt} - C_2\frac{du_2}{dt} - \frac{u_2}{R_6}\right)\\
&\qquad - R_2(i_1 - i_2) = 0,\\
&R_3\left(i_2 - C_1\frac{du_1}{dt} - C_2\frac{du_2}{dt} - \frac{u_2}{R_6}\right)\\
&\qquad - R_4\left(C_1\frac{du_1}{dt} + C_2\frac{du_2}{dt} + \frac{u_2}{R_6}\right) - u_1 = 0,\\
&\frac{du_1}{dt} = -\frac{R_3 + 2R_4}{C_1 R_4}u_1 + \frac{u_2}{C_1 R_5} + \frac{R_3}{C_1(R_3 + R_4)}i_2,\\
&u_1 - R_5\left(C_2\frac{du_2}{dt} + \frac{u_2}{R_6}\right) - u_2 = 0,
\end{aligned} \tag{15a}$$

which can be written in the form

$$\frac{d}{dt}\begin{bmatrix} i_1 \\ i_2 \\ u_1 \\ u_2 \end{bmatrix} = A \begin{bmatrix} i_1 \\ i_2 \\ u_1 \\ u_2 \end{bmatrix} + Be, \tag{15b}$$

where

$$A = \begin{bmatrix}
-\frac{R_1+R_2}{L_1} & \frac{R_2}{L_1} & 0 & 0\\
\frac{R_2}{L_2} & -\frac{R_2(R_3+R_4)+R_3R_4}{(R_3+R_4)L_2} & \frac{R_3}{(R_3+R_4)L_2} & 0\\
0 & \frac{R_3}{C_1(R_3+R_4)} & -\frac{R_3+2R_4}{C_1R_4(R_3+R_4)} & \frac{1}{C_1R_5}\\
0 & 0 & \frac{1}{C_2R_5} & -\frac{R_5+R_6}{R_5R_6C_2}
\end{bmatrix},$$

$$B = \begin{bmatrix} \frac{1}{L_1} & 0 & 0 & 0 \end{bmatrix}^T. \tag{15c}$$

Note that the matrix A satisfies the condition of Theorem 3 and it is a cyclic matrix. As the output $y = y(t)$ of the electrical circuit the voltage on the resistance R_6 is chosen

$$y = u_2 = Cx, \qquad C = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}. \tag{15d}$$

Note that the matrix A is the cyclic Metzler matrix and the matrices $B \in \mathbb{R}_+^{4\times 1}$, $C \in \mathbb{R}_+^{1\times 4}$. Therefore, the electrical circuit is positive with cyclic state matrix. By the condition (5) of Theorem 2 the electrical circuit is asymptotically stable if

$$\frac{R_3}{R_3 + R_4} + \frac{1}{R_5} < \frac{R_3 + 2R_4}{R_4(R_3 + R_4)}. \tag{16}$$

4. FRACTIONAL ELECTRICAL CIRCUITS

Consider the fractional linear electrical circuit [7] described by the equations

$$\frac{d^\alpha x(t)}{dt^\alpha} = Ax(t) + Bu(t), \quad 0 < \alpha < 1, \tag{17a}$$

$$y(t) = Cx(t), \tag{17b}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$ are the state, input and output vectors and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$,

$$\frac{d^\alpha x(t)}{dt^\alpha} = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\dot{x}(\tau)}{(t-\tau)^\alpha}d\tau, \qquad \dot{x}(\tau) = \frac{dx(t)}{dt} \tag{17c}$$

is the Caputo fractional derivative and

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt, \qquad \mathrm{Re}(z) > 0 \tag{17d}$$

is the gamma function [6, 11].

Definition 6. [6] The fractional electrical circuit (17) is called (internally) positive if $x(t) \in \mathbb{R}_+^n$ and $y(t) \in \mathbb{R}_+^p$, $t \geq 0$ for any initial conditions $x(0) \in \mathbb{R}_+^n$ and all inputs $u(t) \in \mathbb{R}_+^m$, $t \geq 0$.

Theorem 6. [6] The fractional electrical (17) is positive if and only if

$$A \in \mathbb{M}_n, \quad B \in \mathbb{R}_+^{n\times m}, \quad C \in \mathbb{R}_+^{p\times n}. \tag{18}$$

Definition 7. [6] The positive fractional electrical circuit (17) (for $u(t) = 0$) is called asymptotically stable (the matrix A is Hurwitz) if

$$\lim_{t\to\infty} x(t) = 0 \quad \text{for any } x(0) \in \mathbb{R}_+^n. \tag{19}$$

Theorem 7. [6] The positive electrical circuit (17) is asymptotically stable if and only if one of the equivalent conditions is satisfied:

1. All coefficients of the characteristic polynomial

$$\det[\mathbb{I}_n s - A] = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0 \tag{20}$$

are positive, i.e. $a_k > 0$ for $k = 0, 1, \ldots, n-1$.

2. All principal minors $\bar{M}_i$, $i = 1, \ldots, n$ of the matrix $-A$ are positive, i.e.

$$\bar{M}_1 = |-a_{11}| > 0, \quad \bar{M}_2 = \begin{vmatrix} -a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{vmatrix} > 0, \quad \ldots, \quad \bar{M}_n = \det[-A] > 0. \tag{21}$$

3. There exists strictly positive vector $\lambda^T = [\, \lambda_1 \quad \ldots \quad \lambda_n \,]$, $\lambda_k > 0$, $k = 1, \ldots, n$ such that

$$A\lambda < 0 \quad \text{or} \quad A^T\lambda < 0. \tag{22}$$

Consider the fractional electrical circuit shown in Fig. 1 with given resistances $R_1, R_2, \ldots, R_n$, inductances $L_1, L_2, \ldots, L_n$ and source voltage $e = e(t)$. In a similar way as for integer order derivative we can write using Kirchhoff's laws similar equations as (12) substituting the first order derivatives by corresponding fractional α-order derivatives of the currents in the coils. The equations can be written in the form

$$\frac{d^\alpha x_L}{dt^\alpha} = A_1 x_L + B_1 e, \qquad y = \bar{C}_1 x_L, \tag{23}$$

where the matrices A_1, B_1, $\bar{C}_1$ are given by (13b) and (13c).
In a similar way as for standard electrical circuits we can prove for the fractional electrical circuits the following theorems.

Theorem 8. The fractional electrical circuit shown in Fig. 1 is a positive one with asymptotically stable cyclic Metzler state matrix A_1 defined by (13b).

Theorem 9. The fractional electrical circuit shown in Fig. 2 is a positive one with asymptotically stable cyclic Metzler state matrix A_2 defined by (14c).

5. CONCLUDING REMARKS

The positive electrical circuits of integer and fractional orders with cyclic Metzler state matrices have been investigated. Some new results concerning positive electrical circuits with the chain structure and cyclic Metzler state matrices have been established (Theorems 4 and 5) and next extended to fractional electrical circuits (Theorems 8 and 9).The considerations can be extended to descriptor linear electrical circuits. An open problem is an extension of these considerations to fractional different orders linear electrical circuits and systems.

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