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Bending of a generalized circular sandwich plate under a concentrated force with consideration of an improved shear deformation theory

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The paper is devoted to an axisymmetric bending problem of a generalized circular sandwich plate with continuous variation of mechanical properties in the thickness direction of the core. The plate is clamped and carries a concentrated force in its center. The improved shear deformation theory of the normal straight line to the neutral surface is elaborated. The deformation of this normal straight line is graphically presented for the exemplary sandwich structures of the plate. Two differential equation of equilibrium of the plate are obtained based on the principle of stationary potential energy. This system of equations is analytically solved and the maximum deflection of the example plates are derived. Moreover, the deformation of the normal strain line and the maximum deflection of the plate are calculated numerically (FEM). Results of these calculations are compared.

Key words: circular sandwich plate, shear deformation theory, axisymmetric bending.

1. Introduction

The theory of sandwich structures, initiated in the mid of 20th century, nowadays is being perfected. Carrera [1] presents a thorough review of the literature involving the use of multilayered plates and shells in modeling using the Reissner Mixed Variational Theorem. Ventsel and Krauthammer [2] emphasize novel analytical and numerical methods for solving linear and nonlinear plate and shell dilemmas and new theories for the design and analysis of thin plate-shell structures. Vinson [3] provides a general introduction to the structural mechanics involved in the field of sandwich structures and a sufficient number of references. Auricchio and Sacco [4] propose a mixed variational formulations for the first-order shear deformation laminate theory that does not require the use of shear correction factors. CARRERA [5] gives a historical review of the theories that have been developed for the analysis of multilayered structures. REDDY [6] provides full coverage of the theories, analytical solutions, and modeling of laminated composite plate and shell structures. YANG and QIAO [7] develop a higher-order impact model to simulate the response of a soft-core sandwich beam subjected to a foreign object impact. This model provides accurate predictions of the generated stresses and impact process and can be used effectively in design analysis of anti-impact structures made of sandwich materials. ZENKOUR [8] uses the generalized shear deformation theory to study the static response of a simply supported functionally graded rectangular plate subjected to a transverse uniform load. The influences played by transversal shear deformation, plate aspect ratio, side-to-thickness ratio, and volume fraction distributions are considered. CARRERA and BRISCHETTO [9] assess a large variety of plate theories to evaluate the bending and vibration of sandwich structures. The accuracy of the plate theories is established with respect to the length-to-thickness-ratio geometrical parameters and to the face-to-core-stiffness-ratio mechanical parameters.

JASION *et al.* [10] study a global and local buckling–wrinkling of the face sheets of sandwich beams and sandwich circular plates. A shear effect is included in a developed mathematical model of displacements. MAGNUCKI *et al.* [11] present the mathematical model of a sandwich circular plate under pure bending. The results obtained by the analytical approach are compared with the ones given by the experimental tests and the finite element method. SENJANOVIĆ *et al.* [12] use the modified Mindlin theory with decomposed flexural and in-plane shear vibrations for vibration analysis of circular plates. THAI *et al.* [13] present a new inverse tangent shear deformation theory for the static, free vibration and buckling analysis of laminated composite and sandwich plates. The proposed formulation requires C^1-continuity. SARANGAN and SINGH [14] develop a new shear deformation theories to analyze the static, buckling and free vibration responses of laminated composite and sandwich plates using the Navier closed form solution technique. ABRATE and DI SCIUVA [15] present an overview of the field of equivalent single layer theories for beams, plates and shells. MEKSI *et al.* [16] introduce a new shear deformation plate theory to illustrate the bending, buckling and free vibration responses of functionally graded material sandwich plates. A new displacement field containing integrals is proposed. The Navier solution technique is adopted to derive analytical solutions for simply supported rectangular sandwich plates. NADERI BENI and BOTSHEKANAN DEHKORDI [17] extend the Carrera Unified Formulation in the polar coordinates for analyzing the sandwich circular plate with the functionally graded material core. The results are provided for different geometries, thickness ratios, boundary conditions as well as deflection, radial displacements, annular and radial stresses, and transverse stresses along the thickness. Zarga *et al.* [18] employ a simple quasi-3D shear deformation theory for thermo-mechanical bending analysis of functionally graded material sandwich plates. The effects of gradient index, geometrical parameters and thermal load on thermo-mechanical bending response of the FG sandwich plates are examined. Boussoula *et al.* [19] perform a thermomechanical flexural analysis of functionally graded material sandwich plates with P-FGM face sheets and E-FGM and symmetric S-FGM core by employing the nth-order shear deformation theory. A novel type of S-FGM sandwich plates, namely, both P-FGM face sheets and a symmetric S-FGM hard core are considered. The effects of volume fraction variation, geometrical parameters and thermal load on thermomechanical flexural behavior of the symmetric FGM sandwich plates are investigated.

Magnucki *et al.* [20] conduct the study of a simply supported beams with bisymmetrical cross-sections under a generalized load. Based on the Zhuravsky shear stress formula, the shear deformation theory of a planar beam cross-section is formulated. Magnucki *et al.* [21] study a simply supported beams under three-point bending with symmetrically varying mechanical properties in the depth direction. The individual shear deformation theory for beams of such features is proposed. Magnucki *et al.* [22] study analytically and numerically an axisymmetric bending problem of the circular plate with symmetrically thickness-wise varying mechanical properties. A nonlinear function of deformation of the straight line normal to the plate neutral surface is assumed. The shear effect is also considered. Karimi and Fallah [23] investigate a nonlinear behavior of functionally graded (FG) sandwich circular sector plates with simply supported radial edges under transverse loading using the first-order shear deformation theory with von Karman geometric nonlinearity. The effects of non-linearity, material constant, lay-up, and boundary conditions on bending of FG sandwich sector plates with FG core/homogenous face sheets and metallic core/FG face sheets are studied. Magnucka-Blandzi *et al.* [24] analyze a simply supported circular plate with symmetrically varying mechanical properties in the thickness direction. The main goal of the study is to develop a mathematical description of both the single-layer and three-layer structure using one formula for the axisymmetric bending and buckling problems with consideration of the shear effect. Two dimensionless functions closely related to the variability of mechanical properties were introduced and the nonlinear hypothesis of deformation of the straight line normal to the plate neutral surface is assumed. Magnucki and Magnucka-Blandzi [25] perform a study of generalization of the analytical model of sandwich structures. The continuous variation of mechanical properties in the thickness direction of the structure wall is proposed. The individual nonlinear theory of deformation of the straight line normal to the neutral surface is developed. Sadiq and Salau [26] carry out the analysis of the deflection response of a structural circular sandwich plate, simply supported or clamped at its boundary, under different loading configurations (uniformly distributed, concentrated, and linearly varying load). MAGNUCKI [27] presents the analytical formulation of the shear deformation theory of homogeneous, sandwich and symmetrically varying mechanical properties beams.

The aim of this paper is the generalization of the classical circular sandwich plate by the application of the improved shear deformation theory that describes accurately the bending behavior of the sandwich plate.

2. Analytical model of the circular sandwich plate

The subject of the studies is a clamped generalized circular sandwich plate of the radius R_1 and the total thickness h with a rigid central part of the radius R_0. The plate is subjected to the concentrated force F (Fig. 1).

Taking into account the paper [25], the variation of Young's modulus of the core is assumed to be continuous in its thickness direction (Fig. 2).

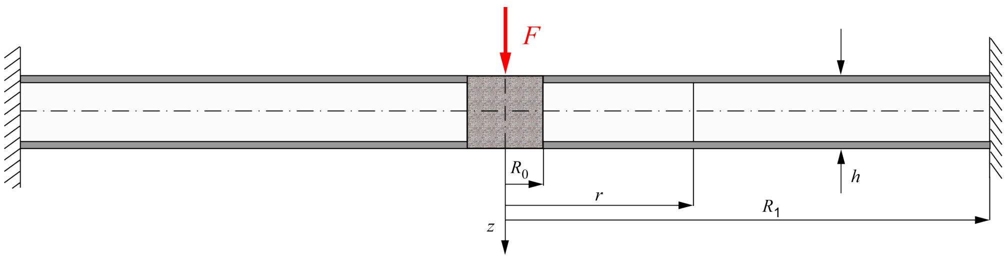


FIG. 1. Scheme of the circular sandwich plate with the load.

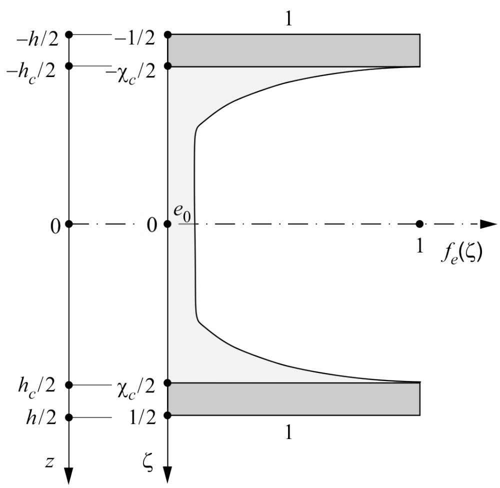


FIG. 2. Scheme of Young's modulus variability in the thickness direction of the core.

The value of Young's modulus E_f of the faces is constant, while in the core is variable

$$E_c(\zeta) = E_f f_e(\zeta), \tag{2.1}$$

where the dimensionless function of elasticity modulus

$$f_e(\zeta) = e_0 + (1 - e_0)\left(\frac{2}{\chi_c}\zeta\right)^{k_e} \tag{2.2}$$

and $e_0 = E_0/E_f$ – dimensionless coefficient, $\chi_c = h_c/h$ – dimensionless thickness of the core, $\zeta = z/h$ – dimensionless coordinate, k_e – exponent – positive even number.

The faces of the plate are thin-walled and the shear effect in them is neglected. The deformation of the straight line normal to the neutral surface of the plate is shown in Fig. 3.

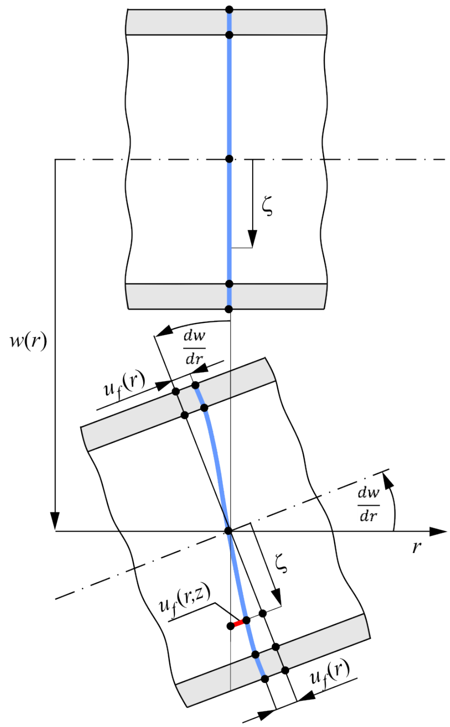


FIG. 3. The scheme of the deformation of the straight normal line.

The longitudinal displacements in accordance with Fig. 3 are as follows:

- the upper face $(-1/2 \le \zeta \le -\chi_c/2)$

$$u(r,\zeta) = -h\left[\zeta\frac{dw}{dr} + \psi(r)\right], \tag{2.3}$$

where: $w(r)$ – defection, $\psi(r) = u_f/h$ – dimensionless displacement function of the faces;

- the core $(-\chi_c/2 \le \zeta \le \chi_c/2)$

$$u(r,\zeta) = -h\left[\zeta\frac{dw}{dr} - f_d(\zeta)\psi(r)\right], \tag{2.4}$$

where $f_d(\zeta)$ – dimensionless function of deformation of the straight line of the core;

- the lower face $(\chi_c/2 \le \zeta \le 1/2)$

$$u(r,\zeta) = -h\left[\zeta\frac{dw}{dr} - \psi(r)\right]. \tag{2.5}$$

Therefore, the strains:

- the upper face $(-1/2 \le \zeta \le -\chi_c/2)$

$$\varepsilon_r^{(uf)}(r,\zeta) = \frac{\partial u}{\partial r} = -h\left[\zeta\frac{d^2w}{dr^2} + \frac{d\psi}{dr}\right], \tag{2.6}$$

$$\varepsilon_\varphi^{(uf)}(r,\zeta) = \frac{u(r,\zeta)}{r} = -h\left[\zeta\frac{dw}{rdr} + \frac{\psi(r)}{r}\right], \tag{2.7}$$

$$\gamma_{rz}^{(uf)}(r,\zeta) = \frac{dw}{dr} + \frac{\partial u}{\partial z} = 0; \tag{2.8}$$

- the core $(-\chi_c/2 \le \zeta \le \chi_c/2)$

$$\varepsilon_r^{(c)}(r,\zeta) = \frac{\partial u}{\partial r} = -h\left[\zeta\frac{d^2w}{dr^2} - f_d(\zeta)\frac{d\psi}{dr}\right], \tag{2.9}$$

$$\varepsilon_\varphi^{(c)}(r,\zeta) = \frac{u(r,\zeta)}{r} = -h\left[\zeta\frac{dw}{rdr} - f_d(\zeta)\frac{\psi(r)}{r}\right], \tag{2.10}$$

$$\gamma_{rz}^{(c)}(r,\zeta) = \frac{dw}{dr} + \frac{\partial u}{\partial z} = \frac{df_d}{d\zeta}\psi(r); \tag{2.11}$$

- the lower face $(\chi_c/2 \le \zeta \le 1/2)$

$$\varepsilon_r^{(lf)}(r,\zeta) = \frac{\partial u}{\partial r} = -h\left[\zeta\frac{d^2w}{dr^2} - \frac{d\psi}{dr}\right], \tag{2.12}$$

$$\varepsilon_\varphi^{(lf)}(r,\zeta) = \frac{u(r,\zeta)}{r} = -h\left[\zeta\frac{dw}{rdr} - \frac{\psi(r)}{r}\right], \tag{2.13}$$

$$\gamma_{rz}^{(lf)}(r,\zeta) = \frac{dw}{dr} + \frac{\partial u}{\partial z} = 0. \tag{2.14}$$

Consequently, the stresses:

- the upper face $(-1/2 \le \zeta \le -\chi_c/2)$

$$\sigma_r^{(uf)}(r,\zeta) = \frac{E_f}{1-\nu^2}\Big[\varepsilon_r^{(uf)}(r,\zeta) + \nu\varepsilon_\varphi^{(uf)}(r,\zeta)\Big], \tag{2.15}$$

$$\sigma_\varphi^{(uf)}(r,\zeta) = \frac{E_f}{1-\nu^2}\Big[\varepsilon_\varphi^{(uf)}(r,\zeta) + \nu\varepsilon_r^{(uf)}(r,\zeta)\Big], \tag{2.16}$$

$$\tau_{rz}^{(uf)}(r,\zeta) = 0; \tag{2.17}$$

- the core $(-\chi_c/2 \le \zeta \le \chi_c/2)$

$$\sigma_r^{(c)}(r,\zeta) = \frac{E_f}{1-\nu^2}\Big[\varepsilon_r^{(c)}(r,\zeta) + \nu\varepsilon_\varphi^{(c)}(r,\zeta)\Big] f_e(\zeta), \tag{2.18}$$

$$\sigma_\varphi^{(c)}(r,\zeta) = \frac{E_f}{1-\nu^2}\Big[\varepsilon_\varphi^{(c)}(r,\zeta) + \nu\varepsilon_r^{(c)}(r,\zeta)\Big] f_e(\zeta), \tag{2.19}$$

$$\tau_{rz}^{(c)}(r,\zeta) = \frac{E_f}{2(1+\nu)}\gamma_{rz}^{(c)}(r,\zeta) f_e(\zeta); \tag{2.20}$$

- the lower face $(\chi_c/2 \le \zeta \le 1/2)$

$$\sigma_r^{(lf)}(r,\zeta) = \frac{E_f}{1-\nu^2}\Big[\varepsilon_r^{(lf)}(r,\zeta) + \nu\varepsilon_\varphi^{(lf)}(r,\zeta)\Big], \tag{2.21}$$

$$\sigma_\varphi^{(lf)}(r,\zeta) = \frac{E_f}{1-\nu^2}\Big[\varepsilon_\varphi^{(lf)}(r,\zeta) + \nu\varepsilon_r^{(lf)}(r,\zeta)\Big], \tag{2.22}$$

$$\tau_{rz}^{(lf)}(r,\zeta) = 0. \tag{2.23}$$

Taking into account the procedure of formulation of the deformation function applied in the paper [20, 27], the dimensionless function of deformation of the straight line of the core is elaborated in the following form:

$$f_d(\zeta) = \frac{1}{C_0}\int \frac{1-\chi_c^2 + (\chi_c^2 - 4\zeta^2)e_0 + 8(1-e_0)J_c(\zeta)}{f_e(\zeta)}\,d\zeta, \tag{2.24}$$

where:

$$C_0 = \int_0^{\chi_c/2} \frac{1-\chi_c^2 + (\chi_c^2 - 4\zeta^2)e_0 + 8(1-e_0)J_c(\zeta)}{f_e(\zeta)}\,d\zeta \quad \text{dimensionless coefficient,}$$

$$J_c(\zeta) = \frac{\chi_c^2}{4(k_e+2)}\left[1 - \left(\frac{2}{\chi_c}\zeta\right)^{k_e+2}\right] \quad \text{dimensionless function.}$$

The elastic strain energy of the plate

$$U_\varepsilon = \pi h \int_{R_0}^{R_1} \Big\{\Phi_\varepsilon^{(uf)}(r) + \Phi_\varepsilon^{(c)}(r) + \Phi_\varepsilon^{(lf)}(r)\Big\} r\,dr, \tag{2.25}$$

where:

$$\Phi_\varepsilon^{(uf)}(r) = \int\limits_{-1/2}^{-\chi_c/2} \Big[\sigma_r^{(uf)}(r,\zeta)\varepsilon_r^{(uf)}(r,\zeta) + \sigma_\varphi^{(uf)}(r,\zeta)\varepsilon_\varphi^{(uf)}(r,\zeta)\Big]\,d\zeta,$$

$$\Phi_\varepsilon^{(c)}(r) = \int\limits_{-\chi_c/2}^{\chi_c/2} \Big[\sigma_r^{(c)}(r,\zeta)\varepsilon_r^{(c)}(r,\zeta) + \sigma_\varphi^{(c)}(r,\zeta)\varepsilon_\varphi^{(c)}(r,\zeta) + \tau_{rz}^{(c)}(r,\zeta)\gamma_{rz}^{(c)}(r,\zeta)\Big]d\zeta,$$

$$\Phi_\varepsilon^{(lf)}(r) = \int\limits_{\chi_c/2}^{1/2} \Big[\sigma_r^{(lf)}(r,\zeta)\varepsilon_r^{(lf)}(r,\zeta) + \sigma_\varphi^{(lf)}(r,\zeta)\varepsilon_\varphi^{(lf)}(r,\zeta)\Big]d\zeta.$$

The expression (2.25) after integration in the thickness direction of the plate and after a simply transformation is as follows:

$$U_\varepsilon = \pi\frac{E_f h^3}{1-\nu^2} \times \int\limits_{R_0}^{R_1} \Big\{C_{ww} f_{ww}(r) - 2C_{w\psi} f_{w\psi}(r) + C_{\psi\psi} f_{\psi\psi}(r) + \frac{1}{2}(1-\nu)\frac{J_3}{h^2}\psi^2(r)\Big\} r\,dr, \tag{2.26}$$

where

$$f_{ww}(r) = \left(\frac{d^2w}{dr^2}\right)^2 + 2\nu\frac{d^2w}{dr^2}\,\frac{dw}{r\,dr} + \left(\frac{dw}{r\,dr}\right)^2,$$

$$f_{\psi\psi}(r) = \left(\frac{d\psi}{dr}\right)^2 + 2\nu\frac{d\psi}{dr}\,\frac{\psi(r)}{r} + \left(\frac{\psi(r)}{r}\right)^2,$$

$$f_{w\psi}(r) = \frac{d^2w}{dr^2}\frac{d\psi}{dr} + \nu\left(\frac{d^2w}{dr^2}\,\frac{\psi(r)}{r} + \frac{dw}{r\,dr}\frac{d\psi}{dr}\right) + \frac{dw}{r\,dr}\,\frac{\psi(r)}{r},$$

$$C_{ww} = \frac{1}{12}\left[1 - (1-e_0)\frac{k_e}{k_e+3}\chi_c^3\right],$$

$$C_{w\psi} = \frac{1}{4}(1-\chi_c^2) + J_1, \qquad C_{\psi\psi} = 1 - \chi_c + J_2,$$

$$J_1 = \int\limits_{-\chi_c/2}^{\chi_c/2} f_d(\zeta) f_e(\zeta)\zeta\,d\zeta, \qquad J_2 = \int\limits_{-\chi_c/2}^{\chi_c/2} f_d^2(\zeta) f_e(\zeta)\,d\zeta,$$

$$J_3 = \frac{1}{C_0^2}\int\limits_{-\chi_c/2}^{\chi_c/2} \frac{[1-\chi_c^2 + (\chi_c^2 - 4\zeta^2)e_0 + 8(1-e_0)J_c(\zeta)]^2}{f_e(\zeta)}d\zeta.$$

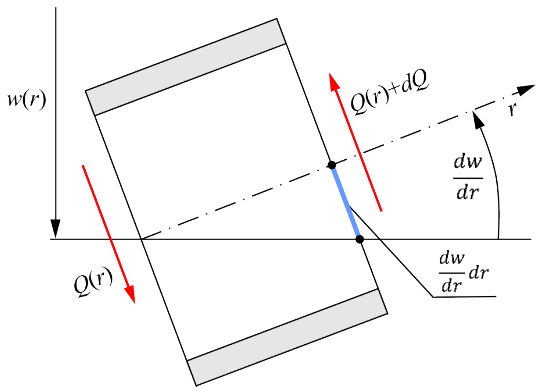


FIG. 4. Scheme of the shear force in the circular plate.

The work of the load formulated according to Fig. 4 is as follows:

$$W = -2\pi \int_{R_0}^{R_1} rQ(r)\,\frac{dw}{dr}\,dr \tag{2.27}$$

and the first variation of the work

$$\delta W = 2\pi \int_{R_0}^{R_1} \frac{d}{dr}[rQ(r)]\delta w\,dr, \tag{2.28}$$

where $Q(r) = \frac{1}{r}\frac{F}{2\pi}$ – the shear force.

Consequently, based on the principle of stationary total potential energy $\delta(U_\varepsilon - W) = 0$, the system of two differential equations of equilibrium of this circular plate is obtained in the following form:

$$\frac{d}{dr}\left\{\frac{1}{r}\frac{d}{dr}\left[r\left(C_{ww}\frac{dw}{dr} - C_{w\psi}\psi(r)\right)\right]\right\} = \frac{1-\nu^2}{2\pi}\frac{1}{r}\frac{F}{E_f h^3}, \tag{2.29}$$

$$\frac{d}{dr}\left\{\frac{1}{r}\frac{d}{dr}\left[r\left(C_{w\psi}\frac{dw}{dr} - C_{\psi\psi}\psi(r)\right)\right]\right\} + \frac{1-\nu}{2}\frac{J_3}{h^2}\psi(r) = 0. \tag{2.30}$$

3. Analytical study – bending of the circular sandwich plate

Integrating Eq. (2.29) one obtains

$$C_{ww}\frac{dw}{dr} - C_{w\psi}\psi(r) = \frac{1-\nu^2}{4\pi}\left(\frac{2}{r}C_2 + rC_1 - \frac{1}{2}r + r\ln\frac{r}{R_1}\right)\frac{F}{E_f h^3}, \tag{3.1}$$

where C_1 and C_2 are integration constants.

The boundary conditions are as follows:

- for $r = R_0$, $\frac{dw}{dr}\big|_{R_0} = 0$ and $\psi(R_0) = 0$, from which $2C_2 + R_0^2 C_1 = \frac{1}{2}R_0^2 - R_0^2 \ln\frac{R_0}{R_1}$,
- for $r = R_1$, $\frac{dw}{dr}\big|_{R_1} = 0$ and $\psi(R_1) = 0$, from which $2C_2 + R_1^2 C_1 = \frac{1}{2}R_1^2$,

therefore $C_1 = \frac{1}{2} + \frac{R_0^2}{R_1^2 - R_0^2}\ln\frac{R_0}{R_1}$, and $C_2 = -\frac{R_0^2 R_1^2}{2(R_1^2 - R_0^2)}\ln\frac{R_0}{R_1}$.

Equation (3.1) with consideration of the above expressions for the integration constants is in the following form:

$$C_{ww}\frac{dw}{R_1 d\xi} = C_{w\psi}\psi(\xi) + \frac{1-\nu^2}{4\pi}\left[\xi\ln\xi - \left(\frac{1}{\xi} - \xi\right)C_R\right]\frac{FR_1}{E_f h^3}, \tag{3.2}$$

where $\xi = \frac{r}{R_1}$ – dimensionless coordinate, $C_R = \frac{\xi_0^2}{1-\xi_0^2}\ln\xi_0$ – coefficient, $\xi_0 = \frac{R_0}{R_1}$.

Equations (2.28) and (3.2) are approximately solved with the use of the assumed following functions:

$$\psi(\xi) = \left[\xi\ln\xi - \left(\frac{1}{\xi} - \xi\right)C_R\right]\psi_0, \tag{3.3}$$

$$w(\xi) = \frac{1}{4}\left[(1 - 2C_R)(1 - \xi^2) + 2(\xi^2 - 2C_R)\ln\xi\right]w_0, \tag{3.4}$$

where ψ_0, w_0 – coefficients.

The function (3.4) satisfies also the boundary condition $w(1) = 0$.

Equation (2.30) with consideration of the functions (3.3) and (3.4) is as follows:

$$\frac{2}{\xi}\left(C_{w\psi}\frac{w_0}{R_1} - C_{\psi\psi}\psi_0\right) + \frac{1-\nu}{2}J_3\left(\frac{R_1}{h}\right)^2\left[\xi\ln\xi - \left(\frac{1}{\xi} - \xi\right)C_R\right]\psi_0 = 0. \tag{3.5}$$

An application of the Galerkin method to this equation gives the following:

$$\int_{\xi_0}^{1}\Re(\xi)\left[\xi\ln\xi - \left(\frac{1}{\xi} - \xi\right)C_R\right]\xi\,d\xi = 0, \tag{3.6}$$

where $\Re(\xi)$ – the left part of Eq. (3.5).

After integration and after simply transformation, one obtains

$$\psi_0 = \frac{C_{w\psi}}{C_{\psi\psi} - \frac{1-\nu}{4}J_3\left(\frac{R_1}{h}\right)^2\frac{J_{G2}}{J_{G1}}}\,\frac{w_0}{R_1}\quad, \tag{3.7}$$

where

$$J_{G1} = \int_{\xi_0}^{1}\left[\xi\ln\xi - \left(\frac{1}{\xi} - \xi\right)C_R\right]d\xi, \qquad J_{G2} = \int_{\xi_0}^{1}\left[\xi\ln\xi - \left(\frac{1}{\xi} - \xi\right)C_R\right]^2\xi\,d\xi.$$

Substituting the functions (3.3) and (3.4) into Eq. (3.2), after simply transformation with consideration the expression (3.7), one obtains:

$$w_0 = \frac{1-\nu^2}{4\pi}\,\frac{1}{C_{ww}-C_s}\,\frac{FR_1^2}{E_f h^3}, \tag{3.8}$$

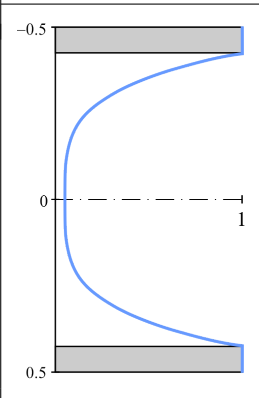
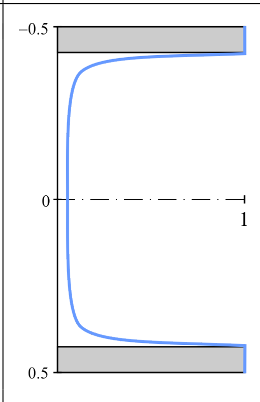
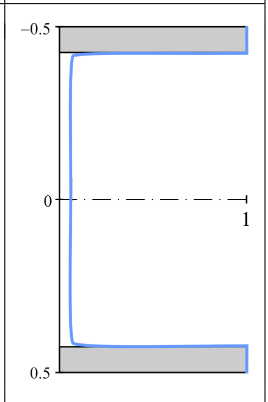
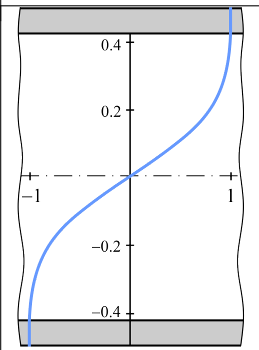
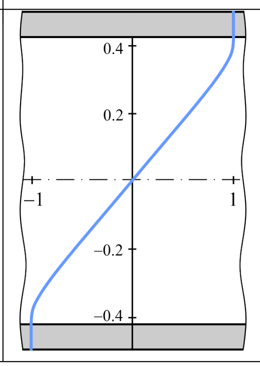
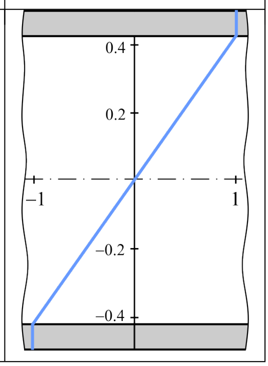
where

$$C_s = \frac{C_{w\psi}^2}{C_{\psi\psi} - \frac{1-\nu}{4} J_3 \left(\frac{R_1}{h}\right)^2 \frac{J_{G2}}{J_{G1}}}.$$

Thus, taking into account the function (3.4), the maximum deflection of the plate is in the following form:

$$w_{\max} = w(\xi_0) = \bar{w}_{\max}\frac{F}{E_f h}, \tag{3.9}$$

Table 1. The results of the analytical study of the exemplary cases of the circular plates.

k_e	4	20	100
C_{ww}	0.0555515	0.0410566	0.0361312
C_s	0.00266677	0.00211769	0.00177185
$\bar{w}_{\max}$	190.40	270.82	306.91
$f_e(\zeta)$			
$f_d(\zeta)$			

where the dimensionless maximum deflection

$$\bar{w}_{\max} = w(\xi_0) = \frac{1-\nu^2}{16\pi} \frac{(1-2C_R)(1-\xi_0^2) + 2(\xi_0^2 - 2C_R)\ln\xi_0}{C_{ww} - C_s} \left(\frac{R_1}{h}\right)^2. \tag{3.10}$$

The detailed calculations are carried out for the exemplary cases of the circular sandwich plates. The data of the three example plates are as follows: radiuses $R_0 = 10\,\mathrm{mm}$, $R_1 = 500\,\mathrm{mm}$, thicknesses $h = 20\,\mathrm{mm}$, $h_c = 17\,\mathrm{mm}$, $h_f = 1.5\,\mathrm{mm}$, material constants $E_f = 72000\,\mathrm{MPa}$, $E_c = 3600\,\mathrm{MPa}$, $\nu_f = \nu_c = \nu = 0.3$. The results of the calculations are specified in Table 1.

4. Numerical FEM study – bending of the circular sandwich plate

Numerical computations of the exemplary cases of the circular sandwich plates are carried out with the ABAQUS finite-element code. The FE model is developed using the shell element S4R for a face sheet, and the solid elements C3D8 for a core. The shell elements are placed at the mid-surface of the face sheet. The interaction between the core and the face sheets is provided with the use of the node-to-node tie constraint. The plate is located in a cylindrical coordinate system (r, φ, z) with the downward directed z axis. The reference point is set in the origin of the cylindrical coordinate system and is connected with face edges and a core surface of the inner side of the plate with the use of a coupling constraint (Fig. 5).

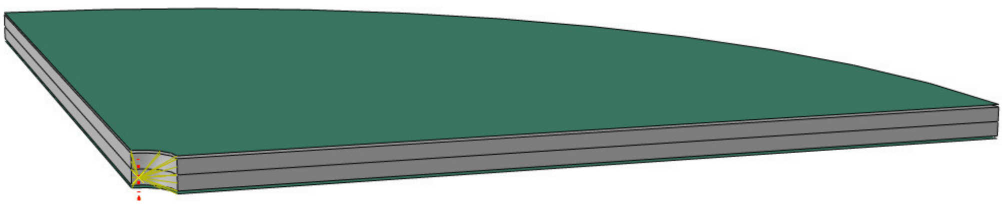


FIG. 5. The finite element model.

The reference point is allowed to move freely only in the z-direction. The remaining DOFs at this point are fixed. The plate is clamped around the outer face edges and the outer core surface. Due to symmetry of the plate and the load only a quarter of plate is considered. The symmetry boundary conditions are used for the region at the symmetry planes. The load is applied in the z-direction at a reference point. The vertical displacement is measured at the inner edge. The core is divided into 51 subsections in order to obtain a good agreement between approximated (discretized) and real Young's modulus. The value of the Young modulus is set in the middle of every subsection. A mesh sensitivity study was conducted with respect to the number of elements to ensure that the mesh was fine enough to give reliable results. The element size used is 4 mm for both a face sheet and a core. The results from the numerical (FEM) analysis were used to validate the analytical model. The comparison between obtained results is collected in Table 2.

Table 2. The comparison between the analytical (AN) and the numerical (FEM) studies of the exemplary cases of the circular plates.

k_e	4	20	100
$\bar{w}_{\max}^{(AN)}$	199.40	270.82	306.91
$\bar{w}_{\max}^{(FEM)}$	205.35	280.48	318.61
$\left\lvert \dfrac{\bar{w}_{\max}^{(FEM)} - \bar{w}_{\max}^{(AN)}}{\bar{w}_{\max}^{(FEM)}} \right\rvert$	2.9%	3.4%	3.7%

The assumed analytical model of the circular plate describes the whole family of plates with or without the rigid part in the center from the homogeneous ones to the classical sandwich ones. The particular case of the considered cases in Table 1 is the classical sandwich plate obtained for constant Young's modulus in the core when $k_e \to \infty$. For this case the shear coefficient $C_s = 0.00167$ and $\bar{w}_{\max} = 319.08$.

Another interesting case is the homogeneous plate without a rigid part in the center. If the shear effect is not included ($C_s = 0$) then the following dimensionless maximum deflection is obtained

$$\bar{w}_{\max} = \frac{1-\nu^2}{16\pi} \frac{1}{C_{ww}} \left(\frac{R_1}{h}\right)^2,$$

where $C_{ww} = \frac{1}{12}$ This formula is consistent with the well-known one in [2].

5. Conclusions

The presented studies of the circular sandwich plates allow to formulate the following conclusions:

- The proposed dimensionless function of deformation of the straight line of the core (2.24) describes well the bending behavior of the analyzed structures.
- Young's modulus variability enables to consider the plates from the homogeneous to the sandwich ones. For the very high value of the k_e parameter the plate can be treated as the classical sandwich plate because the Young modulus is almost constant along the core depth. Hence the limit value of deflection of the considered plates is obtained by the classical sandwich plate (Fig. 6).

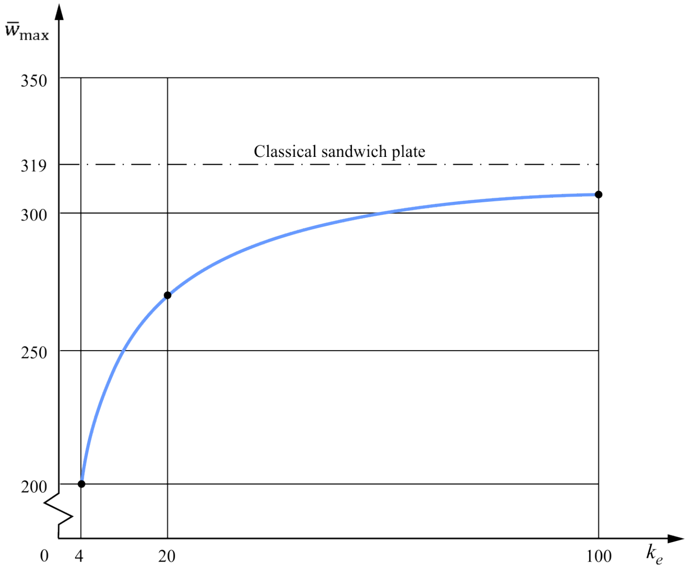


FIG. 6. Dimensionless maximum deflection of the exemplary plates.

- A good agreement between the analytical and the numerical (FEM) results is observed. The maximal percentage difference is less than 3.7%.

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