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On the fully discrete approximations of the MGT two-temperatures thermoelastic problem

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We consider a one-dimensional two-temperatures thermoelastic model. The corresponding variational problem leads to a coupled system which is written in terms of the mechanical velocity, the temperature speed and the inductive temperature. An existence and uniqueness result is recalled. Then, fully discrete approximations are introduced by using the finite element method and the implicit Euler scheme. A priori error estimates are proved and the linear convergence of the approximations is deduced under suitable additional regularity conditions. Finally, some numerical simulations are shown to demonstrate the accuracy of the proposed algorithm and the behavior of the discrete energy.

Key words: two-temperatures thermoelasticity, finite elements, a priori error estimates, numerical simulations.

1. Introduction

The Moore–Gibson–Thompson equation has deserved much attention in the last ten years (see, for instance, [1–7]) among the mathematical community. It proposes a third-order in time equation of the form:

$$c(\tau\dddot{u} + \ddot{u}) = \kappa^* Au + \kappa A\dot{u},$$

where c, τ, κ, κ^* and $\overline{\kappa} = \kappa - \tau\kappa^*$ are positive constants, and A is a positive definite operator. The motivation for this equation came from the mechanics of fluids and, in this situation, the parameters a, τ, κ and κ^* are determined by the properties of the material. Recently, this equation has also been obtained from the heat conduction theory. In fact, as the type III Green–Naghdi heat conduction involves an equation where the thermal waves propagate instantaneously (therefore, the causality principle is violated), it is natural to introduce a relaxation parameter (as it was proposed by Maxwell and Cattaneo for the Fourier law) that transforms it into the so-called MGT equation. Hence, a new thermoelastic theory was proposed [8]. This theory has received much attention in the last two years (see [9–22] among others). Some other recent theories for the heat conduction have been proposed [23, 24]; however, in certain situations they result in ill-posed problems in the sense of Hadamard [25]. In order to avoid this drawback, it was suggested to combine the theories with delay with the two-temperatures theory [26–29] as it was done in [30, 31]. In this case, the constitutive equation for the heat flux takes the form:

$$q_i(\boldsymbol{x}, t+\tau_1) = -\kappa^* \beta_{,i}(\boldsymbol{x}, t+\tau_3) - \kappa T_{,i}(\boldsymbol{x}, t+\tau_2),$$

where $\alpha = \beta - a^2\Delta\beta$, $\theta = T - a^2\Delta T$, being α the thermal displacement, θ the temperature, β the inductive thermal displacement and T the inductive temperature. When we juxtapose this equation with the classical energy equation one obtains a well posed problem. It is worth noting that this theory has not been deeply studied, since it seems very complex. However, some attention has been devoted to the problems we can obtain by taking finite Taylor developments. For instance, when $\tau_2 = \tau_3 < \tau_1$ and considering the approximations:

$$q_i(\boldsymbol{x}, t+\tau) \approx q_i(\boldsymbol{x}, t) + \tau \dot{q}(\boldsymbol{x}, t), \quad \tau = \tau_1 - \tau_2,$$

one obtains the MGT-type equation:

$$c(\tau \dddot{\alpha} + \ddot{\alpha}) = \kappa^* \Delta\beta + \kappa \Delta T.$$

It is worth saying that, in this context, c is the thermal capacity, τ is the relaxation parameter, κ is related with the thermal conductivity and κ^* with the rate thermal conductivity.

Therefore, we can consider the thermoelastic theory associated to this equation [32].

In this paper, we study the one-dimensional version of the above problem. We recall that the general case can be defined by the constitutive equations:

$$\begin{gathered} t_{ji} = 2\mu e_{ij} + \lambda e_{rr}\delta_{ij} + \beta^*\theta\delta_{ij}, \quad \eta = -\beta^* e_{ii} + c\theta, \\ \tau\dot{q}_i + q_i = -(\kappa^*\beta_{,i} + \kappa T_{,i}), \end{gathered} \tag{1.1}$$

where $e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$, and the evolution equations:

$$\rho\ddot{u}_i = t_{ji,j}, \qquad T_0^*\dot{\eta} = -q_{i,i}. \tag{1.2}$$

In Eqs. (1.1) and (1.2), t_{ij} is the stress tensor, e_{ij} is the strain tensor, η is the entropy, λ and μ are the Lamé constants, β^* is the coupling term, c is the thermal capacity and u_i is the displacement vector, ρ is the mass density and T_0^* is the reference temperature.

Let us consider the new variable $\hat{f} = f + \tau \dot{f}$. If we substitute evolution equations (1.2) into constitutive equations (1.3), we obtain the system:

$$\begin{aligned} \rho \ddot{\hat{u}} &= \mu \hat{u}_{xx} + (\lambda + \mu)\hat{u}_{j,ji} + \beta^*_{ij}(\dot{\alpha}_{,j} + \tau \ddot{\alpha}_{,j}), \\ c\ddot{\alpha} + \tau \dddot{\alpha} &= \beta^*_{ij}\dot{\hat{u}}_{i,j} + \kappa \dot{\beta}_{,jj} + \kappa^* \beta_{,jj}, \\ \alpha &= \beta - a\Delta\beta, \end{aligned} \tag{1.3}$$

where we have assumed $T_0^* = 1$ to reduce the calculations. From now on, we omit the hat in the system of equations to simplify the notation.

It is worth noting that the Moore–Gibson–Thompson thermoelasticity has received much attention in the last two years; however, the system we study in this paper corresponds to the Moore–Gibson–Thompson with two temperatures and, as far as we know, there are not contributions concerning this system (neither mathematical or mechanical).

As the proposed theory by this system of equations is very new (it was formulated two years ago), it is needed to clarify its applicability. We believe that mathematical and physical studies are necessary in this sense. Hence, this paper is addressed in this direction. We want to see if the proposed mathematical model can be accepted from a thermomechanical point of view. We develop a numerical study, including an estimation of the a priori numerical error, and we obtain the approximate discrete solutions for the problem. As far as our results seem to agree with the empirical ones, we show an aspect from which this new theory can be considered.

The paper is divided into four sections. In the next section, we describe the model with the required assumptions for its study. We also recall an existence and uniqueness result [32]. Then, in Section 3, the numerical approximation of this problem is introduced, by using the finite element method and the implicit Euler scheme to approximate the spatial variable and to discretize the time derivatives, respectively. A main a priori error estimates result is proved. Finally, some numerical simulations are described in Section 4 to demonstrate the accuracy of the approximations and the behavior of the discrete energy.

2. Thermomechanical model

Let u, α and β be the displacement field, the thermal displacement and the inductive thermal displacement, respectively. Moreover, let us denote by θ and T the temperature and the inductive temperature obtained as $\theta = \dot{\alpha}$ and $T = \dot{\beta}$.

From the new system of equations (1.3) the thermomechanical problem of a one-dimensional thermal rod with two temperatures is written as follows (see [32]).

Problem P. *Find the displacement field* $u : [0,\ell] \times [0,T_f] \to \mathbb{R}$, *the thermal displacement* $\alpha : [0,\ell] \times [0,T_f] \to \mathbb{R}$ *and the inductive thermal displacement* $\beta : [0,\ell] \times [0,T_f] \to \mathbb{R}$ *such that*

$$\left.\begin{aligned} &\rho\ddot{u} = \mu u_{xx} + \beta^*(\dot{\alpha}_x + \tau\ddot{\alpha}_x),\\ &c\ddot{\alpha} + c\tau\dddot{\alpha} = \beta^*\dot{u}_x + \kappa T_{xx} + \kappa^*\beta_{xx},\\ &\alpha = \beta - a\beta_{xx}, \end{aligned}\right\} \quad \text{in } (0,\ell)\times(0,T_f), \tag{2.1}$$

$$\left.\begin{aligned} &u(x,0) = u_0(x), \quad \dot{u}(x,0) = v_0(x),\\ &\beta(x,0) = \beta_0(x), \quad \dot{\beta}(x,0) = T_0(x),\\ &\ddot{\beta}(x,0) = \vartheta_0(x), \end{aligned}\right\} \quad \text{for a.e. } x \in (0,\ell), \tag{2.2}$$

$$\left.\begin{aligned} u(0,t) = \beta(0,t) = 0,\\ u(\ell,t) = \beta(\ell,t) = 0, \end{aligned}\right\} \quad \text{for a.e. } t \in (0,T_f). \tag{2.3}$$

We note that, in Problem P, u_0, v_0, β_0, T_0 and ϑ_0 are initial conditions for the variables. From the above equations, we may define some "artificial initial conditions" for variable α as

$$\begin{aligned} \alpha(x,0) &= \alpha_0(x) = \beta_0(x) - a\beta_{0xx}(x),\\ \dot{\alpha}(x,0) &= \theta_0(x) = T_0(x) - aT_{0xx}(x),\\ \ddot{\alpha}(x,0) &= \xi_0(x) = \vartheta_0(x) - a\vartheta_{0xx}(x) \end{aligned}$$

for a.e. $x \in (0,\ell)$. We note that functions α_0, θ_0 and ξ_0 are not really initial conditions because they are obtained from the real ones β_0, T_0 and ϑ_0.

According to [32] we make the following assumptions on the constitutive coefficients:

$$\rho > 0, \quad c > 0, \quad \mu > 0, \quad \tau > 0, \quad \kappa^* > 0, \quad \kappa > \tau\kappa^*, \quad a > 0. \tag{2.4}$$

In order to obtain the variational formulation of the above thermomechanical problem, let us denote $Y = L^2(0,\ell)$, $V = H_0^1(0,\ell)$ and $E = H_0^2(0,\ell)$. Moreover, let $(\cdot,\cdot)$ and $\|\cdot\|$ be the inner product and the norm defined on $L^2(0,\ell)$, respectively.

Integrating by parts equations (2.1) and using initial conditions (2.2) and boundary conditions (2.3), we obtain the following weak formulation written using the velocity $v = \dot{u}$, the temperature $\theta = \dot{\alpha}$, the temperature speed $\xi = \ddot{\alpha}$ and the inductive temperature $T = \dot{\beta}$.

Problem VP. *Find the mechanical velocity* $v : [0,T_f] \to V$, *the temperature speed* $\xi : [0,T_f] \to Y$ *and the inductive temperature* $T : [0,T_f] \to E$ *such that*

$v(0) = v_0$, $\xi(0) = \xi_0$, $T(0) = T_0$ *and, for a.e.* $t \in (0, T_f)$ *and for all* $w \in V$ *and* r, $l \in Y$,

$$\rho(\dot{v}(t), w) + \mu(u_x(t), w_x) = -\beta^*(\theta(t) + \tau\xi(t), w_x), \tag{2.5}$$

$$c(\xi(t) + \tau\dot{\xi}(t), r) = \beta^*(v_x(t), r) + \kappa(T_{xx}(t), r) + \kappa^*(\beta_{xx}(t), r), \tag{2.6}$$

$$(\theta(t), l) = (T(t) - aT_{xx}(t), l), \tag{2.7}$$

where the displacements, the temperature, the thermal displacements and the inductive thermal displacements are then recovered from the relations:

$$\begin{aligned} u(t) &= \int_0^t v(s)\,\mathrm{d}s + u_0, \qquad \beta(t) = \int_0^t T(s)\,\mathrm{d}s + \beta_0, \\ \theta(t) &= \int_0^t \xi(s)\,\mathrm{d}s + \theta_0, \qquad \alpha(t) = \int_0^t \theta(s)\,\mathrm{d}s + \alpha_0. \end{aligned} \tag{2.8}$$

The following result has been recently proved in [32], which states the existence of a unique solution to Problem VP and an energy decay property.

Theorem 1. *Assume that the coefficients satisfy conditions* (2.4). *If we denote by* $(u, v, \alpha, \theta, \xi, \beta, T)$ *the solution to Problem VP and we suppose that the initial conditions have the following regularity:*

$$u_0 \in H^1(0,\ell), \quad v_0 \in L^2(0,\ell), \quad \alpha_0,\, \theta_0,\, \xi_0 \in L^2(0,\ell), \quad \beta_0,\, T_0, \vartheta_0 \in H^2(0,\ell),$$

then Problem VP admits a unique solution such that:

$$u \in C^1([0,T_f]; H^1(0,\ell)), \quad \alpha \in C^2([0,T_f]; L^2(0,\ell)), \quad \beta \in C^1([0,T_f]; H^2(0,\ell)).$$

Moreover, this solution is polynomially stable of order $-1/2$.

3. Numerical analysis: fully discrete approximations and a priori error estimates

In this section, we study a fully discrete approximation of Problem VP. Firstly, we assume that the interval $[0,\ell]$ is divided into M subintervals $a_0 = 0 < a_1 < \ldots < a_M = \ell$ of length $h = a_{i+1} - a_i = \ell/M$ and so, to approximate the variational spaces E, V and Y, we define the finite dimensional spaces $V^h \subset V$, $E^h \subset E$ and $W^h \subset Y$ given by:

$$\begin{aligned} V^h = \{w^h \in C([0,\ell]);\ w^h_{[a_i,a_{i+1}]} \in P_1([a_i,a_{i+1}]), \quad i = 0,\ldots,M-1, \\ w^h(0) = w^h(\ell) = 0\}, \end{aligned} \tag{3.1}$$

$$\begin{aligned} E^h = \{r^h \in C^1([0,\ell]) \cap H^2(0,\ell);\ r^h_{[a_i,a_{i+1}]} \in P_3([a_i,a_{i+1}]), \\ i = 0,\ldots,M-1, \quad r^h_x(0) = r^h_x(\ell) = r^h(0) = r^h(\ell) = 0\}, \end{aligned} \tag{3.2}$$

$$W^h = \{l^h \in L^2([0,\ell]);\ l^h_{[a_i,a_{i+1}]} \in P_1([a_i,a_{i+1}]), \quad i = 0,\ldots,M-1\}, \tag{3.3}$$

where $P_r([a_i, a_{i+1}])$ represents the space of polynomials of degree less or equal to r in the subinterval $[a_i, a_{i+1}]$; i.e. the finite element space V^h is made of continuous and piecewise affine functions, E^h is made of C^1 and piecewise cubic functions, and W^h is composed of L^2 and piecewise affine functions. Here, $h > 0$ denotes the spatial discretization parameter. Furthermore, let the discrete initial conditions u_0^h, v_0^h, β_0^h, T_0^h and ϑ_0^h:

$$u_0^h = \mathcal{P}_1^h u_0, \quad v_0^h = \mathcal{P}_1^h v_0, \quad \beta_0^h = \mathcal{P}_2^h \beta_0, \quad T_0^h = \mathcal{P}_2^h T_0, \quad \vartheta_0^h = \mathcal{P}_2^h \vartheta_0, \tag{3.4}$$

where $\mathcal{P}_1^h$ and $\mathcal{P}_2^h$ are the classical finite element interpolation operators over V^h and E^h, respectively (see [33]). Moreover, we also consider artificial discrete initial conditions for function α, denoted by α_0^h, θ_0^h and ξ_0^h, given by:

$$\alpha_0^h = \mathcal{P}_3^h \alpha_0, \quad \theta_0^h = \mathcal{P}_3^h \theta_0, \quad \xi_0^h = \mathcal{P}_3^h \xi_0, \tag{3.5}$$

where P_3^h represents the projection operator over the finite element space W^h.

Secondly, we consider a uniform partition of the time interval $[0, T]$, denoted by $0 = t_0 < t_1 < \ldots < t_N = T$, with step size $k = T/N$ and nodes $t_n = nk$ for $n = 0, 1, \ldots, N$. For a continuous function $z(t)$ let $z_n = z(t_n)$ and, given a sequence $\{z_n\}_{n=0}^N$, we denote by $\delta z_n = (z_n - z_{n-1})/k$ its divided differences.

Therefore, using the implicit Euler scheme, the fully discrete approximations of Problem VP are the following.

PROBLEM VP^{hk}. *Find the discrete mechanical velocity $v^{hk} = \{v_n^{hk}\}_{n=0}^N \subset V^h$, the discrete temperature speed $\xi^{hk} = \{\xi_n^{hk}\}_{n=0}^N \subset W^h$ and the discrete inductive temperature $T^{hk} = \{T_n^{hk}\}_{n=0}^N \subset E^h$ such that $v_0^{hk} = v_0^h$, $\xi_0^{hk} = \xi_0^h$, $T_0^{hk} = T_0^h$ and, for all $w^h \in V^h$ and $r^h, l^h \in W^h$, and $n = 1, \ldots, N$,*

$$\rho(\delta v_n^{hk}, w^h) + \mu((u_n^{hk})_x, w_x^h) = -\beta^*(\theta_n^{hk} + \tau \xi_n^{hk}, w_x^h), \tag{3.6}$$

$$c(\xi_n^{hk} + \tau \delta \xi_n^{hk}, r^h) = \beta^*((v_n^{hk})_x, r^h) + \kappa((T_n^{hk})_{xx}, r^h) + \kappa^*((\beta_n^{hk})_{xx}, r^h), \tag{3.7}$$

$$(\theta_n^{hk}, l^h) = (T_n^{hk} - a(T_n^{hk})_{xx}, l^h), \tag{3.8}$$

where the discrete mechanical displacements u_n^{hk}, the discrete temperature θ_n^{hk}, the discrete thermal displacements α_n^{hk} and the discrete inductive thermal displacements β_n^{hk} are now recovered from the relations:

$$\begin{aligned} u_n^{hk} &= k \sum_{j=1}^n v_j^{hk} + u_0^h, \quad & \theta_n^{hk} &= k \sum_{j=1}^n \xi_j^{hk} + \theta_0^h, \\ \alpha_n^{hk} &= k \sum_{j=1}^n \theta_j^{hk} + \alpha_0^h, \quad & \beta_n^{hk} &= k \sum_{j=1}^n T_j^{hk} + \beta_0^h. \end{aligned} \tag{3.9}$$

It is straightforward to show that Problem VP^{hk} admits a unique solution applying the well-known Lax Milgram lemma and using assumptions (2.4).

The aim of this section is to obtain a priori error estimates on the numerical errors. In order to simplify the calculations, we assume that $\tau = 1$ in this section.

First, we derive the error estimates for the velocity field. Therefore, we subtract variational equation (2.5) at time $t=t_n$ for a test function $w=w^h\in V^h\subset V$ and discrete variational equation (3.6) to obtain, for all $w^h \in V^h$,

$$\rho(\dot v_n - \delta v_n^{hk}, w^h) + \mu((u_n - u_n^{hk})_x, w_x^h) = -\beta^*(\theta_n - \theta_n^{hk} + \xi_n - \xi_n^{hk}, w_x^h),$$

and so, we have, for all $w^h \in V^h$,

$$\begin{aligned}\rho(\dot v_n - \delta v_n^{hk}, v_n - v_n^{hk}) &+ \mu((u_n - u_n^{hk})_x, (v_n - v_n^{hk})_x)\\ &+ \beta^*(\theta_n - \theta_n^{hk} + \xi_n - \xi_n^{hk}, (v_n - v_n^{hk})_x)\\ &= \rho(\dot v_n - \delta v_n^{hk}, v_n - w^h) + \mu((u_n - u_n^{hk})_x, (v_n - w^h)_x)\\ &+ \beta^*(\theta_n - \theta_n^{hk} + \xi_n - \xi_n^{hk}, (v_n - w^h)_x).\end{aligned}$$

Taking into account that

$$\begin{aligned}(\dot v_n - \delta v_n^{hk}, v_n - v_n^{hk}) &= (\dot v_n - \delta v_n, v_n - v_n^{hk}) + (\delta v_n - \delta v_n^{hk}, v_n - v_n^{hk}),\\ (\delta v_n - \delta v_n^{hk}, v_n - v_n^{hk}) &\geq \frac{1}{2k}\big\{\|v_n - v_n^{hk}\|^2 - \|v_{n-1} - v_{n-1}^{hk}\|^2\big\},\\ ((u_n - u_n^{hk})_x, (v_n - v_n^{hk})_x) &\geq ((u_n - u_n^{hk})_x, (\dot u_n - \delta u_n)_x)\\ &\quad + \frac{1}{2k}\big\{\|(u_n - u_n^{hk})_x\|^2 - \|(u_{n-1} - u_{n-1}^{hk})_x\|^2\big\},\end{aligned}$$

where $\delta v_n = (v_n - v_{n-1})/k$, $\delta u_n = (u_n - u_{n-1})/k$, using several times Cauchy's inequality

$$ab \leq \epsilon a^2 + \frac{1}{4\epsilon}b^2, \qquad a, b, \epsilon \in \mathbb{R},\ \epsilon > 0, \tag{3.10}$$

it follows that, for all $w^h \in V^h$,

$$\begin{aligned}&\frac{\rho}{2k}\big\{\|v_n-v_n^{hk}\|^2-\|v_{n-1}-v_{n-1}^{hk}\|^2\big\}+\beta^*(\theta_n-\theta_n^{hk}+\xi_n-\xi_n^{hk}, (v_n-v_n^{hk})_x)\\ &\qquad+\frac{\mu}{2k}\big\{\|(u_n-u_n^{hk})_x\|^2-\|(u_{n-1}-u_{n-1}^{hk})_x\|^2\big\}\\ &\leq C\big(\|\dot v_n-\delta v_n\|^2+\|(\dot u_n-\delta u_n)_x\|^2+\|v_n-w^h\|^2_{H^1(0,\ell)}+\|(u_n-u_n^{hk})_x\|^2\\ &\qquad+\|\xi_n-\xi_n^{hk}\|^2+\|v_n-v_n^{hk}\|^2+\|\theta_n-\theta_n^{hk}\|^2+(\delta v_n-\delta v_n^{hk}, v_n-w^h)\big),\end{aligned} \tag{3.11}$$

where, from now on, C is a positive constant assumed to be independent of the discretization parameters h and k, and whose value may change from line to line.

Now, we obtain the error estimates on the temperature speed. Thus, subtracting variational equation (2.6) at time $t = t_n$ for a test function $w = w^h \in W^h \subset W$ and discrete variational equation (3.7), we have, for all $r^h \in W^h$,

$$
\begin{aligned}
c(\xi_n - \xi_n^{hk} + \dot{\xi}_n - \delta\xi_n^{hk}, \xi_n - \xi_n^{hk}) &- \beta^*((v_n - v_n^{hk})_x, \xi_n - \xi_n^{hk}) \\
&- \kappa((T_n - T_n^{hk})_{xx}, \xi_n - \xi_n^{hk}) - \kappa^*((\beta_n - \beta_n^{hk})_{xx}, \xi_n - \xi_n^{hk}) \\
&= c(\xi_n - \xi_n^{hk} + \dot{\xi}_n - \delta\xi_n^{hk}, \xi_n - r^h) - \beta^*((v_n - v_n^{hk})_x, \xi_n - r^h) \\
&- \kappa((T_n - T_n^{hk})_{xx}, \xi_n - r^h) - \kappa^*((\beta_n - \beta_n^{hk})_{xx}, \xi_n - r^h).
\end{aligned}
$$

Keeping in mind that

$$
\begin{aligned}
(\dot{\xi}_n - \delta\xi_n^{hk}, \xi_n - \xi_n^{hk}) &= (\dot{\xi}_n - \delta\xi_n, \xi_n - \xi_n^{hk}) + (\delta\xi_n - \delta\xi_n^{hk}, \xi_n - \xi_n^{hk}), \\
(\delta\xi_n - \delta\xi_n^{hk}, \xi_n - \xi_n^{hk}) &\geq \frac{1}{2k}\big\{\|\xi_n - \xi_n^{hk}\|^2 - \|\xi_{n-1} - \xi_{n-1}^{hk}\|^2\big\},
\end{aligned}
$$

using Cauchy's inequality (3.10) we find that, for all $r^h \in W^h$,

$$
\begin{aligned}
\text{(3.12)} \qquad & \frac{c}{2k}\big\{\|\xi_n - \xi_n^{hk}\|^2 - \|\xi_{n-1} - \xi_{n-1}^{hk}\|^2\big\} - \beta^*((v_n - v_n^{hk})_x, \xi_n - \xi_n^{hk}) \\
& \leq C\big(\|\dot{\xi}_n - \delta\xi_n\|^2 + \|\xi_n - r^h\|^2 + \|(T_n - T_n^{hk})_{xx}\|^2 + \|(\beta_n - \beta_n^{hk})_{xx}\|^2 \\
& \quad + ((\delta u_n - \delta u_n^{hk})_x, \xi_n - r^h) + \|\xi_n - \xi_n^{hk}\|^2 + \|(\dot{u}_n - \delta u_n)_x\|^2 \\
& \quad + (\delta\xi_n - \delta\xi_n^{hk}, \xi_n - r^h)\big),
\end{aligned}
$$

where $\delta\xi_n = (\xi_n - \xi_{n-1})/k$ and we have used conditions (2.4).

Combining estimates (3.11) and (3.12), it follows that

$$
\begin{aligned}
\frac{\rho}{2k}\big\{\|v_n - v_n^{hk}\|^2 - \|v_{n-1} - v_{n-1}^{hk}\|^2\big\} &+ \frac{\mu}{2k}\big\{\|(u_n - u_n^{hk})_x\|^2 - \|(u_{n-1} - u_{n-1}^{hk})_x\|^2\big\} \\
+ \frac{c}{2k}\big\{\|\xi_n - \xi_n^{hk}\|^2 &- \|\xi_{n-1} - \xi_{n-1}^{hk}\|^2\big\} + \beta^*(\theta_n - \theta_n^{hk}, (\delta u_n - \delta u_n^{hk})_x) \\
\leq C\big(\|\dot{v}_n - \delta v_n\|^2 &+ \|\dot{u}_n - \delta u_n\|_{H^1(0,\ell)}^2 + \|v_n - w^h\|_{H^1(0,\ell)}^2 + \|(u_n - u_n^{hk})_x\|^2 \\
+ \|\xi_n - \xi_n^{hk}\|^2 &+ \|v_n - v_n^{hk}\|^2 + (\delta v_n - \delta v_n^{hk}, v_n - w^h) + \|\dot{\xi}_n - \delta\xi_n\|^2 \\
+ \|\xi_n - r^h\|^2 &+ \|(T_n - T_n^{hk})_{xx}\|^2 + (\delta\xi_n - \delta\xi_n^{hk}, \xi_n - r^h) \\
+ ((\delta u_n - \delta u_n^{hk})_x, &\xi_n - r^h) + \|\theta_n - \theta_n^{hk}\|^2 + \|(\beta_n - \beta_n^{hk})_{xx}\|^2\big).
\end{aligned}
$$

Multiplying the above estimates by k and summing up to n, we have

$$
\begin{aligned}
\text{(3.13)} \qquad & \|v_n - v_n^{hk}\|^2 + \|(u_n - u_n^{hk})_x\|^2 + \|\xi_n - \xi_n^{hk}\|^2 \\
& \qquad + Ck\sum_{j=1}^{n}(\theta_j - \theta_j^{hk}, (\delta u_j - \delta u_j^{hk})_x)
\end{aligned}
$$

$$
\begin{aligned}
&\le Ck\sum_{j=1}^{n}\big(\|\dot v_j-\delta v_j\|^2+\|\dot u_j-\delta u_j\|_V^2+((\delta u_j-\delta u_j^{hk})_x,\xi_j-r_j^h)\\
&\quad+\|v_j-w_j^h\|_V^2+\|(u_j-u_j^{hk})_x\|^2+\|\xi_j-\xi_j^{hk}\|^2+\|v_j-v_j^{hk}\|^2+\|\dot\xi_j-\delta\xi_j\|^2\\
&\quad+(\delta v_j-\delta v_j^{hk},v_j-w_j^h)+\|\xi_j-r_j^h\|^2+\|(T_j-T_j^{hk})_{xx}\|^2\\
&\quad+(\delta\xi_j-\delta\xi_j^{hk},\xi_j-r_j^h)+\|\theta_j-\theta_j^{hk}\|^2+\|(\beta_j-\beta_j^{hk})_{xx}\|^2\big)\\
&\quad+C(\|v_0-v_0^h\|^2+\|(u_0-u_0^h)_x\|^2+\|\xi_0-\xi_0^h\|^2).
\end{aligned}
$$

Finally, we get the error estimates on the inductive temperature. Therefore, subtracting variational equation (2.7) at time $t=t_n$ for a test function $l=l^h\in W^h\subset Y$ and discrete variational equation (3.8) we obtain

$$(\theta_n-\theta_n^{hk},l^h)=(T_n-T_n^{hk}-a(T_n-T_n^{hk})_{xx},l^h),\qquad \forall l^h\in W^h,$$

and so, we have, for all $m^h\in E^h$ (because $m_{xx}^h\in W^h$),

$$
\begin{aligned}
&(\theta_n-\theta_n^{hk},(T_n-T_n^{hk})_{xx})-(T_n-T_n^{hk}-a(T_n-T_n^{hk})_{xx},(T_n-T_n^{hk})_{xx})\\
&\qquad=(\theta_n-\theta_n^{hk},(T_n-m^h)_{xx})-(T_n-T_n^{hk}-a(T_n-T_n^{hk})_{xx},(T_n-m^h)_{xx}).
\end{aligned}
$$

Taking into account that:

$$
\begin{aligned}
-(T_n-T_n^{hk},(T_n-T_n^{hk})_{xx})&=((T_n-T_n^{hk})_x,(T_n-T_n^{hk})_x),\\
-(T_n-T_n^{hk},(T_n-m^h)_{xx})&=((T_n-T_n^{hk})_x,(T_n-m^h)_x),
\end{aligned}
$$

using several times Cauchy's inequality (3.10) we obtain, for all $m^h\in E^h$,

$$
\begin{aligned}
&\|(T_n-T_n^{hk})_x\|^2+\|(T_n-T_n^{hk})_{xx}\|^2\\
&\qquad\le C\big(\|(T_n-m^h)_{xx}\|^2+\|\theta_n-\theta_n^{hk}\|^2+\|(T_n-m^h)_x\|^2\big).
\end{aligned}\tag{3.14}
$$

Now, from estimates (3.13) and (3.14) we find that

$$
\begin{aligned}
&\|v_n-v_n^{hk}\|^2+\|(u_n-u_n^{hk})_x\|^2+\|\xi_n-\xi_n^{hk}\|^2+\|(T_n-T_n^{hk})_x\|^2\\
&\quad+\|(T_n-T_n^{hk})_{xx}\|^2+Ck\sum_{j=1}^{n}(\theta_j-\theta_j^{hk},(\delta u_j-\delta u_j^{hk})_x)\\
&\le Ck\sum_{j=1}^{n}\big(\|\dot v_j-\delta v_j\|^2+\|\dot u_j-\delta u_j\|_V^2+\|v_j-w_j^h\|_V^2+\|(u_j-u_j^{hk})_x\|^2\\
&\quad+\|\xi_j-\xi_j^{hk}\|^2+\|v_j-v_j^{hk}\|^2+\|\dot\xi_j-\delta\xi_j\|^2+\|\xi_j-r_j^h\|^2\\
&\quad+((\delta u_j-\delta u_j^{hk})_x,\xi_j-r_j^h)+(\delta v_j-\delta v_j^{hk},v_j-w_j^h)\\
&\quad+\|(T_j-T_j^{hk})_{xx}\|^2+(\delta\xi_j-\delta\xi_j^{hk},\xi_j-r_j^h)\\
&\quad+\|\theta_j-\theta_j^{hk}\|^2+\|(\beta_j-\beta_j^{hk})_{xx}\|^2\big)+C\big(\|T_n-m^h\|_{H^2(0,\ell)}^2+\|\theta_n-\theta_n^{hk}\|^2\big)\\
&\quad+C\big(\|v_0-v_0^h\|^2+\|(u_0-u_0^h)_x\|^2+\|\xi_0-\xi_0^h\|^2\big).
\end{aligned}
$$

Keeping in mind that:

$$\sum_{j=1}^{n}(v_j - v_j^{hk} - (v_{j-1} - v_{j-1}^{hk}), v_j - w_j^h)$$
$$= (v_n - v_n^{hk}, v_n - w_n^h) + (v_0^h - v_0, v_1 - w_1^h) + \sum_{j=1}^{n-1}(v_j - v_j^{hk}, v_j - w_j^h - (v_{j+1} - w_{j+1}^h)),$$

$$\sum_{j=1}^{n}(\xi_j - \xi_j^{hk} - (\xi_{j-1} - \xi_{j-1}^{hk}), \xi_j - r_j^h)$$
$$= (\xi_n - \xi_n^{hk}, \xi_n - r_n^h) + (\xi_0^h - \xi_0, \xi_1 - r_1^h) + \sum_{j=1}^{n-1}(\xi_j - \xi_j^{hk}, \xi_j - r_j^h - (\xi_{j+1} - r_{j+1}^h)),$$

$$\sum_{j=1}^{n}((u_j - u_j^{hk} - (u_{j-1} - u_{j-1}^{hk}))_x, \xi_j - r_j^h) = ((u_n - u_n^{hk})_x, \xi_n - r_n^h)$$
$$+ ((u_0^h - u_0)_x, \xi_1 - r_1^h) + \sum_{j=1}^{n-1}((u_j - u_j^{hk})_x, \xi_j - r_j^h - (\xi_{j+1} - r_{j+1}^h)),$$

$$\sum_{j=1}^{n}(\theta_j - \theta_j^{hk}, (u_j - u_j^{hk} - (u_{j-1} - u_{j-1}^{hk}))_x) = (\theta_n - \theta_n^{hk}, (u_n - u_n^{hk})_x)$$
$$- (\theta_0 - \theta_0^h, (u_0 - u_0^h)_x) - k\sum_{j=1}^{n}(\delta\theta_j - \delta\theta_j^{hk}, (u_j - u_j^{hk})_x),$$

$$\|\alpha_n - \alpha_n^{hk}\|^2 \le C\Big(\|\alpha_0 - \alpha_0^h\|^2 + I_n^1 + k\sum_{j=1}^{n}\|\theta_n - \theta_n^{hk}\|^2\Big),$$

$$\|\theta_n - \theta_n^{hk}\|^2 \le C\Big(\|\theta_0 - \theta_0^h\|^2 + I_n^2 + k\sum_{j=1}^{n}\|\xi_n - \xi_n^{hk}\|^2\Big),$$

$$\|(\beta_n - \beta_n^{hk})_{xx}\|^2 \le C\Big(\|(\beta_0 - \beta_0^h)_{xx}\|^2 + I_n^3 + k\sum_{j=1}^{n}\|(T_n - T_n^{hk})_{xx}\|^2\Big),$$

where I_n^1, I_n^2 and I_n^3 are the integration errors defined as:

$$I_n^1 = \Big\|\int_0^{t_n}\theta(s)\,\mathrm{d}s - k\sum_{j=1}^{n}\theta_j\Big\|^2,$$

$$I_n^2 = \Big\|\int_0^{t_n}\xi(s)\,\mathrm{d}s - k\sum_{j=1}^{n}\xi_j\Big\|^2,$$

$$I_n^3 = \Big\|\int_0^{t_n}T_{xx}(s)\,\mathrm{d}s - k\sum_{j=1}^{n}(T_{xx})_j\Big\|^2,$$

and applying a discrete version of Gronwall's inequality (see, again, [34]) it leads to the following a priori error estimates result.

THEOREM 2. *Let the assumptions of Theorem 1 still hold. If we denote by $(u, v, \alpha, \theta, \xi, \beta, T)$ the solution to problem (2.5)–(2.8) and by $(u^{hk}, v^{hk}, \alpha^{hk}, \theta^{hk}, \xi^{hk}, \beta^{hk}, T^{hk})$ the solution to problem (3.6)–(3.9), then we have the following a priori error estimates, for all $w^h = \{w_j^h\}_{j=0}^N \subset V^h$, and $r^h = \{r_j^h\}_{j=0}^N \subset W^h$, $m^h = \{m_j^h\}_{j=0}^N \subset E^h$,*

$$
\begin{aligned}
&\max_{0\le n\le N}\{\|v_n - v_n^{hk}\|^2 + \|u_n - u_n^{hk}\|^2_{H^1(0,\ell)} + \|\xi_n - \xi_n^{hk}\|^2 + \|\theta_n - \theta_n^{hk}\|^2 \\
&\quad + \|\alpha_n - \alpha_n^{hk}\|^2 + \|T_n - T_n^{hk}\|^2_{H^2(0,\ell)} + \|\beta_n - \beta_n^{hk}\|^2_{H^2(0,\ell)}\} \\
&\le Ck\sum_{j=1}^{N}\big(\|\dot v_j - \delta v_j\|^2 + \|\dot u_j - \delta u_j\|^2_{H^1(0,\ell)} + \|v_j - w_j^h\|^2_{H^1(0,\ell)} + \|\dot\xi_j - \delta\xi_j\|^2 \\
&\quad + \|\xi_j - r_j^h\|^2 + I_j^1 + I_j^2 + I_j^3 + \|\dot\theta_j - \delta\theta_j\|^2\big) \\
&\quad + C\max_{0\le n\le N}\|v_n - w_n^h\|^2 + C\max_{0\le n\le N}\|\xi_n - r_n^h\|^2 + C\max_{0\le n\le N}\|T_n - m_n^h\|^2_{H^2(0,\ell)} \\
&\quad + \frac{C}{k}\sum_{j=1}^{N-1}\big(\|v_j - w_j^h - (v_{j+1} - w_{j+1}^h)\|^2 + \|\xi_j - r_j^h - (\xi_{j+1} - r_{j+1}^h)\|^2\big) \\
&\quad + C\big(\|v_0 - v_0^h\|^2 + \|u_0 - u_0^h\|^2_{H^1(0,\ell)} + \|\xi_0 - \xi_0^h\|^2 + \|\theta_0 - \theta_0^h\|^2 + \|\alpha_0 - \alpha_0^h\|^2 \\
&\quad + \|\beta_0 - \beta_0^h\|^2_{H^2(0,\ell)} + \|T_0 - T_0^h\|^2_{H^2(0,\ell)}\big),
\end{aligned}
$$

where C is a positive constant which does not depend on parameters h and k and we have used the notation $\|\cdot\|_X$ to represent the norm of the Hilbert space X.

REMARK 1. From the above estimates we can obtain the convergence order of the approximations provided by the discrete problem (3.6)–(3.9). So, if we assume the following additional regularity:

$$
\begin{aligned}
&u \in H^3(0,T;Y) \cap H^2(0,T;V) \cap \mathcal{C}^1([0,T];H^2(0,\ell)),\\
&\alpha \in H^4(0,T;Y) \cap C^2([0,T];H^1(0,\ell)) \cap H^3(0,T;V),\\
&\beta \in H^2(0,T;Y) \cap C^1([0,T];H^3(0,\ell)),
\end{aligned}
\tag{3.15}
$$

applying some results on the approximation by finite elements (see [33]) and the estimates (see [34]):

$$
\begin{aligned}
&\frac{C}{k}\sum_{j=1}^{N-1}\big(\|v_j - w_j^h - (v_{j+1} - w_{j+1}^h)\|^2 + \|\xi_j - r_j^h - (\xi_{j+1} - r_{j+1}^h)\|^2\big) \\
&\qquad\qquad \le C(h^2 + k^2)(\|u\|^2_{H^2(0,T;V)} + \|\alpha\|^2_{H^3(0,T;V)}),
\end{aligned}
$$

we obtain the linear convergence of the algorithm. That is, it follows that there exists a positive constant $C>0$, independent of the discretization parameters h and k, such that

$$\max_{0\le n\le N}\big\{\|v_n-v_n^{hk}\|+\|u_n-u_n^{hk}\|_{H^1(0,\ell)}+\|\xi_n-\xi_n^{hk}\|+\|\theta_n-\theta_n^{hk}\| \\ +\|\alpha_n-\alpha_n^{hk}\|+\|T_n-T_n^{hk}\|_{H^2(0,\ell)}+\|\beta_n-\beta_n^{hk}\|_{H^2(0,\ell)}\big\}\le C(h+k).$$

4. Numerical results

In this final section, we show some numerical results solving the discrete problem analyzed in the previous section, including the numerical convergence and the discrete energy decay.

We note that the numerical scheme was implemented on a 1.8 GHz PC using MATLAB, and a typical run ($h=k=0.001$ for a final time $T_f=1$ and length of the beam $\ell=1$) took about 3 seconds of CPU time.

As an academical example, in order to show the accuracy of the approximation we consider problem (2.1)–(2.3) with the data:

$$\ell=1,\quad T_f=0.5,\quad \rho=2,\quad \mu=10,\quad \beta^*=2,$$
$$c=2,\quad \tau=0.5,\quad \kappa=2,\quad \kappa^*=1,\quad a=2.$$

By using the following initial conditions, for all $x\in(0,1)$,

$$u_0(x)=v_0(x)=x^3(x-1)^3,\qquad \beta_0(x)=T_0(x)=\vartheta_0(x)=7x^3(x-1)^3,$$

and considering the (artificial) supply terms, for all $(x,t)\in(0,1)\times(0,1)$,

$$F_1(x,t)=-e^t(2x^6-132x^5+21x^4+5386x^3-7857x^2+3084x-252),$$
$$F_2(x,t)=0,$$
$$F_3(x,t)=-3xe^t(-7x^5+25x^4+599x^3-1245x^2+754x-126),$$

the exact solution to the above problem can be easily calculated and it has the form, for $(x,t)\in[0,1]\times[0,1]$:

$$u(x,t)=e^tx^3(x-1)^3,\qquad \beta(x,t)=7e^tx^3(x-1)^3,$$
$$\alpha(x,t)=\theta(x,t)=-7xe^t(-x^5+3x^4+27x^3-59x^2+36x-6).$$

We note that functions F_1, F_2 and F_3 are used to obtain an easy exact solution. The analysis of this slightly modified problem is done as in the previous sections with some straightforward changes.

Thus, the approximation errors estimated by

$$\max_{0\le n\le N}\big\{\|v_n-v_n^{hk}\|+\|u_n-u_n^{hk}\|_{H^1(0,\ell)}+\|\xi_n-\xi_n^{hk}\|+\|\theta_n-\theta_n^{hk}\| \\ +\|\alpha_n-\alpha_n^{hk}\|+\|T_n-T_n^{hk}\|_{H^2(0,\ell)}+\|\beta_n-\beta_n^{hk}\|_{H^2(0,\ell)}\big\}$$

are presented in Table 1 for several values of the discretization parameters h and k. Moreover, by using the diagonal of this table the evolution of the error depending on the parameter $h+k$ is plotted in Fig. 1. We notice that the convergence of the algorithm is clearly observed, although the linear convergence, stated in Remark 1, is not achieved. This fact has also been observed in the analysis of other two-temperatures problems and maybe it is due to a superconvergence property which arises in this kind of problems.

Table 1. Numerical errors for some h and k.

$h \downarrow k \rightarrow$	0.1	0.05	0.01	0.005	0.001	0.0005	0.0001
0.1	1.455431	1.546087	1.753793	1.809254	1.866552	1.875182	1.882523
0.05	0.388811	0.341761	0.427286	0.449906	0.471635	0.474768	0.477426
0.01	0.402299	0.196183	0.034794	0.018994	0.017558	0.018637	0.019638
0.005	0.408483	0.202239	0.038642	0.018471	0.004642	0.004409	0.005047
0.001	0.410633	0.204409	0.040826	0.020442	0.003975	0.001969	0.000767

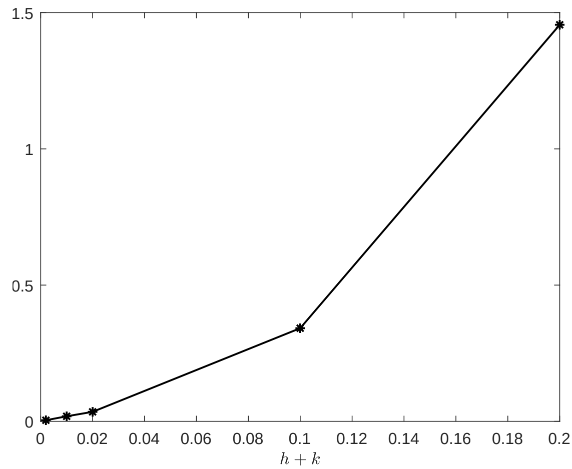

Fig. 1. Asymptotic error.

Now, we consider the evolution of the discrete energy for different values of the diffusion coefficient κ. We assume that there are not supply terms, and we use the final time $T_f = 100$, the data:

$$\ell = 1, \quad \rho = 2, \quad \mu = 1, \quad \beta^* = 0.5, \quad c = 2, \quad \tau = 0.1, \quad \kappa^* = 10, \quad a = 2,$$

and the initial conditions, for all $x \in (0, 1)$,

$$u_0(x) = v_0(x) = x^3(x-1)^3, \qquad \beta_0(x) = T_0(x) = \vartheta_0(x) = x^3(x-1)^3.$$

Taking the discretization parameters $h = 0.002$ and $k = 0.001$, the evolution in time of the discrete energy given by

$$\mathcal{E}^{hk} = \frac{1}{2}\int_0^\ell \Big(\rho(v_n^{hk})^2 + \mu(u_{nx}^{hk})^2 + c(\theta_n^{hk} + \tau\xi_n^{hk})^2 + \kappa^*(\beta_{nx}^{hk} + \tau T_{nx}^{hk})^2 + \tau\overline{\kappa}(T_{nx}^{hk})^2 + a\kappa^*(\beta_{nxx}^{hk} + \tau T_{nxx}^{hk})^2 + \tau a\overline{\kappa}(T_{nxx}^{hk})^2\Big)\,\mathrm{d}x,$$

where we recall that $\overline{\kappa} = \kappa - \tau\kappa^*$, is plotted in Fig. 2 for some values of the diffusion coefficient κ (in both natural and semi-log scales). As we can see, the exponential decay seems to be achieved for every value of parameter κ although the energy decreases in a slower way, when it increases. A possible justification could be that the beam becomes more rigid, in its elastic part, and it is more difficult to stabilize the system.

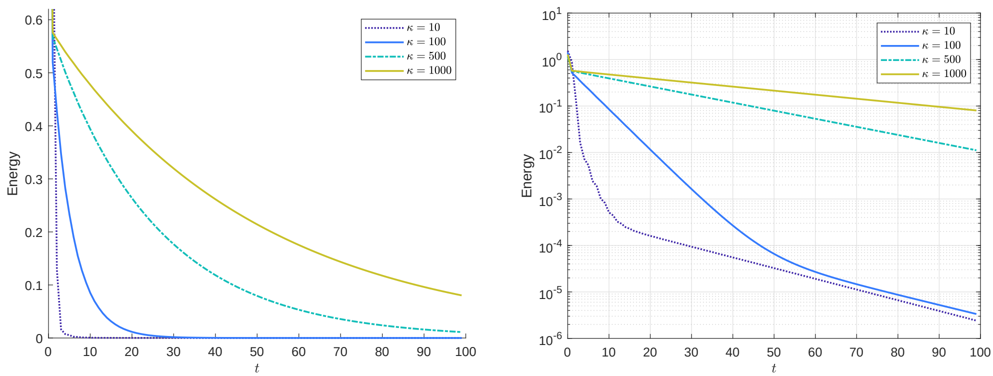

Fig. 2. Evolution in time of the discrete energy depending on the diffusion parameter κ (natural and semi-log scales).

5. Conclusions

In this work, we analyzed, from the numerical point of view, a dynamic thermoelastic problem with two temperatures (the classical and the inductive temperatures). The so-called Moore–Gibson–Thompson equation was used in the modeling of the thermal part. An existence and uniqueness result proved in [32] was recalled. Then, by using the classical finite element method and the implicit Euler scheme we introduced a fully discrete approximation of the resulting variational problem. An a priori error estimates were obtained, after some tedious algebraic manipulations, by using a discrete version of Gronwall's inequality. Finally, some numerical results were performed to demonstrate the behavior of the algorithm (the convergence was shown in the first example), the exponential decay of the discrete energy and the dependence on a diffusion parameter.

The obtained behavior for the solutions is similar to the one found from the empirical point of view. Therefore, the study described previously gives a first positive reason to consider the proposed thermoelastic system from a realistic (mechanical) point of view.

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