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CONTROL AND INFORMATICS

Static anti-windup compensator based on BMI optimisation for discrete-time systems with cut-off constraints

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Abstract. In the paper, a design method of a static anti-windup compensator for systems with input saturations is proposed. First, an anti-windup controller is presented for system with cut-off saturations, and, secondly, the design problem of the compensator is presented to be a non-convex optimization problem easily solved using bilinear matrix inequalities formulation. This approach guarantees stability of the closed-loop system against saturation nonlinearities and optimizes the robust control performance while the saturation is active.

Key words: anti-windup compensation; bilinear matrix inequalities; cut-off constraint.

1. Introduction

Actuator saturation, a different name to control input saturation, is the most common nonlinearity encountered when dealing with control systems, and is given rise by finite possibility to apply calculated control signals due to e.g. safety requirements. One can distinguish two major strands in dealing with this topis. The first approach is based on designing a nominal linear controller and ignoring the actuator saturation. After that, anti-windup compensators are introduced to the system to deal with negative effects of the saturation. A frequent choice is to include some compensator which reduces the discrepancy between the internal controller states and its output. The second approach is to take the saturation constraint(s) from the beginning of the design task into account. This usually calls for nonlinear system analysis or requires some optimization procedures for nonlinear programming tasks to be used. Also, stability criteria, such as the Lyapunov method, can be used here, to impose additional requirements on the closed-loop system.

The paper [1] shed some light on the way to evaluate control quality in anti-windup-compensated (AWC) control systems. The procedures described there allow one to compensate windup phenomenon by means of static AWC for continuous-time control systems. Still, the problem for discrete-time systems has been open. This paper aims to solve it using optimisation techniques.

The optimisation problems formulated henceforth have the form of minimisation of a linear function subject to bilinear matrix inequality constraints (BMIs) [2]. Such problems cannot be solved with the use of algorithms dedicated for linear matrix inequalities (LMIs) [3–5] in any other way as bootstrapping-like procedure that is formed by consecutive iterations some variables are constant and some are computed [1]. Nevertheless, such tasks are solved more effectively [2] with the use of special commercial algorithms, such as PENBMI (PENOPT suite) [6–8], TOMLAB-PENBMI [9] or freeware toolboxes as BMIsolver [10] for some types of BMI constraints. No matter if the problems are in BMI or LMI-type, the most convenient tool modeling their structure is Yalmip [11], and CVX [12, 13] in other cases.

The results presented in the literature [14–16] present synthesis algorithms of static AWCs for continuous-time systems with the use of BMIs for cut-off constraints. This paper presents the corresponding results in a discrete-time case. A simple, non-optimized, static anti-windup compensator deployment is reported, e.g. in [17], where the Authors use it to reduce the windup effect on the performance of the permanent magnet synchronous motor control system.

In [18], a synthesis method to obtain a globally stable solution while maintaining high control performance is presented. The authors also take induced L_2 norm into account and saturating inputs, though in their approach they aim at performing simultaneous design of a controller and a static anti-windup compensator on the basis of LMI conditions. In their paper, the inequality condition binding $\underline{v}$ and $\underline{\eta}$ is used, on the contrary to the approach presented in this paper. Their final optimization problem is of weighed-cost function type, taking both the performance in linear and active-saturation regimes. The result presented in this paper, presents the final optimization task as a single-criterion optimization problem. In addition, the authors of [18] present their results for a single-input single-output (SISO) case only. Similarly, the authors of [19] use conical sector conditions to present their results in an alike setup for a SISO system.

A static AWC design is also presented in [20], where the author considered a piecewise linear system representation, partitioning the state-space description. No optimization procedure is used, just a hyperplane-based partition between saturated and

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unsaturated behaviour to find the parameters of a static AWC to satisfy LMI conditions.

On the contrary to the results presented above, the authors of [21] present their results for the multivariable case, though for a sliding-mode controller with directionality compensation, which is a very interesting area of research, present also in the D.Sc. monograph [22]. They also use LMI conditions to ensure stability of the closed-loop system, though the gains of the AWC obtained there refer to a different structure of the control system, with a specific controller.

A classical results on the static AWC can be traced back to [23], where the authors obtain a bilinear matrix inequality representation, making the problem non-convex. As in the case of this paper, the authors consider the system with a predesigned controller, making the performance in the unsaturated case acceptable. The representation of the constraints is through a deadzone nonlinearity though. Making some assumptions, they reduce (simplify) the problem to a LMI type. In their approach, the H_∞ gain of disturbance-to-output transfer function is minimized.

In the approach considered in this paper, we present a BMI-based derivation (non-convex problem, with no simplifying assumptions made) of the optimal static AWC, making it possible to consider robustness issues (by introduction of polytopic representation of the plants), and to perform calculations for arbitrary controllers, obtained from specific control laws, such as of linear-quadratic type, satisfying some performance requirements in a linear case. In addition, the method allows one to calculate the gains in an off-line manner, where a single choice of the AWC matrix gain $\boldsymbol{\Lambda}$ is done by the optimization algorithm, with the estimate of the performance given by δ. Furthermore, the LFT representation derived in the paper opens possible research directions to consider directional change in controls, by introducing specific saturation functions into the BMI conditions. A single solution obtained from the BMI problem makes the computational burden small, as modern BMI solvers offer very good performance, and are being continuously developed, by using efficient interior-point optimization methods. Of course non-convexity causes some problems from the viewpoint of the accuracy of the solution, though due to the frontiers of optimization still pushed aside, this is not a serious issue of the approach.

The paper is structured as follows: Section 2 presents system description, and its reformulation to the linear fractional form. In Section 3, stability conditions imposed on the closed-loop system are defined in terms of the decision variables of the optimization problem. Section 4 presents an example, and the summary follows in Section 5.

2. Control system description, linear fractional form

It is assumed that a plant and a controller are described with the use of linear continuous-time state-space equations. The following structure description holds for:

- the plant

$$\underline{x}_{p,t+1} = \boldsymbol{A}_p \underline{x}_{p,t} + \boldsymbol{B}_p \underline{u}_t\,, \tag{1}$$

$$\underline{y}_t = \boldsymbol{C}_p \underline{x}_{p,t} + \boldsymbol{D}_p \underline{u}_t\,, \tag{2}$$

- the controller

$$\underline{x}_{c,t+1} = \boldsymbol{A}_c \underline{x}_{c,t} + \boldsymbol{B}_c \underline{e}_t + \underline{\xi}_t\,, \tag{3}$$

$$\underline{v}_t = \boldsymbol{C}_c \underline{x}_{c,t} + \boldsymbol{D}_c \underline{e}_t\,, \tag{4}$$

where the appropriate matrices $\boldsymbol{A}_p \in \mathscr{R}^{n\times n}$, $\boldsymbol{B}_p \in \mathscr{R}^{n\times m}$, $\boldsymbol{C}_p \in \mathscr{R}^{p\times n}$, $\boldsymbol{D}_p \in \mathscr{R}^{p\times m}$, $\boldsymbol{A}_c \in \mathscr{R}^{n_c\times n_c}$, $\boldsymbol{B}_c \in \mathscr{R}^{n_c\times p}$, $\boldsymbol{C}_c \in \mathscr{R}^{m\times n_c}$, $\boldsymbol{D}_c \in \mathscr{R}^{m\times p}$. It is assumed that the vector $\underline{\xi}$ modifies the controller state vector, with n_c referring to its length. The general block diagram (with time indices omitted) of the considered control system is presented in Fig. 1.

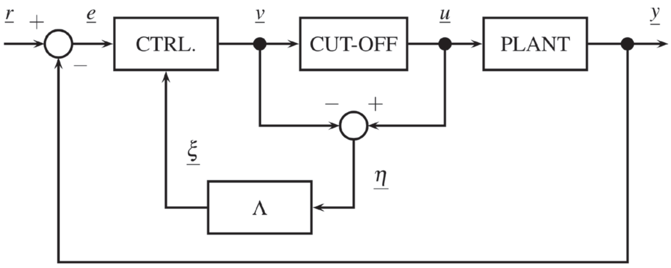

Fig. 1. Block diagram of the proposed control system with static AWC ($\boldsymbol{\Lambda}$)

According to the adopted convention, it holds that: state of the plant $\underline{x}_p \in \mathscr{R}^n$, output of the closed-loop system $\underline{y} \in \mathscr{R}^p$, applied control vector $\underline{u} \in \mathscr{R}^m$, calculated control vector $\underline{v} \in \mathscr{R}^m$, error vector $\underline{e} \in \mathscr{R}^p$, reference vector $\underline{r} \in \mathscr{R}^p$, compensator vector $\underline{\xi} \in \mathscr{R}^{n_c}$, with $\underline{e} = \underline{r} - \underline{y}$.

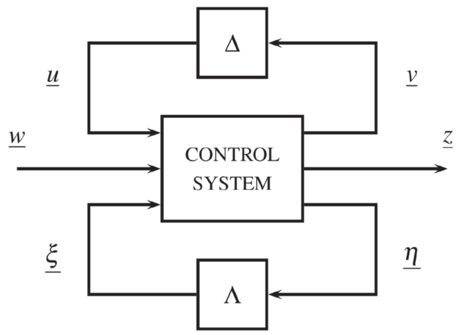

Fig. 2. LFT description of the control system

The cut-off saturation is considered, which in the case of amplitude constraints can be presented as a result of the operation $\boldsymbol{\Delta}_t \underline{v}_t$, where $\boldsymbol{\Delta}_t = \mathrm{diag}\{\Delta_{1,t}, \ldots, \Delta_{m,t}\}$ with $|\Delta_{i,t}| \leq 1$ for $i = 1, \ldots, m$ [24], and diag is used to represent diagonal matrix components.

The control vector $\underline{\xi}$, allowing one to modify controller states is defined as

$$\underline{\xi} = \boldsymbol{\Lambda}\underline{\eta} = \boldsymbol{\Lambda}(\underline{u} - \underline{v})\,. \tag{5}$$

On the basis of (1)–(4), with q as the shift operator, e.g. $qx_t = x_{t+1}$, and having introduced the notation $D = q$ for compatibility purposes with robust control literature, state-space equations for the linear fractional transformation, LFT, description (Fig. 2) can be put in the form [25]

$$
\begin{aligned}
D\underline{x} &= \begin{bmatrix} D\underline{x}_p \\ D\underline{x}_c \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}_p\underline{x}_p + \boldsymbol{B}_p\underline{u} \\ \boldsymbol{A}_c\underline{x}_c + \boldsymbol{B}_c(\underline{w}-\underline{y}) + \underline{\xi} \end{bmatrix} = \\
&= \begin{bmatrix} \boldsymbol{A}_p\underline{x}_p + \boldsymbol{B}_p\underline{u} \\ \boldsymbol{A}_c\underline{x}_c + \boldsymbol{B}_c\underline{w} - \boldsymbol{B}_c\left(\boldsymbol{C}_p\underline{x}_p + \boldsymbol{D}_p\underline{u}\right) + \underline{\xi} \end{bmatrix} = \\
&= \mathscr{A}\,\underline{x} + \mathscr{B}_u\,\underline{u} + \mathscr{B}_w\,\underline{w} + \mathscr{B}_\xi\,\underline{\xi}, \qquad (6)
\end{aligned}
$$

where $\underline{w} = \underline{r}$, $\underline{z} = \underline{e}$ (note this substitution is not necessary, nevertheless it bridges a gap between standard state-space description notation, and robust-control notation, widely used in the papers referring to matrix-inequality-involved problems, see e.g. [4]) and:

$$
\mathscr{A} = \begin{bmatrix} \boldsymbol{A}_p & \boldsymbol{0}^{n\times n_c} \\ -\boldsymbol{B}_c\boldsymbol{C}_p & \boldsymbol{A}_c \end{bmatrix}, \qquad (7)
$$

$$
\mathscr{B}_u = \begin{bmatrix} \boldsymbol{B}_p \\ -\boldsymbol{B}_c\boldsymbol{D}_p \end{bmatrix}, \qquad (8)
$$

$$
\mathscr{B}_w = \begin{bmatrix} \boldsymbol{0}^{n\times p} \\ \boldsymbol{B}_c \end{bmatrix}, \qquad (9)
$$

$$
\mathscr{B}_\xi = \begin{bmatrix} \boldsymbol{0}^{n\times n_c} \\ \boldsymbol{I}^{n_c\times n_c} \end{bmatrix}. \qquad (10)
$$

Eventually, the static AWC is to be proposed for LFT description [26].

In a similar way one can write

$$
\begin{aligned}
\underline{v} &= \boldsymbol{C}_c\underline{x}_c + \boldsymbol{D}_c\underline{e} = \boldsymbol{C}_c\underline{x}_c + \boldsymbol{D}_c(\underline{w}-\underline{y}) = \\
&= \boldsymbol{C}_c\underline{x}_c + \boldsymbol{D}_c\underline{w} - \boldsymbol{D}_c\left(\boldsymbol{C}_p\underline{x}_p + \boldsymbol{D}_p\underline{u}\right) = \\
&= \mathscr{C}_v\underline{x} + \mathscr{D}_{vu}\underline{u} + \mathscr{D}_{vw}\underline{w} + \mathscr{D}_{v\xi}\underline{\xi}, \qquad (11)
\end{aligned}
$$

where:

$$
\mathscr{C}_v = [-\boldsymbol{D}_c\boldsymbol{C}_p, \boldsymbol{C}_c], \qquad \mathscr{D}_{vu} = -\boldsymbol{D}_c\boldsymbol{D}_p, \qquad (12)
$$

$$
\mathscr{D}_{vw} = \boldsymbol{D}_c, \qquad \mathscr{D}_{v\xi} = \boldsymbol{0}^{m\times n_c}. \qquad (13)
$$

The LFT output vector

$$
\begin{aligned}
\underline{z} &= \underline{e} = \underline{w} - \underline{y} = \underline{w} - \boldsymbol{C}_p\underline{x}_p - \boldsymbol{D}_p\underline{u} = \\
&= \mathscr{C}_z\underline{x} + \mathscr{D}_{zu}\underline{u} + \mathscr{D}_{zw}\underline{w} + \mathscr{D}_{z\xi}\underline{\xi}, \qquad (14)
\end{aligned}
$$

with:

$$
\mathscr{C}_z = \left[-\boldsymbol{C}_p, \boldsymbol{0}^{p\times n_c}\right], \qquad \mathscr{D}_{zu} = -\boldsymbol{D}_p, \qquad (15)
$$

$$
\mathscr{D}_{zw} = \boldsymbol{I}^{p\times p}, \qquad \mathscr{D}_{z\xi} = \boldsymbol{0}^{p\times n_c}. \qquad (16)
$$

Because it holds that $\underline{\eta} = \underline{u} - \underline{v}$, one can eliminate $\underline{\xi}$ from LFT form by transforming

$$
\begin{aligned}
\underline{\xi} &= \boldsymbol{\Lambda}\underline{\eta} = \boldsymbol{\Lambda}(\underline{u}-\underline{v}) = \boldsymbol{\Lambda}\underline{u} - \boldsymbol{\Lambda}\underline{v} = \qquad (17)\\
&= \boldsymbol{\Lambda}\underline{u} - \boldsymbol{\Lambda}\mathscr{C}_v\underline{x} - \boldsymbol{\Lambda}\mathscr{D}_{vu}\underline{u} - \boldsymbol{\Lambda}\mathscr{D}_{vw}\underline{w} - \boldsymbol{\Lambda}\mathscr{D}_{v\xi}\underline{\xi},
\end{aligned}
$$

into

$$
\left(\boldsymbol{I} + \boldsymbol{\Lambda}\mathscr{D}_{v\xi}\right)\underline{\xi} = \boldsymbol{\Lambda}\underline{u} - \boldsymbol{\Lambda}\mathscr{C}_v\underline{x} - \boldsymbol{\Lambda}\mathscr{D}_{vu}\underline{u} - \boldsymbol{\Lambda}\mathscr{D}_{vw}\underline{w}, \qquad (18)
$$

$$
\begin{aligned}
\underline{\xi} &= \left(\boldsymbol{I}+\boldsymbol{\Lambda}\mathscr{D}_{v\xi}\right)^{-1}\boldsymbol{\Lambda}\underline{u} - \left(\boldsymbol{I}+\boldsymbol{\Lambda}\mathscr{D}_{v\xi}\right)^{-1}\boldsymbol{\Lambda}\mathscr{C}_v\underline{x} - \\
&- \left(\boldsymbol{I}+\boldsymbol{\Lambda}\mathscr{D}_{v\xi}\right)^{-1}\boldsymbol{\Lambda}\mathscr{D}_{vu}\underline{u} - \left(\boldsymbol{I}+\boldsymbol{\Lambda}\mathscr{D}_{v\xi}\right)^{-1}\boldsymbol{\Lambda}\mathscr{D}_{vw}\underline{w}, \qquad (19)
\end{aligned}
$$

$$
\begin{aligned}
\underline{\xi} &= \boldsymbol{X}\underline{u} - \boldsymbol{X}\mathscr{C}_v\underline{x} - \boldsymbol{X}\mathscr{D}_{vu}\underline{u} - \boldsymbol{X}\mathscr{D}_{vw}\underline{w} = \\
&= -\boldsymbol{X}\mathscr{C}_v\underline{x} + \boldsymbol{X}(\boldsymbol{I}-\mathscr{D}_{vu})\underline{u} - \boldsymbol{X}\mathscr{D}_{vw}\underline{w}, \qquad (20)
\end{aligned}
$$

$$
\boldsymbol{X} = \left(\boldsymbol{I}^{n_c\times n_c} + \boldsymbol{\Lambda}\mathscr{D}_{v\xi}\right)^{-1}\boldsymbol{\Lambda}. \qquad (21)
$$

Having substituted (20) to the previous LFT description of (6), (11) and (14), one obtains:

$$
\begin{aligned}
D\underline{x} &= \mathscr{A}\,\underline{x} + \mathscr{B}_u\underline{u} + \mathscr{B}_w\underline{w} + \mathscr{B}_\xi\underline{\xi} = \\
&= \mathscr{A}\,\underline{x} + \mathscr{B}_u\underline{u} + \mathscr{B}_w\underline{w} + \\
&\quad + \mathscr{B}_\xi\left(-\boldsymbol{X}\mathscr{C}_v\underline{x} + \boldsymbol{X}(\boldsymbol{I}-\mathscr{D}_{vu})\underline{u} - \boldsymbol{X}\mathscr{D}_{vw}\underline{w}\right) = \\
&= \boldsymbol{A}\underline{x} + \boldsymbol{B}_u\underline{u} + \boldsymbol{B}_w\underline{w}, \qquad (22)
\end{aligned}
$$

$$
\begin{aligned}
\underline{v} &= \mathscr{C}_v\underline{x} + \mathscr{D}_{vu}\underline{u} + \mathscr{D}_{vw}\underline{w} + \mathscr{D}_{v\xi}\underline{\xi} = \\
&= \mathscr{C}_v\underline{x} + \mathscr{D}_{vu}\underline{u} + \mathscr{D}_{vw}\underline{w} + \\
&\quad + \mathscr{D}_{v\xi}\left(-\boldsymbol{X}\mathscr{C}_v\underline{x} + \boldsymbol{X}(\boldsymbol{I}-\mathscr{D}_{vu})\underline{u} - \boldsymbol{X}\mathscr{D}_{vw}\underline{w}\right) = \\
&= \boldsymbol{C}_v\underline{x} + \boldsymbol{D}_{vu}\underline{u} + \boldsymbol{D}_{vw}\underline{w}, \qquad (23)
\end{aligned}
$$

$$
\begin{aligned}
\underline{z} &= \mathscr{C}_z\underline{x} + \mathscr{D}_{zu}\underline{u} + \mathscr{D}_{zw}\underline{w} + \mathscr{D}_{z\xi}\underline{\xi} = \\
&= \mathscr{C}_z\underline{x} + \mathscr{D}_{zu}\underline{u} + \mathscr{D}_{zw}\underline{w} + \\
&\quad + \mathscr{D}_{z\xi}\left(-\boldsymbol{X}\mathscr{C}_v\underline{x} + \boldsymbol{X}(\boldsymbol{I}-\mathscr{D}_{vu})\underline{u} - \boldsymbol{X}\mathscr{D}_{vw}\underline{w}\right) = \\
&= \boldsymbol{C}_z\underline{x} + \boldsymbol{D}_{zu}\underline{u} + \boldsymbol{D}_{zw}\underline{w}, \qquad (24)
\end{aligned}
$$

where:

$$
\begin{aligned}
\boldsymbol{A} &= \mathscr{A} - \mathscr{B}_\xi\boldsymbol{X}\mathscr{C}_v, \qquad (25)\\
\boldsymbol{B}_u &= \mathscr{B}_u + \mathscr{B}_\xi\boldsymbol{X}(\boldsymbol{I}-\mathscr{D}_{vu}), \qquad (26)\\
\boldsymbol{B}_w &= \mathscr{B}_w - \mathscr{B}_\xi\boldsymbol{X}\mathscr{D}_{vw}, \qquad (27)\\
\boldsymbol{C}_v &= \mathscr{C}_v - \mathscr{D}_{v\xi}\boldsymbol{X}\mathscr{C}_v, \qquad (28)\\
\boldsymbol{D}_{vu} &= \mathscr{D}_{vu} + \mathscr{D}_{v\xi}\boldsymbol{X}(\boldsymbol{I}-\mathscr{D}_{vu}), \qquad (29)\\
\boldsymbol{D}_{vw} &= \mathscr{D}_{vw} - \mathscr{D}_{v\xi}\boldsymbol{X}\mathscr{D}_{vw}, \qquad (30)\\
\boldsymbol{C}_z &= \mathscr{C}_z - \mathscr{D}_{z\xi}\boldsymbol{X}\mathscr{C}_v, \qquad (31)\\
\boldsymbol{D}_{zu} &= \mathscr{D}_{zu} + \mathscr{D}_{z\xi}\boldsymbol{X}(\boldsymbol{I}-\mathscr{D}_{vu}), \qquad (32)\\
\boldsymbol{D}_{zw} &= \mathscr{D}_{zw} - \mathscr{D}_{z\xi}\boldsymbol{X}\mathscr{D}_{vw}. \qquad (33)
\end{aligned}
$$

One can obtain the value of $\boldsymbol{\Lambda}$ on the basis of $\boldsymbol{X}$ according to the formulae:

$$
\begin{aligned}
\left(\boldsymbol{I}+\boldsymbol{\Lambda}\mathscr{D}_{v\xi}\right)\boldsymbol{X} &= \boldsymbol{\Lambda}, \\
\boldsymbol{X}+\boldsymbol{\Lambda}\mathscr{D}_{v\xi}\boldsymbol{X} &= \boldsymbol{\Lambda}, \\
\boldsymbol{\Lambda}\left(\boldsymbol{I}-\mathscr{D}_{v\xi}\boldsymbol{X}\right) &= \boldsymbol{X}, \\
\boldsymbol{\Lambda} &= \boldsymbol{X}\left(\boldsymbol{I}^{m\times m}-\mathscr{D}_{v\xi}\boldsymbol{X}\right)^{-1}.
\end{aligned} \tag{34}
$$

3. Stability condition

3.1. Mean-square stability. Stability of the closed-loop system will be tested with the use of the small gain theorem in the mean-square sense [27]. Based on the derivations presented in detail in the Appendix, the final matrix-inequality condition takes the form

$$
\begin{bmatrix}
\boldsymbol{A}^T\boldsymbol{P}\boldsymbol{A}-\boldsymbol{P}+\boldsymbol{C}_v{}^T\boldsymbol{\Gamma}\boldsymbol{C}_v & \star & \star \\
\boldsymbol{B}_u{}^T\boldsymbol{P}\boldsymbol{A}+\boldsymbol{D}_{vu}{}^T\boldsymbol{\Gamma}\boldsymbol{C}_v & \boldsymbol{B}_u{}^T\boldsymbol{P}\boldsymbol{B}_u+\boldsymbol{D}_{vu}{}^T\boldsymbol{\Gamma}\boldsymbol{D}_{vu}-\boldsymbol{\Gamma} & \star \\
\boldsymbol{B}_w{}^T\boldsymbol{P}\boldsymbol{A}+\boldsymbol{D}_{vw}{}^T\boldsymbol{\Gamma}\boldsymbol{C}_v & \boldsymbol{B}_w{}^T\boldsymbol{P}\boldsymbol{B}_u+\boldsymbol{D}_{vw}{}^T\boldsymbol{\Gamma}\boldsymbol{D}_{vu} & \boldsymbol{B}_w{}^T\boldsymbol{P}\boldsymbol{B}_w+\boldsymbol{D}_{vw}{}^T\boldsymbol{\Gamma}\boldsymbol{D}_{vw}
\end{bmatrix} \leq 0, \tag{35}
$$

where $\boldsymbol{\Gamma} = \operatorname{diag}\{\boldsymbol{\Gamma}_1,\ldots,\boldsymbol{\Gamma}_m\}$. The condition (35) is synonymous to mean-square stability of the closed-loop system.

3.2. High control performance condition. The compensator is designed with the assumption that whenever the constraints become active, the nonlinear system's behaviour should be as close as possible to the performance of the linear system. According to [4] it is assumed that the high control performance condition can be imposed as the task to achieve the supremum of the induced L_2 norm (time indices omitted again)

$$
\sup_{\|\underline{w}\|_2\neq 0}\frac{\|\underline{z}\|_2}{\|\underline{w}\|_2}. \tag{36}
$$

If for all $\underline{z}$ and $\underline{w}$ it holds that ($\delta\in\mathscr{R}$)

$$
V(\underline{x}_{t+1})-V(\underline{x}_t)+\underline{z}^T\underline{z}-\delta\,\underline{w}^T\underline{w}\leq 0, \tag{37}
$$

the gain (36) estimated by the use of L_2 norm is not greater than $\sqrt{\delta}$.

Following the procedure presented in the Appendix, condition (37) takes the matrix form

$$
\begin{bmatrix}
\boldsymbol{A}^T\boldsymbol{P}\boldsymbol{A}-\boldsymbol{P}+\boldsymbol{C}_z{}^T\boldsymbol{C}_z & \star & \star \\
\boldsymbol{B}_u{}^T\boldsymbol{P}\boldsymbol{A}+\boldsymbol{D}_{zu}{}^T\boldsymbol{C}_z & \boldsymbol{B}_u{}^T\boldsymbol{P}\boldsymbol{B}_u+\boldsymbol{D}_{zu}{}^T\boldsymbol{D}_{zu} & \star \\
\boldsymbol{B}_w{}^T\boldsymbol{P}\boldsymbol{A}+\boldsymbol{D}_{zw}{}^T\boldsymbol{C}_z & \boldsymbol{B}_w{}^T\boldsymbol{P}\boldsymbol{B}_u+\boldsymbol{D}_{zw}{}^T\boldsymbol{D}_{zu} & \boldsymbol{B}_w{}^T\boldsymbol{P}\boldsymbol{B}_w+\boldsymbol{D}_{zw}{}^T\boldsymbol{D}_{zw}-\delta
\end{bmatrix} \leq 0. \tag{38}
$$

3.3. Mean-square stability condition with high control performance requirement. Using the same approach as in the prior derivation, the condition $u_i^T u_i \leq v_i^T v_i$ (for $i = 1,\ldots,m$) is incorporated into (38), and for every t one gets

$$
\begin{bmatrix}
\boldsymbol{A}^T\boldsymbol{P}\boldsymbol{A}-\boldsymbol{P}+\boldsymbol{C}_z{}^T\boldsymbol{C}_z+\boldsymbol{C}_v{}^T\boldsymbol{\Gamma}\boldsymbol{C}_v & \star & \star \\
\boldsymbol{B}_u{}^T\boldsymbol{P}\boldsymbol{A}+\boldsymbol{D}_{zu}{}^T\boldsymbol{C}_z+\boldsymbol{D}_{vu}{}^T\boldsymbol{\Gamma}\boldsymbol{C}_v & \boldsymbol{B}_u{}^T\boldsymbol{P}\boldsymbol{B}_u+\boldsymbol{D}_{zu}{}^T\boldsymbol{D}_{zu}+\boldsymbol{D}_{vu}{}^T\boldsymbol{\Gamma}\boldsymbol{D}_{vu}-\boldsymbol{\Gamma} & \star \\
\boldsymbol{B}_w{}^T\boldsymbol{P}\boldsymbol{A}+\boldsymbol{D}_{zw}{}^T\boldsymbol{C}_z+\boldsymbol{D}_{vw}{}^T\boldsymbol{\Gamma}\boldsymbol{C}_v & \boldsymbol{B}_w{}^T\boldsymbol{P}\boldsymbol{B}_u+\boldsymbol{D}_{zw}{}^T\boldsymbol{D}_{zu}+\boldsymbol{D}_{vw}{}^T\boldsymbol{\Gamma}\boldsymbol{D}_{vu} & \boldsymbol{B}_w{}^T\boldsymbol{P}\boldsymbol{B}_w+\boldsymbol{D}_{zw}{}^T\boldsymbol{D}_{zw}-\delta+\boldsymbol{D}_{vw}{}^T\boldsymbol{\Gamma}\boldsymbol{D}_{vw}
\end{bmatrix} \leq 0. \tag{39}
$$

Using Schur complement the above inequality can be rewritten as

$$
\begin{aligned}
&\begin{bmatrix}
\boldsymbol{A}^T\boldsymbol{P}\boldsymbol{A}-\boldsymbol{P} & \star & \star \\
\boldsymbol{B}_u{}^T\boldsymbol{P}\boldsymbol{A} & \boldsymbol{B}_u{}^T\boldsymbol{P}\boldsymbol{B}_u-\boldsymbol{\Gamma} & \star \\
\boldsymbol{B}_w{}^T\boldsymbol{P}\boldsymbol{A} & \boldsymbol{B}_w{}^T\boldsymbol{P}\boldsymbol{B}_u & \boldsymbol{B}_w{}^T\boldsymbol{P}\boldsymbol{B}_w-\delta
\end{bmatrix} + \\
&+\begin{bmatrix}
\boldsymbol{C}_z{}^T\boldsymbol{C}_z+\boldsymbol{C}_v{}^T\boldsymbol{\Gamma}\boldsymbol{C}_v & \star & \star \\
\boldsymbol{D}_{zu}{}^T\boldsymbol{C}_z+\boldsymbol{D}_{vu}{}^T\boldsymbol{\Gamma}\boldsymbol{C}_v & \boldsymbol{D}_{zu}{}^T\boldsymbol{D}_{zu}+\boldsymbol{D}_{vu}{}^T\boldsymbol{\Gamma}\boldsymbol{D}_{vu} & \star \\
\boldsymbol{D}_{zw}{}^T\boldsymbol{C}_z+\boldsymbol{D}_{vw}{}^T\boldsymbol{\Gamma}\boldsymbol{C}_v & \boldsymbol{D}_{zw}{}^T\boldsymbol{D}_{zu}+\boldsymbol{D}_{vw}{}^T\boldsymbol{\Gamma}\boldsymbol{D}_{vu} & \boldsymbol{D}_{zw}{}^T\boldsymbol{D}_{zw}+\boldsymbol{D}_{vw}{}^T\boldsymbol{\Gamma}\boldsymbol{D}_{vw}
\end{bmatrix} = \\
&=\begin{bmatrix}
\boldsymbol{A}^T\boldsymbol{P}\boldsymbol{A}-\boldsymbol{P} & \star & \star \\
\boldsymbol{B}_u{}^T\boldsymbol{P}\boldsymbol{A} & \boldsymbol{B}_u{}^T\boldsymbol{P}\boldsymbol{B}_u-\boldsymbol{\Gamma} & \star \\
\boldsymbol{B}_w{}^T\boldsymbol{P}\boldsymbol{A} & \boldsymbol{B}_w{}^T\boldsymbol{P}\boldsymbol{B}_u & \boldsymbol{B}_w{}^T\boldsymbol{P}\boldsymbol{B}_w-\delta
\end{bmatrix} - \\
&-\begin{bmatrix}
\boldsymbol{\Gamma}\boldsymbol{C}_v & \boldsymbol{\Gamma}\boldsymbol{D}_{vu} & \boldsymbol{\Gamma}\boldsymbol{D}_{vw} \\
\boldsymbol{C}_z & \boldsymbol{D}_{zu} & \boldsymbol{D}_{zw}
\end{bmatrix}^T
\begin{bmatrix}
-\boldsymbol{\Gamma}^{-1} & \star \\
\boldsymbol{0} & -\boldsymbol{I}
\end{bmatrix} \times \\
&\times\begin{bmatrix}
\boldsymbol{\Gamma}\boldsymbol{C}_v & \boldsymbol{\Gamma}\boldsymbol{D}_{vu} & \boldsymbol{\Gamma}\boldsymbol{D}_{vw} \\
\boldsymbol{C}_z & \boldsymbol{D}_{zu} & \boldsymbol{D}_{zw}
\end{bmatrix} = \\
&=\begin{bmatrix}
\boldsymbol{A}^T\boldsymbol{P}\boldsymbol{A}-\boldsymbol{P} & \star & \star & \star & \star \\
\boldsymbol{B}_u{}^T\boldsymbol{P}\boldsymbol{A} & \boldsymbol{B}_u{}^T\boldsymbol{P}\boldsymbol{B}_u-\boldsymbol{\Gamma} & \star & \star & \star \\
\boldsymbol{B}_w{}^T\boldsymbol{P}\boldsymbol{A} & \boldsymbol{B}_w{}^T\boldsymbol{P}\boldsymbol{B}_u & \boldsymbol{B}_w{}^T\boldsymbol{P}\boldsymbol{B}_w-\delta & \star & \star \\
\boldsymbol{\Gamma}\boldsymbol{C}_v & \boldsymbol{\Gamma}\boldsymbol{D}_{vu} & \boldsymbol{\Gamma}\boldsymbol{D}_{vw} & -\boldsymbol{\Gamma} & \star \\
\boldsymbol{C}_z & \boldsymbol{D}_{zu} & \boldsymbol{D}_{zw} & \boldsymbol{0} & -\boldsymbol{I}
\end{bmatrix} \leq 0.
\end{aligned} \tag{40}
$$

In order to obtain the final form of the matrix inequality conditions, the following notation has been adopted:

$$
\boldsymbol{P}\boldsymbol{A} = \boldsymbol{P}\left(\mathscr{A}-\mathscr{B}_\xi\boldsymbol{X}\mathscr{C}_v\right), \tag{41}
$$

$$
\boldsymbol{P}\boldsymbol{B}_u = \boldsymbol{P}\left(\mathscr{B}_u+\mathscr{B}_\xi\boldsymbol{X}\left(\boldsymbol{I}-\mathscr{D}_{vu}\right)\right), \tag{42}
$$

$$
\boldsymbol{P}\boldsymbol{B}_w = \boldsymbol{P}\left(\mathscr{B}_w-\mathscr{B}_\xi\boldsymbol{X}\mathscr{D}_{vw}\right). \tag{43}
$$

The optimization task which is related to the optimal compensator

$$\begin{array}{ll} \min\limits_{P,X,\Gamma,\delta} & \delta \\ \text{s.t.} & (40) \\ & P > 0, \quad \Gamma > 0, \quad \delta > 0 \end{array} \tag{44}$$

is solved using the algorithms taking BMI conditions into account. The optimal solution corresponds to the minimal value of δ.

4. Simulation results

Two simulation models are taken into consideration. The first one is the TITO (two-input, two-output) plant with:

$$A_p = \begin{bmatrix} 0.9048 & 0 \\ 0 & 0.9048 \end{bmatrix}, \tag{45}$$

$$B_p = \begin{bmatrix} 9.516 & 0 \\ 0 & 9.516 \end{bmatrix}, \tag{46}$$

$$C_p = \begin{bmatrix} 0.4 & -0.5 \\ -0.3 & 0.4 \end{bmatrix}, \tag{47}$$

$$D_p = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{48}$$

and the controller ($n_c = 1$) given by:

$$A_c = 1, \tag{49}$$

$$B_c = \left[7.0710,\ 7.0710\right], \tag{50}$$

$$C_c = \left[0.0318,\ 0.0247\right]^T, \tag{51}$$

$$D_c = \begin{bmatrix} 2.0 & 2.5 \\ 1.5 & 2.0 \end{bmatrix} \tag{52}$$

It is assumed that for $t = 0$ in accordance with [1] a step change in reference vector takes place between $\underline{w} = \underline{0}$ and $\underline{w} = [0.63, 0.79]^T$, and the control vector is saturated using a cut-off function at the level $pm1$, see Fig. 3. As remarked in [1], the shape of the plot $\delta(\mathbf{\Lambda})$ corresponds to a scaled version of the integral of the squared error, ISE performance index plot as a function of $\mathbf{\Lambda}$.

In addition, Fig. 4 shows the results for a cut-off case. As indicated in [1] the level sets of the ISE index have the same shape, and no compensation results in performance deterioration in comparison to the situation where the AWC compensator has been introduced.

In the second case,

$$A_p = \begin{bmatrix} 0.8528 & 0.0019 & -0.0412 & 0.0135 \\ 0.0019 & 0.9173 & 0.0051 & -0.0056 \\ -0.0412 & 0.0051 & 0.8952 & 0.0110 \\ 0.0135 & -0.0056 & 0.0110 & 0.9132 \end{bmatrix},$$

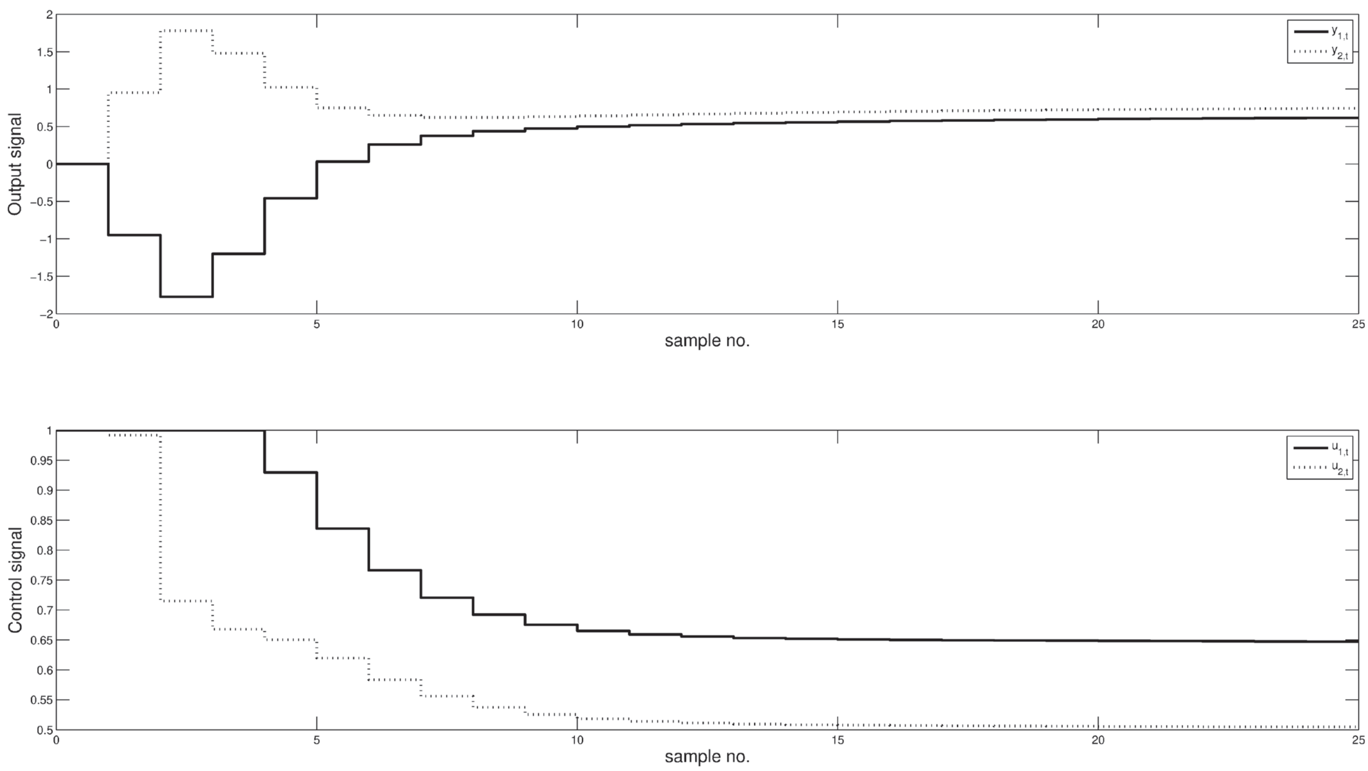

Fig. 3. Tracking performance and constrained control vector (example 1)

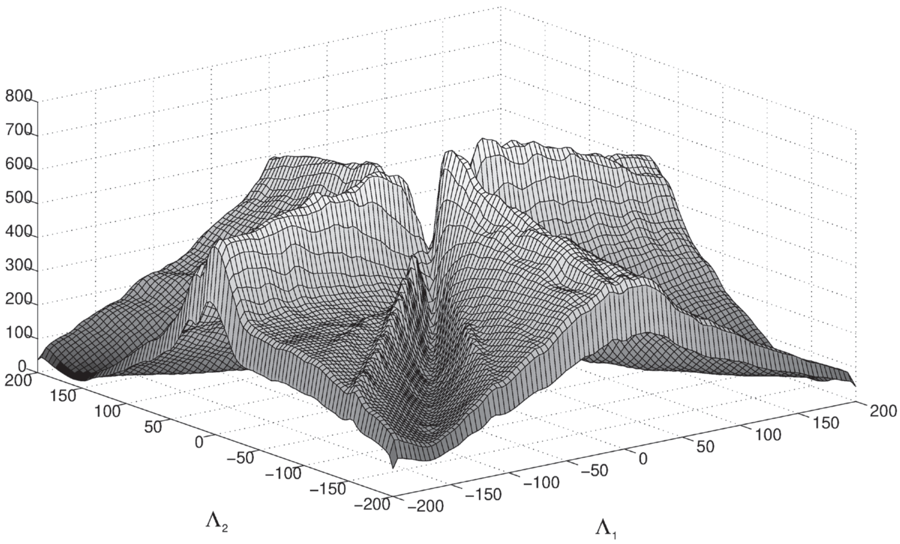

Fig. 4. Values of $\sqrt{\delta}$ with a cut-off control constraint

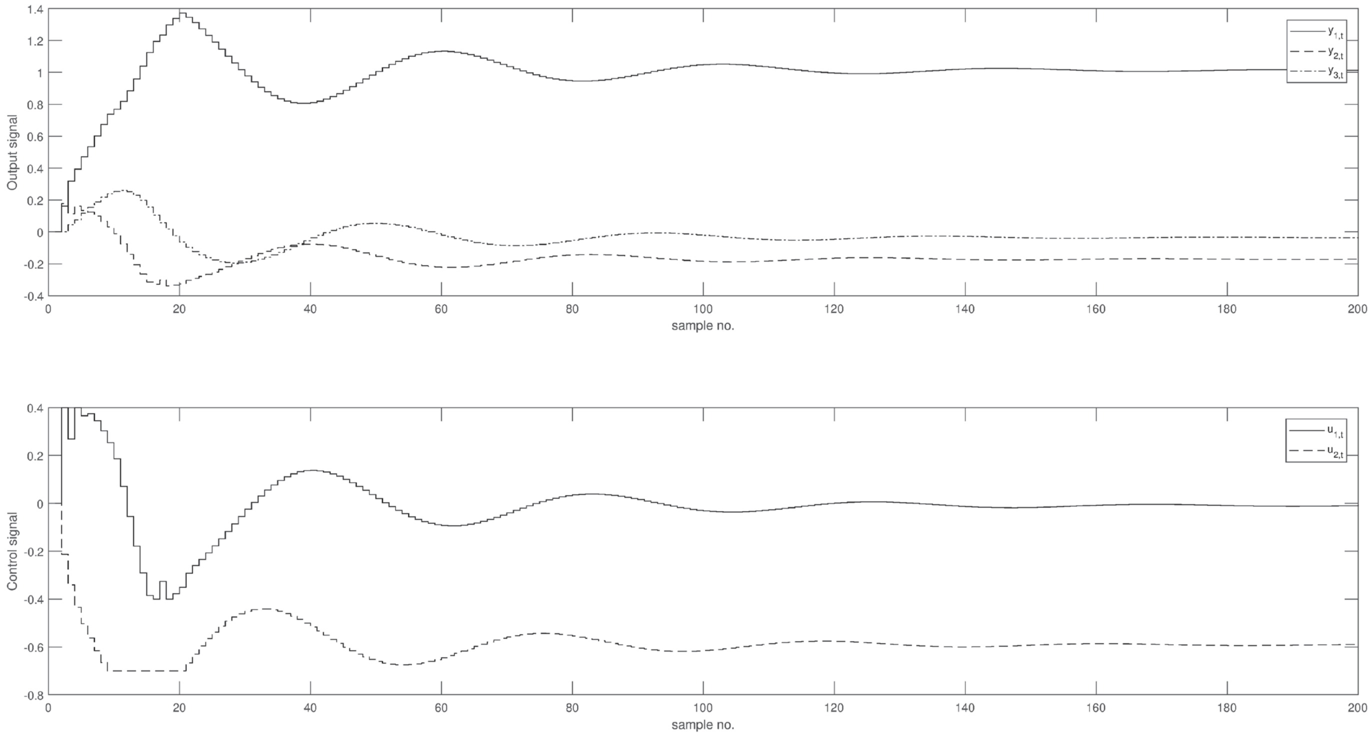

Fig. 5. Tracking performance and constrained control vector (example 2)

$$\boldsymbol{B}_p = \begin{bmatrix} -0.0127 & -0.0794 \\ -0.0146 & -0.0003 \\ -0.0491 & -0.0655 \\ 0.1604 & -0.0010 \end{bmatrix},$$

$$\boldsymbol{C}_p = \begin{bmatrix} -0.1922 & -0.2490 & 1.2347 & -0.4446 \\ -0.2741 & -1.0642 & -0.2296 & -0.1559 \\ 1.5301 & 0 & -1.5062 & 0.2761 \end{bmatrix},$$

$$\boldsymbol{D}_p = \begin{bmatrix} -0.2612 & -1.2507 \\ 0.4434 & 0 \\ 0 & 0 \end{bmatrix},$$

with the nominal controller

$$\boldsymbol{A}_c = \begin{bmatrix} 0.3000 & 0 \\ 0 & 0.6000 \end{bmatrix},$$

$$\boldsymbol{B}_c = \begin{bmatrix} 0.5000 & 0 & 0 \\ 0.1000 & -0.3000 & 1.6000 \end{bmatrix},$$

$$\boldsymbol{C}_c = \begin{bmatrix} 1.0000 & 0 \\ 0 & 0.2500 \end{bmatrix},$$

$$\boldsymbol{D}_c = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and the step change at $t = 0$ is $\underline{w} = [1, 0, 0]^T$, with a cut-off function at both control inputs at the levels ± 0.4 and ± 0.6, respectively. For the tracking performance see Fig. 5.

5. Summary

A method to obtain the optimal compensator feedback matrix has been proposed, in a sense of a supremum of the induced norm, ensuring high control performance and mean-square stability of the closed-loop system. This approach is applicable in all systems where the modification is carried out over the state of the controller, not the calculated control vector directly, as in the other case no optimization algorithm exists to solve the problem (products of the fourth order). Possible solution methods include sequences of BMI problems in bootstrap series, resulting in optimal AWC parameters.

Another possibility is to introduce a nonstationary compensator with a variable feedback matrix, whose parameters $\boldsymbol{\Lambda}$ would be chosen on the basis of a look-up table.

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Appendix

Mean-square stability. The Lyapunov function is sought $V(\underline{x}_t)$ for which $V(\underline{x}_{t+1}) - V(\underline{x}_t) < 0$ holds subject to (time indices omitted) $u_i^T u_i \leq v_i^T v_i$ and for $i = 1, \ldots, m$, for every t ($\underline{x}_t^T = [\underline{x}_{p,t}^T, \underline{x}_{c,t}^T]^T$). Lyapunov function with Eqs. (22) and (24) taken into account satisfies

$$
\begin{aligned}
V(\underline{x}_{t+1}) - V(\underline{x}_t) = \\
&= \underline{x}_{t+1}^T \boldsymbol{P} \underline{x}_{t+1} - \underline{x}_t^T \boldsymbol{P} \underline{x}_t = \\
&= (\boldsymbol{A}\underline{x} + \boldsymbol{B}_u \underline{u} + \boldsymbol{B}_w \underline{w})^T \boldsymbol{P} (\boldsymbol{A}\underline{x} + \boldsymbol{B}_u \underline{u} + \boldsymbol{B}_w \underline{w}) - \underline{x}^T \boldsymbol{P} \underline{x} = \\
&= \left(\underline{x}^T \boldsymbol{A}^T \boldsymbol{P} + \underline{u}^T \boldsymbol{B}_u^T \boldsymbol{P} + \underline{w}^T \boldsymbol{B}_w^T \boldsymbol{P}\right) \times \\
&\quad \times (\boldsymbol{A}\underline{x} + \boldsymbol{B}_u \underline{u} + \boldsymbol{B}_w \underline{w}) - \underline{x}^T \boldsymbol{P} \underline{x} = \\
&= \underline{x}^T \boldsymbol{A}^T \boldsymbol{P} \boldsymbol{A} \underline{x} + \underline{x}^T \boldsymbol{A}^T \boldsymbol{P} \boldsymbol{B}_u \underline{u} + \underline{x}^T \boldsymbol{A}^T \boldsymbol{P} \boldsymbol{B}_w \underline{w} + \\
&\quad + \underline{u}^T \boldsymbol{B}_u^T \boldsymbol{P} \boldsymbol{A} \underline{x} + \underline{u}^T \boldsymbol{B}_u^T \boldsymbol{P} \boldsymbol{B}_u \underline{u} + \underline{u}^T \boldsymbol{B}_u^T \boldsymbol{P} \boldsymbol{B}_w \underline{w} + \\
&\quad + \underline{w}^T \boldsymbol{B}_w^T \boldsymbol{P} \boldsymbol{A} \underline{x} + \underline{w}^T \boldsymbol{B}_w^T \boldsymbol{P} \boldsymbol{B}_u \underline{u} + \underline{w}^T \boldsymbol{B}_w^T \boldsymbol{P} \boldsymbol{B}_w \underline{w} - \underline{x}^T \boldsymbol{P} \underline{x} = \\
&= \underline{x}^T \left[\boldsymbol{A}^T \boldsymbol{P} \boldsymbol{A} - \boldsymbol{P}, \boldsymbol{A}^T \boldsymbol{P} \boldsymbol{B}_u, \boldsymbol{A}^T \boldsymbol{P} \boldsymbol{B}_w\right] \begin{bmatrix} \underline{x} \\ \underline{u} \\ \underline{w} \end{bmatrix} + \\
&\quad + \underline{u}^T \left[\boldsymbol{B}_u^T \boldsymbol{P} \boldsymbol{A}, \boldsymbol{B}_u^T \boldsymbol{P} \boldsymbol{B}_u, \boldsymbol{B}_u^T \boldsymbol{P} \boldsymbol{B}_w\right] \begin{bmatrix} \underline{x} \\ \underline{u} \\ \underline{w} \end{bmatrix} + \\
&\quad + \underline{w}^T \left[\boldsymbol{B}_w^T \boldsymbol{P} \boldsymbol{A}, \boldsymbol{B}_w^T \boldsymbol{P} \boldsymbol{B}_u, \boldsymbol{B}_w^T \boldsymbol{P} \boldsymbol{B}_w\right] \begin{bmatrix} \underline{x} \\ \underline{u} \\ \underline{w} \end{bmatrix},
\end{aligned}
$$

and

$$
\begin{bmatrix} \underline{x} \\ \underline{u} \\ \underline{w} \end{bmatrix}^T
\begin{bmatrix} \boldsymbol{A}^T \boldsymbol{P} \boldsymbol{A} - \boldsymbol{P} & \star & \star \\ \boldsymbol{B}_u{}^T \boldsymbol{P} \boldsymbol{A} & \boldsymbol{B}_u{}^T \boldsymbol{P} \boldsymbol{B}_u & \star \\ \boldsymbol{B}_w{}^T \boldsymbol{P} \boldsymbol{A} & \boldsymbol{B}_w{}^T \boldsymbol{P} \boldsymbol{B}_u & \boldsymbol{B}_w{}^T \boldsymbol{P} \boldsymbol{B}_w \end{bmatrix}
\begin{bmatrix} \underline{x} \\ \underline{u} \\ \underline{w} \end{bmatrix} < 0. \tag{53}
$$

From the condition $u_i^T u_i \leq v_i^T v_i$ for $i = 1, \ldots, m$ and for every t it holds that $v_i^T v_i - u_i^T u_i \geq 0$, thus for $i = 1, \ldots, m$ one can write the constraints in the terms of the rows of appropriate matrices, with the use of (23),

$$
\begin{bmatrix} \underline{x} \\ \underline{u} \\ \underline{w} \end{bmatrix}^T
\begin{bmatrix} \underline{c}_{v,i}{}^T \underline{c}_{v,i} & \star & \star \\ \underline{d}_{vu,i}{}^T \underline{c}_{v,i} & \underline{d}_{vu,i}{}^T \underline{d}_{vu,i} - \boldsymbol{I} & \star \\ \underline{d}_{vw,i}{}^T \underline{c}_{v,i} & \underline{d}_{vw,i}{}^T \underline{d}_{vu,i} & \underline{d}_{vw,i}{}^T \underline{d}_{vw,i} \end{bmatrix} \times
$$
$$
\times \begin{bmatrix} \underline{x} \\ \underline{u} \\ \underline{w} \end{bmatrix} \geq 0. \tag{54}
$$

According to the S-procedure (the name of the procedure dates back to 1979, see [28], where some matrix $\boldsymbol{S}$ was used to represent quadratic forms with symmetry property), the inequality (53) holds if the inequalities (54) hold for $i = 1, \ldots, m$. The formula (53) can now be rewritten to

$$
\begin{aligned}
&-\left(V(\underline{x}_{t+1}) - V(\underline{x}_t)\right) = \\
&- \begin{bmatrix} \underline{x} \\ \underline{u} \\ \underline{w} \end{bmatrix}^T
\begin{bmatrix} \boldsymbol{A}^T \boldsymbol{P} \boldsymbol{A} - \boldsymbol{P} & \star & \star \\ \boldsymbol{B}_u{}^T \boldsymbol{P} \boldsymbol{A} & \boldsymbol{B}_u{}^T \boldsymbol{P} \boldsymbol{B}_u & \star \\ \boldsymbol{B}_w{}^T \boldsymbol{P} \boldsymbol{A} & \boldsymbol{B}_w{}^T \boldsymbol{P} \boldsymbol{B}_u & \boldsymbol{B}_w{}^T \boldsymbol{P} \boldsymbol{B}_w \end{bmatrix}
\begin{bmatrix} \underline{x} \\ \underline{u} \\ \underline{w} \end{bmatrix} > 0,
\end{aligned}
$$

from where having applied the S-procedure [4, 29]

$$
\begin{aligned}
&- \begin{bmatrix} \boldsymbol{A}^T \boldsymbol{P} \boldsymbol{A} - \boldsymbol{P} & \star & \star \\ \boldsymbol{B}_u{}^T \boldsymbol{P} \boldsymbol{A} & \boldsymbol{B}_u{}^T \boldsymbol{P} \boldsymbol{B}_u & \star \\ \boldsymbol{B}_w{}^T \boldsymbol{P} \boldsymbol{A} & \boldsymbol{B}_w{}^T \boldsymbol{P} \boldsymbol{B}_u & \boldsymbol{B}_w{}^T \boldsymbol{P} \boldsymbol{B}_w \end{bmatrix} \geq \\
&\geq \sum_{i=1}^{m} \left(\boldsymbol{\Gamma}_m \begin{bmatrix} \underline{c}_{v,i}{}^T \underline{c}_{v,i} & \star & \star \\ \underline{d}_{vu,i}{}^T \underline{c}_{v,i} & \underline{d}_{vu,i}{}^T \underline{d}_{vu,i} - \boldsymbol{I} & \star \;]! \\ \underline{d}_{vw,i}{}^T \underline{c}_{v,i} & \underline{d}_{vw,i}{}^T \underline{d}_{vu,i} & \underline{d}_{vw,i}{}^T \underline{d}_{vw,i} \end{bmatrix} \right) \\
&= \begin{bmatrix} \boldsymbol{C}_v{}^T \boldsymbol{\Gamma} \boldsymbol{C}_v & \star & \star \\ \boldsymbol{D}_{vu}{}^T \boldsymbol{\Gamma} \boldsymbol{C}_v & \boldsymbol{D}_{vu}{}^T \boldsymbol{\Gamma} \boldsymbol{D}_{vu} - \boldsymbol{\Gamma} & \star \\ \boldsymbol{D}_{vw}{}^T \boldsymbol{\Gamma} \boldsymbol{C}_v & \boldsymbol{D}_{vw}{}^T \boldsymbol{\Gamma} \boldsymbol{D}_{vu} & \boldsymbol{D}_{vw}{}^T \boldsymbol{\Gamma} \boldsymbol{D}_{vw} \end{bmatrix}
\end{aligned} \tag{55}
$$

one obtains (35).

High control performance condition The expression $\underline{z}^T \underline{z} - \delta\, \underline{w}^T \underline{w}$ taking (24) into account can be rewritten using the notation as below:

$$
\begin{aligned}
v &= \underline{z}^T \underline{z} - \delta\, \underline{w}^T \underline{w} = \\
&= (\boldsymbol{C}_z \underline{x} + \boldsymbol{D}_{zu} \underline{u} + \boldsymbol{D}_{zw} \underline{w})^T (\boldsymbol{C}_z \underline{x} + \boldsymbol{D}_{zu} \underline{u} + \boldsymbol{D}_{zw} \underline{w}) - \delta \underline{w}^T \underline{w} = \\
&= \begin{bmatrix} \underline{x} \\ \underline{u} \\ \underline{w} \end{bmatrix}^T
\begin{bmatrix} \boldsymbol{C}_z{}^T \boldsymbol{C}_z & \star & \star \\ \boldsymbol{D}_{zu}{}^T \boldsymbol{C}_z & \boldsymbol{D}_{zu}{}^T \boldsymbol{D}_{zu} & \star \\ \boldsymbol{D}_{zw}{}^T \boldsymbol{C}_z & \boldsymbol{D}_{zw}{}^T \boldsymbol{D}_{zu} & \boldsymbol{D}_{zw}{}^T \boldsymbol{D}_{zw} - \delta \end{bmatrix}
\begin{bmatrix} \underline{x} \\ \underline{u} \\ \underline{w} \end{bmatrix},
\end{aligned} \tag{56}
$$

and then the inequality (37) corresponds to (38).